Bol. Soc. Paran. Mat. (3s.) **v. 38** 1 (2020): 151–163.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i1.38427

Variational Analysis to Fourth-order Impulsive Differential Equations

Saeid Shokooh

ABSTRACT: Applying two critical point theorems, we prove the existence of at least three solutions for a one-dimensional fourth-order impulsive differential equation with two real parameters.

Key Words: Ricceri variational principle, Three solutions, Fourth-order equations.

Contents

1. Introduction

Many dynamical systems describing models in applied sciences have an *impulsive* dynamical behaviour due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to *impulsive differential equations*; they describe various processes of the real world described by models that are subject to *sudden changes* in their states. Essentially, impulsive differential equations correspond to a *smooth evolution* that may *change instantaneously* or even *abruptly*, as happens in various applications that describe mechanical or natural phenomena. These changes correspond to impulses in the smooth system, such as for example in the model of a mechanical clock. Impulsive differential equations also study models in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, chaos theory. Associated with this development, a theory of impulsive differential equations has been given extensive attention. For an introduction of the basic theory of impulsive differential equations in $\mathbb{R}^n$, see [3], [8], and [13]. Some classical tools have been used to study such problems in the literature, such as the coincidence degree theory of Mawhin, the method of upper and lower solutions with the monotone iterative technique, and some fixed point theorems in cones (see [7,9,12]).

Recently, the existence and multiplicity of solutions for impulsive boundary value problems by using variational methods and critical point theory has been considered and here we cite the papers [1,2,10,14,15,16,17,18,19].

2010 *Mathematics Subject Classification:* 35B38, 34B15, 58E050.
Submitted July 30, 2017. Published November 18, 2017

Typeset by $\mathcal{B}^{S}_{\mathcal{M}}\mathcal{P}$ style.
© Soc. Paran. de Mat.

In this paper, motivated by the above facts and the recent paper [4], we consider the fourth-order boundary value problem with impulsive effects

$$\begin{cases} u''''(t) - (p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)), \qquad t \neq t_j,\ t \in [0,1], \\ \Delta(u''(t_j)) = \mu I_{1j}(u'(t_j)), \qquad j = 1, 2, \ldots, m, \\ -\Delta(u'''(t_j)) = \mu I_{2j}(u(t_j)), \qquad j = 1, 2, \ldots, m, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda \in]0, +\infty[$, $\mu \in]0, +\infty[$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $p, q \in L^\infty([0,1])$, $I_{1j}, I_{2j} \in C(\mathbb{R};\mathbb{R})$ for $1 \leq j \leq m$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, and the operator Δ is defined as $\Delta(u(t_j)) := u(t_j^+) - u(t_j^-)$, where $u(t_j^+)$ and $u(t_j^-)$ denote the right and the left limits, respectively, of u at t_j.

By using variational methods, we show the existence of three solutions for this problem. More precisely, by choosing μ in a suitable way and under a growth condition on the nonlinear term we prove that (1.1) has at least three solutions for every λ lying in a precise interval. In particular, we obtain two main theorems. In the first one (Theorem 3.1) we require on the antiderivative of f both a growth more then quadratic in a suitable interval and a growth less then quadratic at infinity, and at the same time, on the impulses I_{1j} and I_{1j}, two asymptotic conditions are required. In the second one (Theorem 3.5) we establish the existence of at least three positive solutions uniformly bounded without asymptotic conditions on f, I_{1j} and I_{2j}.

2. Preliminaries

We now state two critical point theorems which are the main tools for the proofs of our results. The first result has been obtained in [6] and it is a more precise version of Theorem 3.2 of [5]. The second one has been established in [5].

Theorem 2.1 ([6, Theorem 2.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x) \leq r} \Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,

(ii) for each λ in

$$\Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right[,$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([5, Theorem 3.2]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist two positive constants $r_1, r_2 > 0$ and $\bar{x} \in X$ with $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$ such that

(j) $\frac{\sup_{\Phi(x)\le r_1} \Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jj) $\frac{\sup_{\Phi(x)\le r_2} \Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jjj) for each λ in

$$\Lambda^*_{r_1,r_2} := \Big]\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\Big\{\frac{r_1}{\sup_{\Phi(x)\le r_1}\Psi(x)}, \frac{r_2}{2\sup_{\Phi(x)\le r_2}\Psi(x)}\Big\}\Big[$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$, and such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$, one has $\inf_{t\in[0,1]} \Psi(tx_1+(1-t)x_2) \ge 0$.

