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L_r-biharmonic hypersurfaces in $\mathbb{E}^4$

Akram Mohammadpouri and Firooz Pashaie

abstract: An Euclidean hypersurface defined by isometric immersion $x : M^n \to \mathbb{E}^{n+1}$ is said to be biharmonic if the map x satisfies the condition $\Delta^2 x = 0$, where Δ is the Laplace operator on M^n. Based on a well-known conjecture of Bang-Yen Chen, the only biharmonic submanifolds in $\mathbb{E}^{n+1}$ are the minimal ones. In this paper, we consider an extension of biharmonicity condition as $L_r^2 x = 0$ on hypersurfaces of 4-dimensional Euclidean space $\mathbb{E}^4$, where L_r is the linearized operator from the first variation of $(r+1)$-th mean curvature of M^n and specially we have $L_0 = \Delta$. We prove that any L_2-biharmonic hypersurface in $\mathbb{E}^4$ with constant 2-th mean curvature is 2-minimal. We also prove that any L_1-biharmonic hypersurface in $\mathbb{E}^4$ with constant mean curvature is 1-minimal.

Key Words: Linearized operator L_r , L_r-biharmonic hypersurfaces, r-minimal.

Contents

1. Introduction

Biharmonic functions as the solution of some well-known partial differential equations frequently appear in mathematical physics. Especially, when it becomes very difficult to find harmonic maps, sometimes the biharmonic ones are helpful. From geometric points of view, the role of biharmonic surfaces in elasticity and fluid mechanics can be considered as a physical motivation for the theory of biharmonicity. In eighteen decade, B.Y.Chen has investigated the properties of biharmonic submanifolds in the Euclidean spaces (with position vector filed $M^n \to \mathbb{E}^{n+k}$ satisfying the condition $\Delta^2 x = 0$, where Δ is the Laplace operator). He introduced some open problems and conjectures in [4], among them, a longstanding conjecture says that a biharmonic submanifold in a Euclidean space is minimal. Chen himself has proved the conjecture for surfaces in $\mathbb{E}^3$. Later on, I. Dimitrić in his doctoral thesis has verified Chen conjecture in several different cases such as special curves, submanifolds of constant mean curvature and also, hypersurfaces of the Euclidean spaces with at most two distinct principal curvatures ([6]). T. Hasanis

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and T. Vlachos ([8]) verified the conjecture for hypersurfaces in $\mathbb{E}^4$. Having assumed the completeness, Akutagawa and Maeta ([1]) gave an affirmative answer to the global version of Chen conjecture for biharmonic submanifolds in Euclidean spaces. Recently, in [7], it has been shown that the only biharmonic hypersurfaces in Euclidean spaces with three distinct principal curvatures are minimal ones. A condition equivalent to biharmonicity on an Euclidean hypersurface can be expressed by $\Delta H = 0$, where H is the ordinary mean curvature vector field on the hypersurface.

On a hypersurface M^n in $\mathbb{E}^{n+1}$, The Laplace operator Δ stands for the linearized operator of the first variation of the mean curvature arising from normal variations of M^n ([14]), and in general, the advanced operator L_k (where, $L_0 = \Delta$), standing for the linearized operator of the first variation of $(k+1)$-th mean curvature arising from normal variations of M^n, is defined by the explicit formula $L_k(f) = tr(P_k \circ \nabla^2 f)$ for $k = 0, 1, 2, \ldots, n-1$, and $f \in C^\infty(M)$, where P_k denotes the k-th Newton transformation associated to the second fundamental from of M^n and $\nabla^2 f$ is the hessian of f ([13]). From this point of view, as an extension of finite type theory, S.M.B. Kashani ([9]) introduced the notion of L_r-finite type hypersurface in the Euclidean space, which has been followed in the first authors doctoral thesis. One can see our results in the last section of the last chapter of second edition of Chen's book ([3]).

