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# A Class of Strongly Close-to-Convex Functions

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Abstract: In this paper, we study a class of strongly close-to-convex functions $f(z)$ analytic in the unit disk $\mathbb{U}$ with $f(0)=0, f'(0)=1$ satisfying for some convex function $g(z)$ the condition that

$$\frac{zf'(z)}{g(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^m$$

$$(-1 \leq A \leq 1, -1 \leq B \leq 1 \ (A \neq B), 0 < m \leq 1; z \in \mathbb{U}).$$

We obtain for functions belonging to this class, the coefficient estimates, bounds, certain results based on an integral operator and radius of convexity. We also deduce a number of useful special cases and consequences of the various results which are presented in this paper.

Key Words: Convex functions, Close-to-convex functions, Subordination, Differential subordination, Bounds.

## Contents



## 1. Introduction

Let $\mathcal{H}(\mathbb{U})$ represent a linear space of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, $n \in \mathbb{N}$, let

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots\right\}$$

and denote by $\mathcal{A}$ the special class $\mathcal{H}[0,1]$ whose members are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ \ (z \in \mathbb{U}). \tag{1.1}$$

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Further, we denote by $\mathcal{S}$, a class of functions $f \in \mathcal{A}$ which are univalent. Also, let $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the subclasses of $\mathcal{S}$ whose members are starlike and convex in $\mathbb{U}$ satisfying, respectively, the conditions:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \text{ and } \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0 \ \ (z \in \mathbb{U}).$$

A class of close-to-convex functions $f \in \mathcal{A}$ satisfying for some $h \in \mathcal{S}^*$ and for some $\alpha \in (-\pi/2, \pi/2)$ the condition:

$$\Re\left\{\frac{zf'(z)}{e^{i\alpha}h(z)}\right\} > 0 \ \ (z \in \mathbb{U}),$$

is denoted by $\mathcal{C}$. A class $\mathcal{C}'$ of close-to-convex functions $f \in \mathcal{A}$ satisfying for some $g \in \mathcal{K}$ the condition:

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \ \ (z \in \mathbb{U}) \tag{1.2}$$

was defined in [1] (see also [13]). A class of strongly close-to-convex functions $f \in \mathcal{A}$ of order $m$ satisfying for some $g \in \mathcal{K}$ the condition:

$$\left|\arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < m\frac{\pi}{2} \ \ (0 < m \leq 1; \ z \in \mathbb{U})$$

is denoted by $\mathcal{C}'_m$, where the class $\mathcal{C}'_1$ is equivalent to the class $\mathcal{C}'$.

For two functions $p$, $q \in \mathcal{H}(\mathbb{U})$, we say $p$ is subordinate to $q$ in $\mathbb{U}$ and write $p \prec q$ in $\mathbb{U}$, if there exists a Schwarz function $\omega$, analytic in $\mathbb{U}$ with $\omega(0) = 0$, and $|\omega(z)| \leq |z| < 1$ $(z \in \mathbb{U})$ such that $p(z) = q(\omega(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $q$ is univalent in $\mathbb{U}$, then we have the following equivalence (see for details [3,9]):

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}).$$

We denote by $\mathcal{P}$ the class of functions $p \in \mathcal{H}[1,1]$ with $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$, and let $\mathcal{Q}$ be a class of univalent functions $\phi \in \mathcal{P}$ such that $\phi(\mathbb{U})$ is convex and symmetrical with respect to the real axis.

We note that a function $f \in \mathcal{A}$ which satisfies for some $g \in \mathcal{K}$ the condition that

$$\frac{zf'(z)}{g(z)} \prec \phi(z) \ \ (\phi \in \mathcal{Q}; \ z \in \mathbb{U}) \tag{1.3}$$

belongs to the class $\mathcal{C}'$ if the function $\phi(z) = \frac{1+z}{1-z}$ $\left(\text{or } \frac{1-z}{1+z}\right)$, and it belongs to the class $\mathcal{C}'_m$ if the function $\phi(z) = \left(\frac{1+z}{1-z}\right)^m$ $\left(\text{or } \left(\frac{1-z}{1+z}\right)^m\right)$.

For the purpose of this paper, we consider the function $\phi$ which is defined by

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^m$$

$$\left(-1 \leq A \leq 1, -1 \leq B \leq 1 \ (A \neq B), 0 < m \leq 1; \ z \in \mathbb{U}\right),$$

and for this $\phi$, we define a class $\Omega_m(A,B)$ of functions $f \in \mathcal{A}$ which satisfies for some $g \in \mathcal{K}$ the above subordination (1.3).

It may be observed that

(i) for $-1 \leq A \leq 1, -1 \leq B \leq 1 \ (A \neq B), 0 < m_1 \leq m_2 \leq 1; z \in \mathbb{U}$,

$$\left(\frac{1+Az}{1+Bz}\right)^{m_1} \prec \left(\frac{1+Az}{1+Bz}\right)^{m_2} \prec \frac{1+Az}{1+Bz},$$

(see Figure 1 for $A = -.7, \ B = .8, \ m_1 = .25, \ m_2 = .7$)

and

(ii) for $i = 1, 2, \ -1 \leq A_i \leq 1, -1 \leq B_i \leq 1 \ (A_i \neq B_i), 0 < m \leq 1; z \in \mathbb{U}$,

$$\left(\frac{1+A_1 z}{1+B_1 z}\right)^{m} \prec \left(\frac{1+A_2 z}{1+B_2 z}\right)^{m} \prec \left(\frac{1+z}{1-z}\right)^{m},$$

when intervals with end points $A_1, B_1$ are contained in the intervals with end points $A_2, B_2$ (see Figure 2 for $A_1 = -.25, \ B_1 = .6, \ A_2 = -.7, \ B_2 = .8, \ m = .7$).