Then for each $\lambda \in \Lambda^*_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Let us introduce some notation which will be used later. Assume that

$$\min\left\{\frac{p^-}{\pi^2}, \frac{q^-}{\pi^4}, \frac{p^-}{\pi^2} + \frac{q^-}{\pi^4}\right\} > -1, \tag{2.1}$$

where $p^- := \operatorname{ess\,inf}_{x\in[0,1]} p(x)$ and $q^- := \operatorname{ess\,inf}_{x\in[0,1]} q(x)$. Moreover, set

$$\sigma := \min\left\{\frac{p^-}{\pi^2}, \frac{q^-}{\pi^4}, \frac{p^-}{\pi^2} + \frac{q^-}{\pi^4}, 0\right\},$$
$$\delta := \sqrt{1+\sigma}.$$

Define

$$\begin{aligned} H_0^1([0,1]) &:= \Big\{u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0\Big\}, \\ H^2([0,1]) &:= \Big\{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\Big\}. \end{aligned}$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm defined as follows:

$$\|u\|_X := \left(\int_0^1 |u''(t)|^2\, dt\right)^{1/2}.$$

We recall the following Poincaré type inequalities (see, for instance, [11, Lemma 2.3]):

$$\|u'\|^2_{L^2([0,1])} \le \frac{1}{\pi^2}\|u''\|^2_{L^2([0,1])}, \tag{2.2}$$

$$\|u\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u''\|^2_{L^2([0,1])} \tag{2.3}$$

for all $u \in X$. Therefore, taking into account (2.1)-(2.3), the norm

$$\|u\| = \left(\int_0^1 (|u''(t)|^2 + p(t)|u'(t)|^2 + q(t)|u(t)|^2)dt\right)^{1/2}$$

is equivalent to the usual norm, and, in particular,

$$\|u''\|_{L^2([0,1])} \le \frac{1}{\delta}\|u\|. \tag{2.4}$$

For the norm in $C^1([0,1])$,

$$\|u\|_\infty := \max\left\{\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}|u'(t)|\right\},$$

we have the following relation.

Proposition 2.3. Let $u \in X$. Then

$$\|u\|_\infty \le \frac{1}{2\pi\delta}\|u\|. \tag{2.5}$$

Proof: Taking (2.2) and (2.4) into account, the conclusion follows from the well-known inequality $\|u\|_\infty \le \frac{1}{2}\|u'\|_{L^2([0,1])}$. □

Here and in the sequel $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(F1) (a) the mapping $t \longmapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$;

(b) the mapping $x \longmapsto f(t,x)$ is continuous for almost every $t \in [0,1]$;

(c) for every $\varrho > 0$ there exists a function $l_\varrho \in L^1([0,1])$ such that

$$\sup_{|x|\le\varrho} |f(t,x)| \le l_\varrho(t)$$

for almost every $t \in [0,1]$.

Corresponding to f we introduce the function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ as follows

$$F(t,x) := \int_0^x f(t,\xi)\, d\xi$$

for all $(t, x) \in [0, 1] \times \mathbb{R}$.

We say that $u \in C([0, 1])$ is a *classical solution* of problem (1.1), if it satisfies the equation in (1.1) a.e. on $[0, 1] \setminus \{t_1, t_2, \ldots, t_m\}$, the limits $u''(t_j^+), u''(t_j^-), u'''(t_j^+)$ and $u'''(t_j^-)$, $1 \le j \le m$, exist, satisfy two impulsive conditions in (1.1) and the boundary condition $u(0) = u(1) = u''(0) = u''(1) = 0$.
A *weak solution* of problem (1.1) is a function $u \in X$ such that the equality

$$\int_0^1 \big(u''(t)v''(t) + p(t)u'(t)v'(t) + q(t)u(t)v(t)\big)\, dt =$$
$$-\mu \sum_{j=1}^m I_{2j}(u(t_j))v(t_j) - \mu \sum_{j=1}^m I_{1j}(u'(t_j))v'(t_j) + \lambda \int_0^1 f(t, u(t))v(t)\, dt$$

holds for all $v \in X$.
By the same argument as in the proof of [4, Lemma 2.3], we can prove the following lemma.