It seems interesting to generalize the definition of biharmonic hypersurface by replacing $\triangle$ by the operator L_r. We define the L_r-biharmonicity condition and study on L_r-version of Chen conjecture. In [11], we solved the problem for hypersurfaces in $\mathbb{E}^{n+1}$ with at most two distinct principal curvatures. In special case, we showed that every L_1-biharmonic surface in $\mathbb{E}^3$ is flat and every L_2-biharmonic hypersurface in $\mathbb{E}^4$ with at most two distinct principal curvatures is 2-minimal. So, the next step is the study of L_r-biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^4$. Recently, in [12], we proved that any L_1-biharmonic hypersurfaces in $\mathbb{E}^{n+1}$ with at most three distinct principal curvatures and constant mean curvature are 1-minimal. In this paper, we prove that each L_1-biharmonic hypersurface in $\mathbb{E}^4$ with constant mean curvature is 1-minimal. Also, we show that every L_2-biharmonic hypersurface in $\mathbb{E}^4$ with constant 2-th mean curvature is 2-minimal. Indeed, we follow Defevers techniques to prove our results ([5]). Here are our main theorems.

Theorem 1.1. *Every L_1-biharmonic hypersurface in $\mathbb{E}^4$ with contant mean curvature and three distinct principal curvatures is 1-minimal.*

Theorem 1.2. *Every L_2-biharmonic hypersurface in $\mathbb{E}^4$ with constant 2-th mean curvature and three distinct principal curvatures is 2-minimal.*

2. Preliminaries

In this section, we recall some prerequisites from [2]. Let $x : M^3 \to \mathbb{E}^4$ be an isometric immersion of a Riemannian 3-dimensional manifold M into the Euclidean space $\mathbb{E}^4$ with the Gauss map N. We denote ∇^0 and ∇ the Levi-Civita connections on $\mathbb{E}^4$ and M^3, respectively. The Gauss and Weingarten formulae on the hypersurface are given by

$$\nabla^0_X Y = \nabla_X Y + < SX, Y > N, \ SX = -\nabla^0_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^3)$, where $S : \mathfrak{X}(M^3) \to \mathfrak{X}(M^3)$ is the shape operator (or Weingarten endomorphism) of M^3 with respect to the Gauss map N.

As is well-known, for every point $p \in M^3$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^3, and its eigenvalues $\lambda_1(p)$, $\lambda_2(p)$ and $\lambda_3(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3) = t^3 + a_1 t^2 + a_2 t + a_3,$$

where the coefficients of $Q_S(t)$ are given by

$$a_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \ a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \ a_3 = -\lambda_1\lambda_2\lambda_3.$$

The r-th mean curvature H_r or mean curvature of order r of M^3 in $\mathbb{E}^4$ is defined by

$$\binom{3}{r} H_r = (-1)^r a_r,$$

with $H_0 = 1$.

If $H_{r+1} \equiv 0$ then we say that M^3 is a r-minimal hypersurface, a 0-minimal hypersurface is nothing but a minimal hypersurface in $\mathbb{E}^4$. The r-th Newton transformation of M^3 is the operator $P_r : \mathfrak{X}(M^3) \to \mathfrak{X}(M^3)$ defined by

$$P_r = \sum_{j=0}^{r} (-1)^j \binom{3}{r-j} H_{r-j} S^j = (-1)^r \sum_{j=0}^{r} a_{r-j} S^j.$$

In particular,

$$P_0 = I, \ P_1 = 3HI - S, \ P_2 = 3H_2 I - S \circ P_1, \ P_3 = H_3 I - S \circ P_2.$$

Note that by Cayley-Hamilton theorem we have $P_3 = 0$. Let us recall that, for every point $p \in M^3$, each $P_r(p)$ is also a self-adjoint linear operator on the tangent hyperplane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized. If $\{e_1, e_2, e_3\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \lambda_2(p), \lambda_3(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r} = \sum_{\substack{i_1 < \cdots < i_r \\ i_j \neq i}}^{3} \lambda_{i_1} \cdots \lambda_{i_r} \text{ for every } i = 1, 2, 3 \text{ and } k = 1, 2.$$

In particular,

$$\begin{aligned} &\mu_{1,1} = \lambda_2 + \lambda_3,\ \mu_{2,1} = \lambda_1 + \lambda_3,\ \mu_{3,1} = \lambda_1 + \lambda_2, \\ &\mu_{1,2} = \lambda_2\lambda_3,\ \mu_{2,2} = \lambda_1\lambda_3,\ \mu_{3,2} = \lambda_1\lambda_2. \end{aligned} \tag{2.1}$$

We have the following formula for the Newton transformations, [2].