In Figures 1 and 2 below, the regions in blue, red and yellow are, respectively, the images of the subordinated functions occurring in (i) and (ii) above under the unit disk.

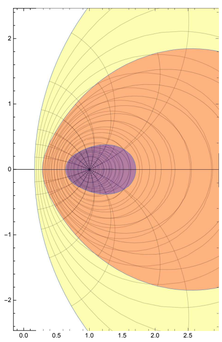


Figure 1: (i)

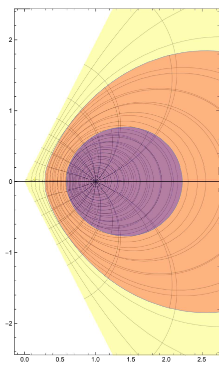


Figure 2: (ii)

We further observe that $\Omega_m(1,-1) = \mathcal{C}'_m$, and the class $\Omega_{1/2}(1,0) = \mathcal{L}\mathcal{C}'$ is associated with the right-half of the Lemniscate of Bernoulli $\partial\mathcal{L}$ (see [14]) enclosing the region:

$$\mathcal{L} := \{w \in \mathbb{C} : \Re\{w\} > 0, |w^2 - 1| < 1\},$$

where

$$\mathcal{L} \subset \left\{ w \in \mathbb{C} : |\arg\ w| < \frac{\pi}{4} \right\}.$$

By denoting $\Omega_1(A, B)$ by $\Omega(A, B)$, we observe that for $-1 \leq B < A \leq 1$, the class $\Omega(A, B) = \mathcal{C}'(A, B)$ is a Janowski type class [6] of close-to-convex functions. In particular, the class $\Omega(1, -1) = \mathcal{C}'$ and for $0 \leq \alpha < 1, 0 < \beta \leq 1$, the class $\Omega((2\alpha - 1)\beta, \beta)$ was studied in [10].

In this paper, we obtain various results based on coefficient estimates, bounds, integral operator and radius of convexity for functions belonging to the class $\Omega_m(A, B)$. We also point out some useful cases and consequences of the main results.

## 2. Coefficient Estimates

**Theorem 2.1.** *Let $-1 \leq A \leq 1, -1 \leq B \leq 1\ (A \neq B), 0 < m \leq 1$ and let $f \in \mathcal{A}$ be of the form (1.1). If $f \in \Omega_m(A, B)$, then*

$$|a_n| \leq \frac{m\,|A - B|\,(n-1) + 1}{n} \quad (n \geq 2). \tag{2.1}$$

**Proof:** Let the function $g \in \mathcal{K}$ be of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n\ z^n \quad (z \in \mathbb{U}) \tag{2.2}$$

and let

$$p(z) := \frac{zf'(z)}{g(z)} = 1 + \sum_{n=1}^{\infty} p_n\ z^n \quad (z \in \mathbb{U}). \tag{2.3}$$

On using the series expansions of $f(z)$ and $g(z)$ from (1.1) and (2.2) in (2.3), we obtain that

$$1 + \sum_{n=2}^{\infty} n\ a_n\ z^{n-1} = \left(1 + \sum_{n=2}^{\infty} b_n\ z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n\ z^n\right), \tag{2.4}$$

and upon equating the coefficients of $z^{n-1}$ $(n \geq 2)$ on both the sides of (2.4), we get

$$n\ a_n = p_{n-1} + \sum_{k=2}^{n-1} b_k\ p_{n-k} + b_n. \tag{2.5}$$

By the hypothesis, we have

$$1 + \sum_{n=1}^{\infty} p_n\ z^n \prec \left(\frac{1 + Az}{1 + Bz}\right)^m = 1 + m(A - B)z + ...,$$

which by a well-known result of Rogosinski [12] on subordination shows that

$$|p_n| \leq m\,|A - B| \quad (n \in \mathbb{N}). \tag{2.6}$$

On applying (2.6) and the condition for convex functions ([3, p. 45]): $|b_n| \leq 1$ $(n \geq 2)$ to the equation (2.5), we obtain the coefficient inequality (2.1). □

In particular, from Theorem 2.1, we obtain the following result for strongly close-to-convex functions of order $m$.

**Corollary 2.2.** *Let $0 < m \leq 1$ and let $f \in \mathcal{A}$ be of the form (1.1). If $f \in \mathcal{C}'_m$, then*

$$|a_n| \leq \frac{2m\,(n-1)+1}{n} \quad (n \geq 2)\,. \tag{2.7}$$

## 3. Bounds

In finding the bounds, we need the following lemmas.