Lemma 2.4. The function $u \in X$ is a weak solution of problem (1.1) if and only if u is a classical solution of problem (1.1).

Lemma 2.5. *Assume that*

(H1) there exist constants $\alpha_i, \beta_i > 0$ and $\sigma_i \in [0, 1[$, $i = 1, 2$, such that

$$|I_{ij}(x)| \le \alpha_i + \beta_i |x|^{\sigma_i} \quad \textit{for all } x \in \mathbb{R},\ i = 1, 2 \textit{ and } j = 1, 2, \ldots, m.$$

Then, for any $u \in X$, we have

$$\Big|\sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(x)\, dx\Big| \le m\Big(\alpha_1 \|u\|_\infty + \frac{\beta_1}{\sigma_1 + 1}\|u\|_\infty^{\sigma_1+1}\Big) \tag{2.6}$$

and

$$\Big|\sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(x)\, dx\Big| \le m\Big(\alpha_2 \|u\|_\infty + \frac{\beta_2}{\sigma_2 + 1}\|u\|_\infty^{\sigma_2+1}\Big). \tag{2.7}$$

Proof: Thanks to (H1), we deduce

$$\Big|\int_0^{u'(t_j)} I_{1j}(x)\, dx\Big| \le \alpha_1 |u'(t_j)| + \frac{\beta_1}{\sigma_1 + 1}|u'(t_j)|^{\sigma_1+1}$$

and

$$\Big|\int_0^{u(t_j)} I_{2j}(x)\, dx\Big| \le \alpha_2 |u(t_j)| + \frac{\beta_2}{\sigma_2 + 1}|u(t_j)|^{\sigma_2+1}.$$

Thus (2.6) and (2.7) are obtained. □

Finally, put

$$k := \frac{27\delta^2\pi^2}{1024\left(1 + \frac{\|p\|_\infty}{\pi^2} + \frac{\|q\|_\infty}{\pi^4}\right)}, \qquad \Gamma_{i,c} := \frac{\alpha_i}{c} + \big(\frac{\beta_i}{\sigma_i + 1}\big)c^{\sigma_i - 1},$$

where α_i, β_i and σ_i, $i = 1, 2$, are given by (H1) and c is a positive constant.

3. Main results

We state our main results as follows:

Theorem 3.1. *Suppose that* (F1) and (H1) *are satisfied. Furthermore, assume that there exist two positive constants* c *and* d *with* $c < \frac{32}{3\sqrt{3}\pi}d$ *such that*

(A1) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in \big([0,\frac{3}{8}] \cup [\frac{5}{8},1]\big) \times [0,d]$*;*

(A2)

$$\frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2} < k\frac{\int_{3/8}^{5/8} F(t,d)\,dt}{d^2};$$

(A3)

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} \leq \frac{\pi^2}{4}\frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2}.$$

Then for every λ *in*

$$\Lambda := \left]\frac{2\delta^2\pi^2 d^2}{k\int_{3/8}^{5/8} F(t,d)\,dt}, \frac{2\delta^2\pi^2 c^2}{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}\right[,$$

there exists

$$\rho := \frac{1}{2m}\min\left\{\frac{2\delta^2\pi^2c^2 - \lambda\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2\Gamma_{1,c}},\right.$$

$$\frac{2\delta^2\pi^2c^2 - \lambda\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2\Gamma_{2,c}},$$

$$\frac{k\lambda\int_{3/8}^{5/8} F(t,d)\,dt - 2\delta^2\pi^2d^2}{d^2\Gamma_{1,(d/\sqrt{k})}},$$

$$\left.\frac{k\lambda\int_{3/8}^{5/8} F(t,d)\,dt - 2\delta^2\pi^2d^2}{d^2\Gamma_{2,(d/\sqrt{k})}}\right\}$$

such that for each $\mu \in [0,\rho[$ *the problem* (1.1) *has at least three distinct classical solutions.*