$$\begin{aligned} &a)\ tr(P_r) = c_r H_r, \\ &b)\ tr(S \circ P_r) = c_r H_{r+1}, \\ &c)\ tr(S^2 \circ P_1) = 9HH_2 - 3H_3, \\ &d)\ tr(S^2 \circ P_2) = 3HH_3, \end{aligned} \tag{2.2}$$

where $r = 1, 2$, $c_1 = 6$ and $c_0 = 3$. Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M^3) \to C^\infty(M^3)$ given by $L_r(f) = tr(P_r \circ \nabla^2 f)$, where, $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$< \nabla^2 f(X), Y >=< \nabla_X(\nabla f), Y >,\ X, Y \in \mathfrak{X}(M^3).$$

Therefore by considering the local orthonormal frame $\{e_1, e_2, e_3\}$, $L_r(f)$ is given by

$$L_r(f) = \mu_{1,r}(\nabla_{e_1} e_1 f - e_1 e_1 f) + \mu_{2,r}(\nabla_{e_2} e_2 f - e_2 e_2 f) + \mu_{3,r}(\nabla_{e_3} e_3 f - e_3 e_3 f). \tag{2.3}$$

3. L_r-biharmonic hypersurfaces in $\mathbb{E}^4$

Let $x : M^3 \to \mathbb{E}^4$ be a connected orientable hypersurface immersed into the Euclidean space, with Gauss map N. Then M^3 is called a L_r-biharmonic hypersurface if and only if $L_r^2 x = 0$ or equivalently, $L_r(H_{r+1}N) = 0$, $r = 1, 2$ (see [2]).

By definition of the L_r-biharmonic hypersurface, it is clear that r-minimal immersions are trivially r-biharmonic. By using formula for $L_r^2 x$ of [2] and the identifying normal and tangent parts of the L_r-biharmonic condition $L_r^2 x = 0$, one obtains necessary and sufficient conditions for M^3 to be L_r-biharmonic in $\mathbb{E}^4$, namely

$$L_r H_{r+1} = tr(S^2 \circ P_r) H_{r+1} \tag{3.1}$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2}\binom{3}{r+1} H_{r+1} \nabla H_{r+1}. \tag{3.2}$$

Example 3.1. *Let M^3 be the rotational hypersurface in $\mathbb{E}^4$ parameterized by*

$$x(u_1, u_2, v) = vY(u_1, u_2) + \int \frac{a\ dv}{\sqrt{v - a^2}} \eta_4,\ v > 0, \tag{3.3}$$

where a is nonzero constant, $\eta_4 = (0, 0, 0, 1) \in \mathbb{E}^4$ and $Y(u_1, u_2)$ is defined by

$$Y(u_1, u_2) = (Y_1(u_1, u_2), Y_2(u_1, u_2), Y_3(u_1, u_2), 0),$$

where

$$Y_1 = \cos u_1,\ Y_2 = \sin u_1 \cos u_2,\ Y_3 = \sin u_1 \sin u_2 \cos u_3.$$

M^3 is the 1-minimal hypersurface (see example 3.1 of [10]), and hence by using (3.1) and (3.2), we get that M^3 is L_1-biharmonic in $\mathbb{E}^4$.

3.1. L_1-biharmonic hypersurfaces

We now turn to the question whether there are non r-minimal L_r-biharmonic hypersurfaces of $\mathbb{E}^4$. In [11], we proved that every L_r-biharmonic hypersurface of $\mathbb{E}^4$ with at most two principal curvatures is in fact r-minimal. We remark that in this paper all three principal curvatures have to be mutually different. Otherwise, M^3 would be a L_r-biharmonic hypersurface with at most two different principal curvatures.

Proof of Theorem 1.1: Let $x : M^3 \to \mathbb{E}^4$ be an isometrically immersed L_1-biharmonic Euclidean hypersurface, first we show that H_2 is constant. Let us consider the open set $\mathfrak{U} = \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$, our objective is to show that $\mathfrak{U}$ is empty.