**Lemma 3.1.** *[9, p. 7] Let $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, \ldots)$, then*
*(i)* ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t$
$(\Re(c) > \Re(b) > 0)\,.$
*(ii)* ${}_2F_1(a, b; c; z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$

**Lemma 3.2.** *Let $-1 \leq A \leq 1, -1 \leq B \leq 1$ $(A \neq B), 0 < m \leq 1$. If $f \in \Omega_m(A, B)$, then for some $g \in \mathcal{K}$ and for $|z| = r$ $(0 < r < 1)$:*

$$\left(\frac{1 - ABr^2 - |A-B|\,r}{1 - B^2r^2}\right)^m \leq \left|\frac{zf'(z)}{g(z)}\right| \leq \left(\frac{1 - ABr^2 + |A-B|\,r}{1 - B^2r^2}\right)^m. \tag{3.1}$$

**Proof:** Let $f \in \Omega_m(A, B)$, then

$$p(z) := \frac{zf'(z)}{g(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^m \tag{3.2}$$

$$(-1 \leq A \leq 1, -1 \leq B \leq 1 \ (A \neq B), 0 < m \leq 1; z \in \mathbb{U})\,.$$

Hence, for $|z| = r$ $(0 < r < 1)$ and assuming the principal values of the powers, we get

$$\left|(p(z))^{1/m} - \frac{1 - ABr^2}{1 - B^2r^2}\right| \leq \frac{|A-B|\,r}{1 - B^2r^2}, \tag{3.3}$$

which implies that

$$\frac{1 - ABr^2 - |A-B|\,r}{1 - B^2r^2} \leq |p(z)|^{1/m} \leq \frac{1 - ABr^2 + |A-B|\,r}{1 - B^2r^2}. \tag{3.4}$$

This proves Lemma 3.2. □

**Remark 3.3.**

*(i) If for $-1 \leq B < A \leq 1, 0 < m \leq 1, f \in \Omega_m(A, B)$, then for some $g \in \mathcal{K}$ and for $|z| = r$ $(0 < r < 1)$:*

$$\left(\frac{1 - Ar}{1 - Br}\right)^m \leq \left|\frac{zf'(z)}{g(z)}\right| \leq \left(\frac{1 + Ar}{1 + Br}\right)^m. \tag{3.5}$$

*(ii) If for* $-1 \le A < B \le 1, 0 < m \le 1,\ f \in \Omega_m(A,B)$, *then for some* $g \in \mathcal{K}$ *and for* $|z| = r\ (0 < r < 1)$:

$$\left(\frac{1+Ar}{1+Br}\right)^m \le \left|\frac{zf'(z)}{g(z)}\right| \le \left(\frac{1-Ar}{1-Br}\right)^m. \tag{3.6}$$

**Theorem 3.4.** *Let* $-1 \le B < A \le 1, 0 < m \le 1$. *If* $f \in \Omega_m(A,B)$, *then for* $|z| = r\ (0 < r < 1)$:

$$\left(\frac{1-Ar}{1-Br}\right)^m \frac{1}{1+r} \le |f'(z)| \le \left(\frac{1+Ar}{1+Br}\right)^m \frac{1}{1-r} \tag{3.7}$$

*and*

$$k \le |f(z)| \le K, \tag{3.8}$$

*where*

$$k = \begin{cases} \sum_{l=0}^{\infty} \left\{ \binom{m}{l} \left(\frac{A}{B}\right)^{m-l} \left(\frac{1-A/B}{1-Br}\right)^l \sum\limits_{n=0}^{\infty} (-1)^n \frac{r^{n+1}}{n+1}\ {}_2F_1\left(l, 1; 2+n; \frac{Br}{Br-1}\right) \right\} & \text{if } B \neq 0, \\ (1-Ar)^m \sum\limits_{n=0}^{\infty} (-1)^n \frac{r^{n+1}}{n+1}\ {}_2F_1\left(-m, 1; 2+n; \frac{Ar}{Ar-1}\right) & \text{if } B = 0. \end{cases}$$

*and*

$$K = \begin{cases} \sum_{l=0}^{\infty} \left\{ \binom{m}{l} \left(\frac{A}{B}\right)^{m-l} \left(\frac{1-A/B}{1+Br}\right)^l \sum\limits_{n=0}^{\infty} \frac{r^{n+1}}{n+1}\ {}_2F_1\left(l, 1; 2+n; \frac{Br}{1+Br}\right) \right\} & \text{if } B \neq 0, \\ (1+Ar)^m \sum\limits_{n=0}^{\infty} \frac{r^{n+1}}{n+1}\ {}_2F_1\left(-m, 1; 2+n; \frac{Ar}{Ar+1}\right) & \text{if } B = 0. \end{cases}$$

*Equalities in* (3.7) *and* (3.8) *occur for the function given by*

$$f(z) = \int_0^z \left(\frac{1+Axu}{1+Bxu}\right)^m \frac{1}{1-xu} \mathrm{d}u \quad (-1 \le B < A \le 1, 0 < m \le 1), \tag{3.9}$$

*if* $z = r\ (0 < r < 1)$, *a real value and* $x = \pm 1$.