Proof: First, we observe that due to (A2) the interval Λ is non-empty and, consequently, one has $\rho > 0$. Now, fix λ and μ as in the conclusion. Our aim is to apply Theorem 2.1. For each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2,$$

$$\Psi(u) := \int_0^1 F(t,u(t))\,dt - \frac{\mu}{\lambda}\sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(x)\,dx - \frac{\mu}{\lambda}\sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(x)\,dx,$$

and put

$$E_{\lambda,\mu}(u) := \Phi(u) - \lambda\Psi(u), \qquad u \in X.$$

Using the property of f and the continuity of I_{ij}, $j = 1, 2, \ldots, m$ and $i = 1, 2$, we obtain that $\Phi, \Psi \in C^1(X, \mathbb{R})$ and for any $v \in X$, we have

$$\Phi'(u)(v) = \int_0^1 \big(u''(t)v''(t) + p(t)u'(t)v'(t) + q(t)u(t)v(t)\big)\,dt$$

and

$$\Psi'(u)(v) = \int_0^1 f(t, u(t))v(t)\,dt + -\frac{\mu}{\lambda}\sum_{j=1}^{m} I_{1j}(u'(t_j))v'(t_j) - \frac{\mu}{\lambda}\sum_{j=1}^{m} I_{2j}(u(t_j))v(t_j).$$

So, with standard arguments, we deduce that the critical points of the functional $E_{\lambda,\mu}$ are the weak solutions of problem (1.1) and so they are classical. We will verify (i) and (ii) of Theorem 2.1. Put $r = 2(\delta\pi c)^2$. Taking (2.5) into account, for every $u \in X$ such that $\Phi(u) \le r$, one has $\max_{t\in[0,1]} |u(t)| \le c$. Consequently, from Lemma 2.5, it follows that

$$\begin{aligned}\sup_{\Phi(u)\le r} \Psi(u) \le \int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt + \frac{\mu}{\lambda} m\Big(\alpha_1 c \\ + \frac{\beta_1}{\sigma_1+1}c^{\sigma_1+1}\Big) + \frac{\mu}{\lambda} m\Big(\alpha_2 c + \frac{\beta_2}{\sigma_2+1}c^{\sigma_2+1}\Big);\end{aligned}$$

that is,

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} \le \frac{1}{2\delta^2\pi^2}\Big[\frac{\int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}{c^2} + \frac{\mu}{\lambda} m\Gamma_{1,c} + \frac{\mu}{\lambda} m\Gamma_{2,c}\Big].$$

Hence, bearing in mind that $\mu < \rho$, one has

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.1}$$

Put

$$\bar{v}(t) = \begin{cases} -\frac{64d}{9}(t^2 - \frac{3}{4}t), & t \in [0, \frac{3}{8}[, \\ d, & t \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(t^2 - \frac{5}{4}t + \frac{1}{4}), & t \in]\frac{5}{8}, 1]. \end{cases}$$

Clearly $\bar{v} \in X$. Moreover, taking (2.2), (2.3) and (2.4) into account, one has

$$\frac{4096}{27}\delta^2 d^2 \le \|\bar{v}\|^2 \le \frac{4096}{27}\Big(1 + \frac{\|p\|_\infty}{\pi^2} + \frac{\|q\|_\infty}{\pi^4}\Big)d^2 = \frac{4\delta^2\pi^2}{k}d^2. \tag{3.2}$$

So, from $c < \frac{32}{3\sqrt{3}\pi}d$, we obtain $r < \Phi(\bar{v})$. Moreover, again from the previous inequality, we have

$$\Phi(\bar{v}) < \frac{2\delta^2\pi^2}{k}d^2.$$

Now, due to Lemma 2.5, (A1), (2.5) and (3.2) one has

$$\begin{aligned}\Psi(\bar v) &\geq \int_{3/8}^{5/8} F(t,d)\,dt - \frac{\mu}{\lambda} m\Big(\alpha_1\|\bar v\|_\infty + \frac{\beta_1}{\sigma_1+1}\|\bar v\|_\infty^{\sigma_1+1}\Big)\\ &- \frac{\mu}{\lambda} m\Big(\alpha_2\|\bar v\|_\infty + \frac{\beta_2}{\sigma_2+1}\|\bar v\|_\infty^{\sigma_2+1}\Big)\\ &\geq \int_{3/8}^{5/8} F(t,d)\,dt - \frac{\mu}{\lambda}\frac{m\,d^2}{k}\Gamma_{1,(d/\sqrt{k})} - \frac{\mu}{\lambda}\frac{m\,d^2}{k}\Gamma_{2,(d/\sqrt{k})}.\end{aligned}$$