We assume that $\{e_1, e_2, e_3\}$ be a local orthonormal frame of principal directions of the shape operator S on $\mathfrak{U}$ such that $Se_i = \lambda_i e_i$ $(i = 1, 2, 3)$. Then we have $P_2 e_i = \mu_{i,2} e_i$, for every i. We get

$$H_2 = \frac{1}{3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3). \tag{3.4}$$

From (3.2) (using the inductive definition of P_2) we get

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2 \ \textit{on}\ \mathfrak{U}. \tag{3.5}$$

On the other hand, we have

$$\nabla H_2 = \sum_{i=1}^{3} e_i(H_2) e_i. \tag{3.6}$$

From (3.6) and (3.5) we get

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2} H_2) = 0 \text{ on } \mathfrak{U}, \tag{3.7}$$

for $i = 1, 2, 3$. There exists at least one i $(1 \leq i \leq 3)$ that $e_i(H_2) \neq 0$, so, we can assume that $e_1(H_2) \neq 0$, then we have,

$$\mu_{1,2} = \frac{9}{2} H_2 \text{ (locally) on } \mathfrak{U}, \tag{3.8}$$

which gives $\lambda_2\lambda_3 = \frac{9}{2} H_2$.

Now, we claim that

$$e_2(H_2) = e_3(H_2) = 0. \tag{3.9}$$

Because, if $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, we get from (3.7) and (3.8) that $\mu_{1,2} = \mu_{2,2}$ or $\mu_{1,2} = \mu_{3,2}$, which gives $\lambda_3(\lambda_2 - \lambda_1) = 0$ or $\lambda_2(\lambda_1 - \lambda_3) = 0$. But, by assumption $\lambda_i' s$ are mutually distinct, then we get $\lambda_3 = 0$ or $\lambda_2 = 0$, which implies $H_2 = 0$ on $\mathcal{U}$, that contradicts with the definition of $\mathcal{U}$.

Now, in order to prove the main claim, $\mathcal{U} = \emptyset$, firstly, we show that $e_2(\lambda_3) = e_3(\lambda_2) = 0$.

We take into account that, since H is constant and in view of (3.4) and (3.9), one has that $e_i(\lambda_1) = 0$ for $i > 1$.
We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \;\; i, j = 1, 2, 3. \tag{3.10}$$

The compatibility conditions $\nabla_{e_k} < e_i, e_i >= 0$ and $\nabla_{e_k} < e_i, e_j >= 0$ imply respectively that

$$\omega_{ki}^i = 0, \;\; \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.11}$$

for $i \neq j$ and $i, j, k = 1, 2, 3$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.12}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.13}$$

for distinct $i, j, k = 1, 2, 3$. We take into account the action of S on the basis $\{e_1, e_2, e_3\}$, and use the Codazzi equations. The relations

$$\begin{aligned}
&< (\nabla_{e_1} S)e_2, e_1 >=< (\nabla_{e_2} S)e_1, e_1 >, \; < (\nabla_{e_2} S)e_3, e_3 >=< (\nabla_{e_3} S)e_2, e_3 >, \\
&< (\nabla_{e_1} S)e_3, e_3 >=< (\nabla_{e_3} S)e_1, e_3 >, \; < (\nabla_{e_2} S)e_3, e_2 >=< (\nabla_{e_3} S)e_2, e_2 >, \\
&< (\nabla_{e_1} S)e_2, e_3 >=< (\nabla_{e_2} S)e_1, e_3 >, \; < (\nabla_{e_1} S)e_3, e_2 >=< (\nabla_{e_3} S)e_1, e_2 >, \\
&< (\nabla_{e_2} S)e_3, e_1 >=< (\nabla_{e_3} S)e_2, e_1 >, \; [e_2, \; e_3](H_2) = 0,
\end{aligned} \tag{3.14}$$

imply that

$$\begin{aligned}
&\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0, \\
&\omega_{21}^2 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \; \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}, \\
&\omega_{23}^2 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \; \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}.
\end{aligned} \tag{3.15}$$

Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0,\ \nabla_{e_2} e_1 = \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2} e_2,\ \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} e_3,\\
&\nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2-\lambda_1} e_1,\ \nabla_{e_3} e_2 = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3} e_3,\\
&\nabla_{e_1} e_3 = 0,\ \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2} e_2,\ \nabla_{e_3} e_3 = \frac{e_1(\lambda_3)}{\lambda_3-\lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3-\lambda_2} e_2.
\end{aligned} \tag{3.16}$$

Now, the Gauss equation for $< R(e_2, e_3)e_1,\ e_2 >$ and $< R(e_2, e_3)e_1,\ e_3 >$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.17}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right). \tag{3.18}$$

We also have the Gauss equation for $< R(e_1, e_2)e_1,\ e_2 >$ and $< R(e_3, e_1)e_1,\ e_3 >$, which give the following relations