**Proof:** Let $f \in \Omega_m(A,B)$, then from (3.5) and upon using $|z| = r\ (0 < r < 1)$, we get

$$\left(\frac{1-Ar}{1-Br}\right)^m \frac{1}{r} |g(z)| \le |f'(z)| \le \left(\frac{1+Ar}{1+Br}\right)^m \frac{1}{r} |g(z)|. \tag{3.10}$$

Since $g \in \mathcal{K}$, we have ( [5, p. 16])

$$\frac{r}{1+r} \le |g(z)| \le \frac{r}{1-r} \quad (|z| = r\ (0 < r < 1)), \tag{3.11}$$

and therefore, from (3.10) and (3.11), we get the desired result given by (3.7). Now for some real $\theta$, let $z = re^{i\theta}\ (0 < r < 1)$, then we have

$$f(z) = \int_0^r f'(te^{i\theta}) e^{i\theta} \mathrm{d}t,$$

and on using the upper bound from (3.7), we get

$$|f(z)| \leq \int_0^r \left(\frac{1+At}{1+Bt}\right)^m \frac{1}{1-t}\mathrm{d}t. \tag{3.12}$$

By using the *binomial expansion:*

$$(a+b)^r = \sum_{l=0}^{\infty} \binom{r}{l} a^{r-l}\, b^l, \quad \binom{r}{l} = \frac{\Gamma(r+1)}{\Gamma(r-l+1)\, l!} \ (r \in \mathbb{R}),$$

we have (in case $B \neq 0$):

$$\begin{aligned} \left(\frac{1+At}{1+Bt}\right)^m &= \left(\frac{A}{B} + \frac{1-A/B}{1+Bt}\right)^m \\ &= \sum_{l=0}^{\infty} \left\{ \binom{m}{l} \left(\frac{A}{B}\right)^{m-l} \left(\frac{1-A/B}{1+Bt}\right)^l \right\}. \end{aligned}$$

Hence, from (3.12), we obtain

$$\begin{aligned} &\int_0^r \left(\frac{1+At}{1+Bt}\right)^m \frac{1}{1-t}\mathrm{d}t \\ &= \sum_{l=0}^{\infty} \left\{ \binom{m}{l} \left(\frac{A}{B}\right)^{m-l} \left(1-\frac{A}{B}\right)^l \int_0^r (1+Bt)^{-l} \frac{1}{1-t}\mathrm{d}t \right\}, \end{aligned} \tag{3.13}$$

where on using (i) and (ii) of Lemma 3.1, we get

$$\begin{aligned} \int_0^r (1+Bt)^{-l} \frac{1}{1-t}\mathrm{d}t &= \sum_{n=0}^{\infty} \int_0^r (1+Bt)^{-l}\, t^n \mathrm{d}t \\ &= \sum_{n=0}^{\infty} r^{n+1} \int_0^1 (1+Bru)^{-l}\, u^n \mathrm{d}u \\ &= \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} (1+Br)^{-l} \ {}_2F_1\left(l, 1; 2+n; \frac{Br}{1+Br}\right). \end{aligned}$$

Thus, we get (in case $B \neq 0$)

$$\begin{aligned} &|f(z)| \\ &\leq \sum_{l=0}^{\infty} \left\{ \binom{m}{l} \left(\frac{A}{B}\right)^{m-l} \left(\frac{1-A/B}{1+Br}\right)^l \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} \ {}_2F_1\left(l, 1; 2+n; \frac{Br}{1+Br}\right) \right\}. \end{aligned} \tag{3.14}$$

In case $B = 0$, then from (3.12) and using (i) and (ii) of Lemma 3.1, we obtain

$$\begin{aligned} |f(z)| &\leq \int_0^r (1+At)^m \frac{1}{1-t}\mathrm{d}t = \sum_{n=0}^{\infty} \int_0^r (1+At)^m \ t^n \mathrm{d}t \\ &= \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} (1+Ar)^m \ {}_2F_1\left(-m, 1; 2+n; \frac{Ar}{Ar+1}\right). \end{aligned} \tag{3.15}$$

The bounds (3.14) and (3.15) establish the upper bound of the result (3.8). To obtain the lower bound of $|f(z)|$ in (3.8), we consider a point $z_0$ ($|z_0| = r < 1$) such that $|f(z)| \geq |f(z_0)|$ ($\forall z : |z| = r$). Let $\gamma$ be an arc in $\mathbb{U}$ which is mapped by the function $w = f(z)$ onto a line segment $L$ joining origin to the point $f(z_0)$ and lying entirely in the image of $\mathbb{U}$ under $f$. Thus, making use of the lower bound of $|f'(z)|$ from (3.7), we get

$$|f(z)| \geq |f(z_0)| = \int_L |dw| = \int_\gamma |f'(z)||\mathrm{d}z| \geq \int_0^r \left(\frac{1-At}{1-Bt}\right)^m \frac{1}{1+t}\mathrm{d}t. \tag{3.16}$$

Adopting similar calculations as employed in finding the upper bounds from (3.12), we get the desired result (3.8) for the lower bounds from (3.16). This proves the result of Theorem 3.4. □

For $m = 1$, we obtain a much simplified form from Theorem 3.4 which is contained in the following corollary.