So, we obtain

$$\frac{\Psi(\bar v)}{\Phi(\bar v)} \geq \frac{k\int_{3/8}^{5/8} F(t,d)\,dt - \frac{\mu}{\lambda} m d^2\Gamma_{1,(d/\sqrt{k})} - \frac{\mu}{\lambda} m d^2\Gamma_{2,(d/\sqrt{k})}}{2(\delta\pi d)^2}.$$

Since $\mu < \rho$, one has

$$\frac{\Psi(\bar v)}{\Phi(\bar v)} > \frac{1}{\lambda}\,. \tag{3.3}$$

Therefore, from (3.1) and (3.3), condition (i) of Theorem 2.1 is fulfilled. Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, due to (A3), we have

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} < \Big(\frac{\pi^4\delta^2}{2}\Big)\frac{1}{\lambda}.$$

So, we can fix $\varepsilon > 0$ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} < \varepsilon < \Big(\frac{\pi^4\delta^2}{2}\Big)\frac{1}{\lambda}\,.$$

Then, there exists a positive constant h such that

$$F(t,\xi) \leq \varepsilon|\xi|^2 + h, \quad \forall t\in[0,1],\ \xi\in\mathbb{R}\,.$$

Taking into account Lemma 2.5, (2.3), (2.4) and (2.5), it follows that

$$\begin{aligned}&\Phi(u) - \lambda\Psi(u)\\ &\geq \frac{1}{2}\|u\|^2 - \lambda\varepsilon\|u\|^2_{L^2[0,1]} - \lambda h - \mu m\Big[\frac{\alpha_1}{2\delta\pi}\|u\| + \frac{\beta_1}{\sigma_1+1}\Big(\frac{1}{2\delta\pi}\Big)^{\sigma_1+1}\|u\|^{\sigma_1+1}\Big]\\ &- \mu m\Big[\frac{\alpha_2}{2\delta\pi}\|u\| + \frac{\beta_2}{\sigma_2+1}\Big(\frac{1}{2\delta\pi}\Big)^{\sigma_2+1}\|u\|^{\sigma_2+1}\Big]\\ &\geq \Big(\frac{1}{2} - \lambda\varepsilon\frac{1}{\delta^2\pi^4}\Big)\|u\|^2 - \lambda h - \mu m\Big[\frac{\alpha_1}{2\delta\pi}\|u\| + \frac{\beta_1}{\sigma_1+1}\Big(\frac{1}{2\delta\pi}\Big)^{\sigma_1+1}\|u\|^{\sigma_1+1}\Big]\\ &- \mu m\Big[\frac{\alpha_2}{2\delta\pi}\|u\| + \frac{\beta_2}{\sigma_2+1}\Big(\frac{1}{2\delta\pi}\Big)^{\sigma_2+1}\|u\|^{\sigma_2+1}\Big]\end{aligned}$$

for all $u \in X$. So, the functional $\Phi - \lambda\Psi$ is coercive. Now, the conclusion of Theorem 2.1 can be used. It follows that, for every

$$\lambda \in \Lambda \subseteq \left]\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X, which are the weak solutions of the problem (1.1). This completes the proof. $\square$

Corollary 3.2. *Suppose that* (H1) *holds. Let $\theta \in L^1([0,1])$ be a non-negative and non-zero function and let $l : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\theta_0 := \int_{3/8}^{5/8}\theta(t)dt$, $\|\theta\|_1 := \int_0^1 \theta(t)\,dt$ and $L(\xi) = \int_0^\xi l(x)dx$ for all $\xi \in \mathbb{R}$, and assume that there exist two positive constants c and d with $c < \frac{32}{3\sqrt{3}\pi}d$ such that*

(A1') $L(\xi) \ge 0$ *for all* $\xi \in [0,d]$*;*

(A2')

$$\frac{\max_{|\xi|\le c} L(\xi)}{c^2} < \frac{27\pi^2\theta_0}{1024\|\theta\|_1}\frac{L(d)}{d^2};$$