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right) + \left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)^2 = \lambda_1\lambda_2, \tag{3.19}$$

$$e_1\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right) + \left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)^2 = \lambda_1\lambda_3. \tag{3.20}$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}. \tag{3.21}$$

On the other hand, we consider the L_1-biharmonic equation (3.1). It follows from (2.3) and (3.9) that

$$-\mu_{1,1} e_1 e_1(H_2) + \left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right) e_1(H_2) - 9H_2^2(H - \frac{3}{2}\lambda_1) = 0. \tag{3.22}$$

By differentiating (3.22) along on e_2 respectively e_3, and using (3.17), (3.18) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right) = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.23}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right) = \frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2} - \frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{3.24}$$

Using (3.16), we find that

$$[e_1,\ e_2] = \frac{e_1(\lambda_2)}{\lambda_2-\lambda_1} e_2. \tag{3.25}$$

Applying both sides of the equality (3.25) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (3.23), (3.19), (3.20), and (3.21), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.26}$$

(3.26) shows that

$$\begin{aligned} e_2(\lambda_3) &= 0 \text{ or} \\ \frac{e_1(\lambda_3)}{\lambda_3-\lambda_1} &= \frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \end{aligned} \tag{3.27}$$

Suppose that

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}, \tag{3.28}$$

we will derive a contradiction. By differentiating on both sides of (3.28) along on e_1, in view of (3.19), (3.20), gives $\lambda_2=\lambda_3$, this is a contradiction.

Hence, we conclude that $e_2(\lambda_3)=0$.

Analogously, using (3.16), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. Applying both sides of this equality on $\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}$, using (3.24), (3.19), (3.20), and (3.21), we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0. \tag{3.29}$$

In a similar way as above, one can show that $e_3(\lambda_2)$ necessarily has to vanish.

Hence, we conclude that both

$$e_2(\lambda_3)=0 \text{ and } e_3(\lambda_2)=0. \tag{3.30}$$

In view of (3.30), the Gauss equation for $< R(e_2,e_3)e_1,e_3 >$, gives the following relation

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{3.31}$$

By differentiating (3.31) along on e_1, using (3.19) and (3.20) gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.32}$$

Hence, we conclude that $\lambda_2\lambda_3=0$, therefore $H_2=0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M^3. If $H_2\neq 0$, by using (3.1) and (2.2)(c) we obtain that H_3 is constant. Therefore all the mean curvatures H_i are constant functions, this is equivalent to M^3 is isoparametric. An isoparametric hypersurface of Euclidean space can have at most two distinct principal curvatures ([15]), which is a contradiction. So $H_2=0$. This finishes the proof.

3.2. L_2-biharmonic hypersurfaces

Now, we consider the case $r = 2$.

Proof of Theorem 1.2: Let $x : M^3 \to \mathbb{E}^4$ be an isometrically immersed L_2-biharmonic Euclidean hypersurface, first we show that H_3 is constant. Let us consider the open set $\mathcal{U} = \{p \in M^3 : \nabla H_3^2(p) \neq 0\}$, our objective is to show that $\mathcal{U}$ is empty. By using formulaes (3.1), (3.2) and (2.2)(d) on $\mathcal{U}$ we get

$$(SoP_2)\nabla H_3 = -\frac{1}{2}H_3\nabla H_3, \tag{3.33}$$

$$L_2H_3 = 3HH_3^2. \tag{3.34}$$

But by the Cayley-Hamilton theorem we have $P_3 = 0$, so

$$SoP_2 = H_3I, \ (SoP_2)\nabla H_3 = H_3\nabla H_3,$$

which jointly with (3.33) yields $\nabla H_3^2 = 0$ on $\mathcal{U}$, which is a contradiction.

If $H_3 \neq 0$, by using (3.34) we obtain that the mean curvature is constant. Therefore all the mean curvatures H_i are constant functions, this is equivalent to M^3 is isoparametric. An isoparametric hypersurface of Euclidean space can have at most two distinct principal curvatures ([15]), which is a contradiction. So, $H_3 = 0$. This finishes the proof.

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Akram Mohammadpouri,
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz,
Iran.
E-mail address: pouri@tabrizu.ac.ir

and

Firooz Pashaie,
Department of Mathematics,
Faculty of Basic Sciences,
University of Maragheh,
Maragheh,
Iran.
E-mail address: f_pashaie@maragheh.ac.ir