**Corollary 3.5.** *Let $-1 \leq B < A \leq 1$. If $f \in \mathcal{C}'(A, B)$, then*

$$\frac{1-Ar}{(1-Br)(1+r)} \leq |f'(z)| \leq \frac{1+Ar}{(1+Br)(1-r)} \quad (|z| = r < 1) \tag{3.17}$$

*and*

$$h \leq |f(z)| \leq H, \tag{3.18}$$

*where*

$$h = \begin{cases} \frac{1+A}{1+B}\log(1+r) + \frac{A-B}{(1+B)B}\log(1-Br), & B \neq 0, -1, \\ A\log\frac{1}{1+r} + (1+A)\frac{r}{1+r}, & B = -1, \\ (1+A)\log(1+r) - Ar, & B = 0, \end{cases} \tag{3.19}$$

*and*

$$H = \begin{cases} \frac{1+A}{1+B}\log\frac{1}{1-r} - \frac{A-B}{(1+B)B}\log(1+Br), & B \neq 0, -1, \\ A\log(1-r) + (1+A)\frac{r}{1-r}, & B = -1, \\ (1+A)\log\frac{1}{1-r} - Ar, & B = 0. \end{cases} \tag{3.20}$$

*Equalities in the results occur for the function given by (3.9) if $m = 1$, $z = r$ $(0 < r < 1)$, a real value and $x = \pm 1$.*

**Proof:** If we choose $m = 1$, then from (3.7), we get the result (3.17), and from (3.12) and (3.13), we get (in case $B \neq 0$)

$$|f(z)| \leq \frac{A}{B}\int_0^r \frac{\mathrm{d}t}{1-t} + \left(1 - \frac{A}{B}\right)\int_0^r \frac{\mathrm{d}t}{(1+Bt)(1-t)}. \tag{3.21}$$

Hence, when $B \neq 0$ and $B \neq -1$, we obtain

$$|f(z)| \leq \frac{A}{B}\log\frac{1}{1-r} + \frac{1-A/B}{1+B}\left[\log(1+Bt) + \log\frac{1}{1-r}\right]$$

which yields the upper bound (3.20) of (3.18) if $B \neq 0, -1$. If $B = -1$, then from (3.21), we get the upper bound (3.20) of (3.18), and if $B = 0$, then from (3.12) for $m = 1$, we get the upper bound (3.20) of (3.18). Similarly, in view of (3.16), if $m = 1$, then we get (in case $B \neq 0$)

$$|f(z)| \geq \frac{A}{B}\int_0^r \frac{\mathrm{d}t}{1+t} + \left(1-\frac{A}{B}\right)\int_0^r \frac{\mathrm{d}t}{(1-Bt)(1+t)}. \tag{3.22}$$

Consequently for $B \neq 0$ and $B \neq -1$, we have

$$|f(z)| \geq \frac{A}{B}\log(1+r) + \frac{1-A/B}{1+B}\left[\log\frac{1}{1-Br} + \log(1+r)\right],$$

which yields the lower bound (3.19) of (3.18) if $B \neq 0, -1$. If $B = -1$, then from (3.22), we get the lower bound (3.19) of (3.18) and in case $B = 0$, from (3.16) for $m = 1$, we get the lower bound (3.19) of (3.18). This completes the proof of Corollary 3.5. □

In view of the result (3.6) mentioned in Remark 3.3 (ii) (and following the proof of Corollary 3.5), we obtain the following result:

**Corollary 3.6.** *Let $-1 \leq A < B \leq 1$. If $f \in \Omega(A,B)$, then*

$$\frac{1+Ar}{(1+Br)(1+r)} \leq |f'(z)| \leq \frac{1-Ar}{(1-Br)(1-r)} \quad (|z| = r < 1) \tag{3.23}$$

*and*

$$l \leq |f(z)| \leq L, \tag{3.24}$$

*where*

$$l = \begin{cases} \frac{1-A}{1-B}\log(1+r) + \frac{A-B}{(1-B)B}\log(1+Br), & B \neq 0, 1, \\ A\log(1+r) + (1-A)\frac{r}{1+r}, & B = 1, \\ (1-A)\log(1+r) + Ar, & B = 0, \end{cases} \tag{3.25}$$

*and*

$$L = \begin{cases} \frac{1-A}{1-B}\log\frac{1}{1-r} - \frac{A-B}{(1-B)B}\log(1-Br), & B \neq 0, 1, \\ A\log\frac{1}{1-r} + (1-A)\frac{r}{1-r}, & B = 1, \\ (1-A)\log\frac{1}{1-r} + Ar, & B = 0. \end{cases} \tag{3.26}$$

*Equalities in the results occur for the function given by*

$$f(z) = \int_0^z \frac{1+Axu}{(1+Bxu)(1+xu)}\mathrm{d}u,$$

*if $z = r$ $(0 < r < 1)$, a real value and $x = \pm 1$.*

**Remark 3.7.** *In view of the lower bounds $h$ and $l$ of $|f(z)|$ for the classes $\mathcal{C}'(A,B)$ and $\Omega(A,B)$, given, respectively, by (3.19) and (3.25), we remark that the disc of maximum radius $h$ given by (3.19) for restricted values of $B$ such that $-1 \leq B < A \leq 1$ is contained in the image domain $f(\mathbb{U}_r)$ if $f \in \mathcal{C}'(A,B)$, where $\mathbb{U}_r = \{z \in \mathbb{C} : |z| = r < 1\}$ and the disc of maximum radius $l$ given by (3.25) for restricted values of $B$ such that $-1 \leq A < B \leq 1$ is contained in the image domain $f(\mathbb{U}_r)$ if $f \in \Omega(A,B)$.*

In view of the above Remark 3.7, we note from the lower bounds $h$ and $l$, that by letting $r \to 1$ in the Corollaries 3.5 and 3.6 (given, respectively, by (3.19) and (3.25)), we have the following properties giving the omission values for the classes $\mathcal{C}'(A,B)$ and $\Omega(A,B)$.