(A3') $\limsup_{|\xi|\to+\infty} L(\xi)/\xi^2 \le 0.$

Then for every

$$\lambda \in \left]\frac{2048d^2}{27\theta_0 L(d)}, \frac{2\pi^2c^2}{\|\theta\|_1 \max_{|\xi|\le c} L(\xi)}\right[,$$

there exists

$$\rho := \frac{1}{2m}\min\left\{\frac{2\pi^2c^2 - \lambda\|\theta\|_1\max_{|\xi|\le c}L(\xi)}{c^2\Gamma_{1,c}}, \frac{2\pi^2c^2 - \lambda\|\theta\|_1\max_{|\xi|\le c}L(\xi)}{c^2\Gamma_{2,c}},\right.$$
$$\left.\frac{\frac{27\lambda\pi^2\theta_0}{1024}L(d) - 2\pi^2d^2}{d^2\Gamma_{1,(d/\sqrt{k})}}, \frac{\frac{27\lambda\pi^2\theta_0}{1024}L(d) - 2\pi^2d^2}{d^2\Gamma_{2,(d/\sqrt{k})}}\right\}$$

such that for each $\mu \in [0,\rho[$ the problem

$$\begin{cases} u''''(t) = \lambda\theta(t)l(u(t)), & t \ne t_j,\ t \in [0,1], \\ \Delta(u''(t_j)) = \mu I_{1j}(u'(t_j)), & j = 1,2,\dots,m, \\ -\Delta(u'''(t_j)) = \mu I_{2j}(u(t_j)), & j = 1,2,\dots,m, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{3.4}$$

has at least three classical solutions.

The proof of the above corollary follows from Theorem 3.1 by choosing $f(t,x) = \theta(t)l(x)$ for all $(t,x) \in [0,1]\times\mathbb{R}$ and taking into account that $k = 27\pi^2/1024$.

Remark 3.3. Clearly, if l is non-negative then assumption (A1') is verified and (A2') becomes

$$\frac{L(c)}{c^2} < \frac{27\pi^2}{1024}\frac{\theta_0}{\|\theta\|_1}\frac{L(d)}{d^2}.$$

Now, we state a result without asymptotic conditions on I_{ij}.

Lemma 3.4. *Suppose that* (F1) *is satisfied. Moreover, assume that* $f(t,x) \geq 0$ *for all* $(t,x) \in [0,1] \times \mathbb{R}$ *and* $I_{ij}(x) \leq 0$ *for all* $x \in \mathbb{R}$, $j = 1,\dots,m$ *and* $i = 1,2$. *If* u *is a classical solution of* (1.1) *then* $u(t) \geq 0$ *for all* $t \in [0,1]$.

Proof: If u is a classical solution of (1.1) then

$$\begin{aligned}&\int_0^1 u''''(t)v(t) - \int_0^1 (p(t)u'(t))'v(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt\\&- \lambda\int_0^1 f(t,u(t))v(t)\,dt = 0\end{aligned}$$

for all $v \in X$. Let $v(t) = \max\{-u(t), 0\}$ for all $t \in [0,1]$; clearly $v \in X$ and we have

$$\begin{aligned}0 &= \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} u''''(t)v(t)\,dt - \int_0^1 (p(t)u'(t))'v(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt\\
&\quad - \lambda \int_0^1 f(t,u(t))v(t)\,dt\\
&= \sum_{j=0}^{m} u'''(t)v(t)\Big|_{t_j}^{t_{j+1}} - \sum_{j=0}^{m} u''(t)v'(t)\Big|_{t_j}^{t_{j+1}} + \int_0^1 u''(t)v''(t)\,dt\\
&\quad - \int_0^1 (p(t)u'(t))'v(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt - \lambda\int_0^1 f(t,u(t))v(t)\,dt\\
&= -\sum_{j=0}^{m} \Delta u'''(t_j)v(t_j) + \sum_{j=0}^{m} \Delta u''(t_j)v'(t_j) + \int_0^1 u''(t)v''(t)\,dt\\
&\quad + \int_0^1 p(t)u'(t)v'(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt - \lambda\int_0^1 f(t,u(t))v(t)\,dt\\
&= \sum_{j=0}^{m} I_{2j}(u(t_j))v(t_j) + \sum_{j=0}^{m} I_{1j}(u'(t_j))v'(t_j) - \int_0^1 (v''(t))^2\,dt\\
&\quad - \int_0^1 p(t)(v'(t))^2\,dt - \int_0^1 q(t)(v(t))^2\,dt - \lambda\int_0^1 f(t,u(t))v(t)\,dt\\
&\leq -\|v\|^2.\end{aligned}$$

So $v(t) = 0$ for $t \in [0,1]$. □

Put

$$\Im_{i,c} := \sum_{j=1}^{m} \min_{|\xi|\leq c} \int_0^{\xi} I_{ij}(x)\,dx \quad \text{for all } c > 0,\ i = 1,2.$$

Our other main result is as follows.