**Corollary 3.8.** *Let for $-1 \le B < A \le 1, f \in \mathcal{C}'(A,B)$ and let $w \in \mathbb{C}$ be such that $f(z) \ne w$ $(z \in \mathbb{U})$. Then $|w| > r_1$, where*

$$r_1 = \begin{cases} \frac{1+A}{1+B}\log 2 + \frac{A-B}{(1+B)B}\log(1-B) & (B \ne 0, -1), \\ \frac{1+A}{2} - A\log 2 & (B = -1), \\ (1+A)\log 2 - A & (B = 0). \end{cases} \tag{3.27}$$

*That is, if for $-1 \le B < A \le 1$, a function $f \in \mathcal{C}'(A,B)$ is not assuming any value $w$, then that $w$ must be somewhere outside the closed disk of radius $r_1$ given by (3.27).*

**Corollary 3.9.** *Let for $-1 \le A < B \le 1, f \in \Omega(A,B)$ and let $w \in \mathbb{C}$ be such that $f(z) \ne w$ $(z \in \mathbb{U})$. Then $|w| > r_2$, where*

$$r_2 = \begin{cases} \frac{1-A}{1-B}\log 2 + \frac{A-B}{(1-B)B}\log(1+B) & (B \ne 0, 1), \\ \frac{1-A}{2} + A\log 2 & (B = 1), \\ (1-A)\log 2 + A & (B = 0). \end{cases} \tag{3.28}$$

That is, if for $-1 \le A < B \le 1$, a function $f \in \Omega(A,B)$ is not assuming any value $w$, then that $w$ must be somewhere outside the closed disk of radius $r_2$ given by (3.28).

**Theorem 3.10.** *Let $-1 \le A \le 1, -1 \le B \le 1$ $(A \ne B), 0 < m \le 1$. If $f \in \Omega_m(A,B)$, then (for $z = re^{i\theta} \in \mathbb{U}$)*

$$|\arg f'(z)| \le m \sin^{-1} \frac{|B-A|\, r}{1 - ABr^2} + \sin^{-1} r. \tag{3.29}$$

*The result is sharp for real $A$ and $B$ such that $|A| = 1 = |B|$ $(A \ne B)$.*

**Proof:** Let the function $f \in \Omega_m(A,B)$, then from (3.3), we have (for $|z| = r$ $(0 < r < 1)$):

$$\frac{1}{m}\left|\arg \frac{zf'(z)}{g(z)}\right| \le \sin^{-1} \frac{|B-A|\, r}{1 - ABr^2} \in [0, \pi/2). \tag{3.30}$$

As $g \in \mathcal{K}$, it follows from [11] that for $|z| = r$ $(0 < r < 1)$:

$$\arg \frac{g(z)}{z} \le \sin^{-1} r \in [0, \pi/2),$$

and hence from (3.30), we get the result (3.29). Sharpness of the result can be observed for the function $f(z)$ such that

$$\left(\frac{zf'(z)}{g(z)}\right)^{1/m} = \frac{1+xz}{1-xz} \quad \left(0 < m \le 1, x = \frac{ir}{z}, r < 1\right) \tag{3.31}$$

with

$$g(z)=\frac{z}{1-yz},\ y=\frac{r}{z}\left(r+i\sqrt{1-r^2}\right)\ (r<1). \tag{3.32}$$

Since, for this function, if for real $A$ and $B$ $|A|=1=|B|$ $(A\neq B)$, then we have $|B-A|=2$, $AB=-1$ and

$$\left|\frac{1}{m}\arg\frac{zf'(z)}{g(z)}\right|=\sin^{-1}\frac{2r}{1+r^2}\ \text{and}\ \left|\arg\frac{g(z)}{z}\right|=\sin^{-1}r. \tag{3.33}$$

□

## 4. Integral operator

In proving our next Theorem 4.3, we use the following lemmas.

**Lemma 4.1.** *[9, Theorem 3.1a, p. 70] Let $h$ be convex in $\mathbb{U}$ and let $P:\mathbb{U}\to\mathbb{C}$, with $\Re P(z)>0$. If $p\in\mathcal{H}[a,n]$ with $\Re a>0$, then*

$$p(z)+P(z)zp'(z)\prec h(z)\Rightarrow p(z)\prec h(z).$$

**Lemma 4.2.** *Let $K$ map the disk $\mathbb{U}$ onto a (possibly many sheeted) starlike region. Also, let $J$ be analytic in $\mathbb{U}$ with $J(0)=K(0)=0$, then*

$$\frac{J'(z)}{K'(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^m\Rightarrow\frac{J(z)}{K(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^m$$

$$(-1\leq A\leq 1,-1\leq B\leq 1\ (A\neq B),0<m\leq 1;z\in\mathbb{U}).$$

**Proof:** Let

$$p(z)=\frac{J(z)}{K(z)}\ (z\in\mathbb{U}), \tag{4.1}$$

then $p\in\mathcal{H}[1,1]$. On differentiating (4.1) logarithmically we easily obtain that

$$\frac{J'(z)}{K'(z)}=p(z)+P(z)zp'(z)\prec\left(\frac{1+Az}{1+Bz}\right)^m,$$

where

$$\Re(P(z))=\Re\left(\frac{K(z)}{zK'(z)}\right)>0\quad(z\in\mathbb{U}).$$

Now using Lemma 4.1, we get the desired result. □

**Theorem 4.3.** *Let for $c>0$, an integral operator $\mathcal{I}_c:\mathcal{A}\to\mathcal{A}$ be defined by*

$$\mathcal{I}_cf(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\mathrm{d}t,\ z\in\mathbb{U}.$$