Theorem 3.5. *Suppose that* (F1) *is satisfied. Furthermore, assume that there exist three positive constants* c_1*,* c_2 *and* d *with* $\frac{3\sqrt{3}\pi}{32}c_1 < d < \sqrt{\frac{k}{2}}c_2$ *such that*

(B1) $f(t,x) \geq 0$ *for all* $(t,x) \in [0,\, 1] \times [0, c_2]$*;*

(B2)

$$\frac{\int_0^1 F(t,c_1)\,dt}{c_1^2} < \frac{2}{3}k\frac{\int_{3/8}^{5/8} F(t,d)\,dt}{d^2};$$

(B3)

$$\frac{\int_0^1 F(t,c_2)\,dt}{c_2^2} < \frac{k}{3}\frac{\int_{3/8}^{5/8} F(t,d)\,dt}{d^2}.$$

Let

$$\Lambda' := \left]\frac{3\delta^2\pi^2 d^2}{k\int_{3/8}^{5/8} F(t,d)\,dt}, \delta^2\pi^2 \min\left\{\frac{2c_1^2}{\int_0^1 F(t,c_1)\,dt}, \frac{c_2^2}{\int_0^1 F(t,c_2)\,dt}\right\}\right[.$$

Then for every $\lambda \in \Lambda'$ *and for every negative continuous function* I_{ij}*,* $j = 1, \ldots, m$*,* $i = 1, 2$*, there exists*

$$\rho^* := \frac{1}{2}\min\left\{\frac{\lambda\int_0^1 F(t,c_1)\,dt - 2\delta^2\pi^2 c_1^2}{\Im_{1,c_1}}, \frac{\lambda\int_0^1 F(t,c_1)\,dt - 2\delta^2\pi^2 c_1^2}{\Im_{2,c_1}},\right.$$
$$\left.\frac{\lambda\int_0^1 F(t,c_2)\,dt - \delta^2\pi^2 c_2^2}{\Im_{1,c_2}}, \frac{\lambda\int_0^1 F(t,c_2)\,dt - \delta^2\pi^2 c_2^2}{\Im_{1,c_2}}\right\}$$

such that for each $\mu \in]0, \rho^*[$ *the problem* (1.1) *has at least three classical solutions* u_ν*,* $\nu = 1, 2, 3$*, such that* $0 < \|u_\nu\|_\infty \leq c_2$*.*

Proof: Without loss of generality, we can assume $f(t,x) \geq 0$ for all $(t,x) \in [0,1] \times \mathbb{R}$. Fix λ, I_{ij} and μ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 3.1. Put $\bar{v}$ as in Theorem 3.1, $r_1 = 2\delta^2\pi^2 c_1^2$ and $r_2 = 2\delta^2\pi^2 c_2^2$. Therefore, one has $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$ and since $\mu < \rho^*$, one has

$$\begin{aligned}
\frac{1}{r_1}\sup_{\Phi(u)<r_1} \Psi(u) &\leq \frac{1}{2\delta^2\pi^2 c_1^2}\left(\int_0^1 F(t,c_1)\,dt - \frac{\mu}{\lambda}\Im_{1,c_1} - \frac{\mu}{\lambda}\Im_{2,c_2}\right) \\
&< \frac{1}{\lambda} < \frac{k}{3\delta^2\pi^2}\frac{\int_{3/8}^{5/8} F(t,d)\,dt}{d^2} \\
&\leq \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}
\end{aligned}$$

and

$$\begin{aligned}\frac{2}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &\leq \frac{1}{\delta^2\pi^2 c_2^2}\Big(\int_0^1 F(t,c_2)\,dt-\frac{\mu}{\lambda}\Im_{1,c_2}-\frac{\mu}{\lambda}\Im_{2,c_2}\Big)\\ &<\frac{1}{\lambda}<\frac{k}{3\delta^2\pi^2}\frac{\int_{3/8}^{5/8}F(t,d)\,dt}{d^2}\\ &\leq\frac{2}{3}\frac{\Psi(\bar v)}{\Phi(\bar v)}\,.\end{aligned}$$

Therefore, conditions (j) and (jj) of Theorem 2.2 are satisfied. Finally, let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions for the problem (1.1). Hence, owing to Lemma 3.4, we obtain $u_1(t)\geq 0$ and $u_2(t)\geq 0$ for all $t\in[0,1]$. So, one has $\Psi(su_1+(1-s)u_2)\geq 0$ for all $s\in[0,1]$. From Theorem 2.2 the functional $\Phi-\lambda\Psi$ has at least three distinct critical points which are weak solutions of (1.1). This completes the proof. □

References

1. G.A. Afrouzi, A. Hadjian and S. Shokooh, Infinitely many solutions for a Dirichlet boundary value problem with impulsive condition, *U.P.B. Sci. Bull., Series A,* **77** (2015), 9–22.
2. G.A. Afrouzi, S. Shokooh and A. Hadjian, Existence of solutions to Dirichlet impulsive differential equations through a local minimization principle, *Electron. J. Differential Equations,* **147** (2014), 1–11.
3. M. Benchohra, J. Henderson and S. Ntouyas, *Theory of Impulsive Differential Equations*, Contemporary Mathematics and Its Applications, vol. 2, Hindawi Publishing Corporation, New York, 2006.
4. G. Bonanno, B. Bella and J. Henderson, Existence of solutions to second-order boundary-value problems with small perturbations of impulses, *Electron. J. Differential Equations,* **126** (2013), 1–14.
5. G. Bonanno and P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations,* **244** (2008), 3031–3059.
6. G. Bonanno and S.A. Marano, On the structure of the critical set of non-differentiable functionals with a weak compactness condition, *Appl. Anal.,* **89** (2010), 1–10.
7. J. Chen, C.C. Tisdell and R. Yuan, On the solvability of periodic boundary value problems with impulse, *J. Math. Anal. Appl.,* **331** (2007), 902–912.
8. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Impulsive Differential Equations and Inclusions*, World Scientific, Singapore, 1989.
9. J. Mawhin, *Topological Degree and Boundary Value Problems for Nonlinear Differential Equations*, Topological methods for ordinary differential equations, Lect. Notes Math., no. 1537, Springer, Berlin, 1993, 74–142.
10. J.J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.,* **70** (2009), 680–690.
11. L.A. Peletier, W.C. Troy, R. C. A. M. Van der Vorst; Stationary solutions of a fourth order nonlinear diffusion equation, (Russian) Translated from the English by V. V. Kurt. *Differentsialnye Uravneniya* **31** (1995), 327-337. English translation in *Differential Equations,* **31** (1995), 301–314.
12. D. Qian and X. Li, Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.,* **303** (2005), pp. 288–303.

13. A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

14. J. Sun, H. Chen and L. Yang, Variational methods to fourth-order impulsive differential equations, *J. Appl. Math. Comput.,* **35** (2011), 323–340.

15. Y. Tian and W.G. Ge, Applications of variational methods to boundary value problem for impulsive differential equations, *Proc. Edinb. Math. Soc.,* **51** (2008), 509–527.

16. W. Wang and J. Shen, Eigenvalue problems of second order impulsive differential equations, *Comput. Math. Appl.,* **62** (2011), 142–150.

17. W. Wang and X. Yang, Multiple solutions of boundary-value problems for impulsive differential equations, *Math. Methods Appl. Sci.,* **34** (2011), 1649–1657.

18. J. Xiao and J.J. Nieto, Variational approach to some damped Dirichlet nonlinear impulsive differential equations, *J. Franklin Inst.,* **348** (2011), 369–377.

19. J. Xie and Z. Luo, Solutions to a boundary value problem of a fourth-order impulsive differential equation, *Boundary Value Problems,* **154** (2013), 1–18.

Saeid Shokooh,
Department of Mathematics, Faculty of Sciences,
Gonbad Kavous University, Gonbad Kavous, Iran.
E-mail address: shokooh@gonbad.ac.ir