*If for $-1\leq A\leq 1,-1\leq B\leq 1\ (A\neq B),0<m\leq 1$ and for $g\in\mathcal{K}$, $f\in\Omega_m(A,B)$, then $\mathcal{I}_cg\in\mathcal{K}$ implies that $\mathcal{I}_cf\in\Omega_m(A,B)$.*

**Proof:** Let for some $g \in \mathcal{K}$, a function $K$ be defined by

$$K(z) = \int_0^z t^{c-1} g(t) \mathrm{d}t \ \ (z \in \mathbb{U}),$$

then, we have

$$\Re \left\{ \frac{zK'(z)}{K(z)} \right\} = \Re \left\{ \frac{z\left(\mathcal{I}_c g(z)\right)'}{\mathcal{I}_c g(z)} \right\} + c > 0 \ \ (z \in \mathbb{U}),$$

since $\mathcal{K} \subset \mathcal{S}^*$. Also, we have

$$\left(z^c \mathcal{I}_c f(z)\right)' = (c+1) z^{c-1} f(z),$$

which implies that

$$cz^c \mathcal{I}_c f(z) + z^{c+1} \left(\mathcal{I}_c f(z)\right)' = (c+1) z^c f(z). \tag{4.2}$$

Using differentiation and some elementary calculations, we obtain from (4.2) that

$$\left( \frac{z^{c+1}}{c+1} \left(\mathcal{I}_c f(z)\right)' \right)^{\prime c} f'(z).$$

Hence, by the hypothesis, we have

$$\frac{\left( \frac{z^{c+1}}{c+1} \left(\mathcal{I}_c f(z)\right)' \right)'}{K'(z)} = \frac{zf'(z)}{g(z)} \prec \left( \frac{1+Az}{1+Bz} \right)^m \quad (z \in \mathbb{U}),$$

and by Lemma 4.2, we get

$$\frac{z^{c+1} \left(\mathcal{I}_c f(z)\right)'}{(c+1) K(z)} \prec \left( \frac{1+Az}{1+Bz} \right)^m \quad (z \in \mathbb{U})$$

or

$$\frac{z \left(\mathcal{I}_c f(z)\right)'}{\mathcal{I}_c g(z)} \prec \left( \frac{1+Az}{1+Bz} \right)^m \quad (z \in \mathbb{U}).$$

This proves the result. □

## 5. Convexity Property

We need the following result of Bernardi [2, Theorem 2, p. 115] to prove our next Theorem 5.2.

**Lemma 5.1.** *[2] Let $p \in \mathcal{P}$, then for $|z| = r, 0 \leq r < 1$ and for any complex number $\delta$, $\Re(\delta) = \beta \geq 0$ :*

$$\left| \frac{zp'(z)}{p(z)+\delta} \right| \leq \frac{2r}{(1-r)\left[1+\beta+(1-\beta)r\right]}.$$

**Theorem 5.2.** *Let* $-1 \leq A \leq 1, -1 \leq B \leq 1$ $(A \neq B)$ *and* $0 < m \leq 1$. *If a function* $f \in \Omega_m(A, B)$, *then the radius of convexity* $r_0 \, (0 < r_0 < 1)$ *of* $f$ *is the unique positive root of the polynomial:*

$$\begin{aligned} P(m, A, B, r) &= (1-r)(1+Ar)(1+Br) \\ &\quad -rm(1+r)\left[(1+A)(1+Br)+(1+B)(1+Ar)\right]. \end{aligned} \tag{5.1}$$

**Proof:** For the Schwarz function $\omega$ with $\omega(0) = 0$, and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ satisfying

$$\frac{zf'(z)}{g(z)} = \left(\frac{1 + A\omega(z)}{1 + B\omega(z)}\right)^m \quad (z \in \mathbb{U}),$$

we define $q \in \mathcal{P}$ such that $\omega(z) = \frac{q(z)-1}{q(z)+1}$ $(z \in \mathbb{U})$. Thus, in terms of $q$ we write

$$\frac{zf'(z)}{g(z)} = \left(\frac{(1+A)\,q(z) + 1 - A}{(1+B)\,q(z) + 1 - B}\right)^m \quad (z \in \mathbb{U}). \tag{5.2}$$

On taking the logarithmic derivative, equation (5.2) gives (after some elementary calculations)

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + m\left[\frac{zq'(z)}{q(z) + \frac{1-A}{1+A}} - \frac{zq'(z)}{q(z) + \frac{1-B}{1+B}}\right]. \tag{5.3}$$

Since, $g \in \mathcal{K}$, we have

$$\Re\left\{\frac{zg'(z)}{g(z)}\right\} \geq \frac{1}{1+r} \quad (|z| = r < 1).$$

Therefore, on applying Lemma 5.1 in (5.3), we obtain

$$\begin{aligned} &\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \\ \geq\ & \frac{1}{1+r} - \frac{2rm}{1-r}\left[\frac{1}{1 + \frac{1-A}{1+A} + \left(1 - \frac{1-A}{1+A}\right)r} + \frac{1}{1 + \frac{1-B}{1+B} + \left(1 - \frac{1-B}{1+B}\right)r}\right] \\ =\ & \frac{1}{1+r} - \frac{rm}{1-r}\left[\frac{1+A}{1+Ar} + \frac{1+B}{1+Br}\right] \\ =\ & \frac{1}{1+r} - \frac{rm}{1-r}\left[\frac{(1+A)(1+Br) + (1+B)(1+Ar)}{(1+Ar)(1+Br)}\right] \\ =\ & \frac{(1-r)(1+Ar)(1+Br) - rm(1+r)\left[(1+A)(1+Br) + (1+B)(1+Ar)\right]}{(1-r^2)(1+Ar)(1+Br)} \\ =\ & \frac{P(m, A, B, r)}{(1-r^2)(1+Ar)(1+Br)}, \end{aligned}$$

and this proves Theorem 5.2. Since, $P(m,A,B,0)=1>0$ and

$$P(m,A,B,1)=-4m\,(1+A)\,(1+B)\leq 0,$$

the polynomial $P(m,A,B,r)$ has a positive real root $r_0$ in $(0,1)$ which is the smallest of the roots. The inequality $\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$ is valid for $|z|=r<r_0$, and hence, the radius of convexity is $r_0$. □

**Remark 5.3.** *For $B=-1$ and for $m=1$, we obtain*

$$P(1,A,-1,r)=:P(A,r)=(1-r)\left(1-2r-(1+2A)\,r^2\right).$$

*Thus, the radius of convexity $r_0\,(0<r_0<1)$ for the class $\Omega\,(A,-1)$ is the unique positive root of the polynomial $P(A,r)$, given by*

$$r_0=\begin{cases}\frac{-1+\sqrt{2+2A}}{1+2A} & \text{when} \quad A\neq -1/2,\\ \frac{1}{2} & \text{when} \quad A=-1/2.\end{cases}$$

In view of the Remark 5.3, we get following result.

**Corollary 5.4.** *Let for $0\leq\alpha<1$ and for $g\in\mathcal{K}$, a function $f\in\mathcal{A}$ satisfies $\frac{zf'(z)}{g(z)}\prec\frac{1+(1-2\alpha)z}{1-z}\quad(z\in\mathbb{U})$, then it is convex for $|z|<r_0<1$, where $r_0$ is given by*

$$r_0=\begin{cases}\frac{-1+2\sqrt{1-\alpha}}{3-4\alpha} & \text{when} \quad \alpha\neq 3/4,\\ \frac{1}{2} & \text{when} \quad \alpha=3/4.\end{cases}$$

For $\alpha=0$, Corollary 5.4 proves the following result of [1, Theorem 4, p. 65].

**Corollary 5.5.** *Let a function $f\in\mathcal{C}'$, then it is convex for $|z|<1/3$.*

If $m=1$ and $B=0$ in Theorem 5.2, then we get following result.

**Corollary 5.6.** *Let for $0<A\leq 1$ and for $g\in\mathcal{K}$, a function $f\in\mathcal{A}$ satisfies $\left|\frac{zf'(z)}{g(z)}-1\right|<A\quad(z\in\mathbb{U})$, then it is convex for $|z|<r_0<1$, where $r_0$ is the unique positive root of the polynomial:*

$$Q(A,r)=1-3r-(3A+2)\,r^2-Ar^3.$$

**Remark 5.7.** *In view of the Corollary 5.6, we remark that if for $g\in\mathcal{K}$, a function $f\in\mathcal{A}$ satisfies $\left|\frac{zf'(z)}{g(z)}-1\right|<1\quad(z\in\mathbb{U})$, then $f$ is convex for $|z|<r_0$, where $r_0=.236$ (approximately).*

## 6. Concluding Remarks and Observations

In our present investigation, we have considered some important and useful geometric properties of a presumably new class $\Omega_m(A,B)$ in the open unit disk $\mathbb{U}$. The results (see Theorems 2.1 to 5.2) depict such geometric properties which lead to various interesting results presented in this paper (see also [1] and [10]).

We deem it proper to point out some of the known special cases which arise from the results proved above. Thus, if we set

$$m = 1, \quad A = \beta(2\alpha - 1), \quad B = \beta \quad (0 \leq \alpha < 1, 0 < \beta \leq 1),$$

then our results Theorem 2.1, Corollary 3.6 and Theorem 4.3 (for $c = 1$) coincide with the results of Peng [10, Theorems 1, 2 and 4, p. 1451-55 ] for the class $\Omega((2\alpha-1)\beta, \beta)$. Also, on choosing $m = 1$, $A = 1$, $B = -1$, our results Corollary 2.2, Corollary 3.5 and Theorem 3.10 coincide with the results proved earlier by Abdel-Gawad and Thomas [1, Theorems 3, 1 and 2, pp. 61-63] for the class $\mathcal{C}'$.

Lastly, in view of Lemma 4.2, we infer the following sufficient condition for the class $\Omega_m(A,B)$:

$$\frac{(zf'(z))'}{g'(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^m \quad \Rightarrow f \in \Omega_m(A,B)$$

$$(-1 \leq A \leq 1, -1 \leq B \leq 1 \ (A \neq B), 0 < m \leq 1; z \in \mathbb{U}).$$


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