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Nano Topology Induced by Lattices

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ABSTRACT: This paper is to study the nano topology within the context of lattices. In lattice, there is a special class of join-congruence relation which is defined with respect to an ideal. We have defined the nano approximations of a set with respect to an ideal of a lattice. Also some properties of the approximations of a set in a lattice with respect to ideals are studied. On the other hand,the lower and upper approximations have also been studied within the context various algebraic structures.

Key Words: Nano topology, Lattices, Ideal of a lattice.

Contents

1. Introduction

Nano topology[3] explored by Thivagar et al. can be described as a collection of nano approximations, a non-empty finite universe and empty set for which equivalence classes are buliding blocks. On the other hand, the lower and upper approximations have also been studied within the context various algebraic structures. Lattice is a partially ordered set in which all finite subsets have a least upper bound and greatest lower bound. Dedekind worked on lattice theory in the 19th century. The motivation of this paper is to discuss the algebraic properties of nano topology induced by ideals in lattices. In a lattice, there is a special class of join-congruence relation which is defined with respect to an ideal.

2. Preliminaries

The following recalls requisite ideas and preliminaries necessitated in the sequel of our work.

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Definition 2.1[3]: Let $\mathcal{U}$ be a non-empty finite set of objects called the universe $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ named as the indiscerniblity relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.
(i) The Lower appproximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup\limits_{x \in \mathcal{U}} \{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.
(ii) The Upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by
$U_R(X) = \bigcup\limits_{X \in \mathcal{U}} \{R(x) : R(x) \cap X \neq \emptyset\}$
(iii) The Boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not -X with respect to R and it is denoted by
$B_R(X) = U_R(X) - L_R(X)$

Definition 2.2:[3] Let $\mathcal{U}$ be the universe,R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{U}$ and $\tau_R(X)$satisfies the following axioms.
(i) $\mathcal{U}$ and $\emptyset \in \tau_R(X)$
(ii) The union of elements of any subcollection $\tau_R(X)$ is in $\tau_R(X)$.
(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.That is $\tau_R(X)$ forms a topology $\mathcal{U}$ called as the nano topology on $\mathcal{U}$ with respect to X. $(\mathcal{U}, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets. A set A is said to be nano closed if its complement is nano open.

Definition 2.3:[1] A relation defined on a set which is reflexive, anti-symmetric and transitive is called a Partially ordering on S. A set S with a partial ordering on ρ defined on it is called a poset denoted by (S,ρ)

Definition 2.4:[1] A Lattice is a poset in which any two elements have greatest lower bound and least upper bound.

Definition 2.5:[1] Let L be a lattice, a non-empty subset $\mathcal{I}$ of L is called an ideal if

(i) a,b $\in \mathcal{I} \Rightarrow a \vee b \in \mathcal{I}$

(ii) a$\in L, b \in \mathcal{I}$ and a$\leq b \Rightarrow a \in \mathcal{I}$

Definition 2.6:[1] An equivalence relation θ on L is called a join-congruence and meet congruence if, for all $a, b, c, d \in L$, $a \cong b(mod\theta)$ and $c \cong d(mod\theta)$ imply $a \vee c \cong b \vee d(mod\theta)$ and $ac \cong bd(mod\theta)$,respectively. θ is a congruence relation if it is both a join congruence and a meet congruence.

Theorem 2.7:[1] Let $\mathfrak{I}$ be an ideal of L and define a relation $\theta_{\mathfrak{I}}$ on L by for all $a, b, c \in L$, a $\cong b(mod\theta_{\mathfrak{I}}) \Leftrightarrow$ there exist d $\in \mathfrak{I}$ such that $a \vee d = b \vee d$. Then the following statements hold:

(i) : $\theta_{\mathfrak{I}}$ is a join-congruence on L

(ii) : $\theta_{\mathfrak{I}}$ is a congruence on L iff L is distributive.

3. Nano topology in Lattices

In this section we have framed Nano topology over Lattices by means of a special class of join congruence relation which is defined with respect to an ideal.

Definition 3.1: Let the Lattice L be the universe, $\theta_{\mathfrak{I}}$ be a congruence relation with respect to the ideal $\mathfrak{I}$ on L and A be a non-empty subset of L. Then the sets,

(i) $\underline{\theta_{\mathfrak{I}}}(A) = \{x \in L : [x]_{\theta_{\mathfrak{I}}} \subseteq A\}$

(ii) $\overline{\theta_{\mathfrak{I}}}(A) = \{x \in L : [x]_{\theta_{\mathfrak{I}}} \cap A \neq \emptyset\}$

(iii) $B_{\theta_{\mathfrak{I}}}(A) = \overline{\theta_{\mathfrak{I}}}(A)$ - $\underline{\theta_{\mathfrak{I}}}(A)$

Definition 3.2: Let the Lattice L be the universe, $\theta_{\mathfrak{I}}$ be a congruence relation with respect to the ideal $\mathfrak{I}$ on L
$\tau_{\theta_{\mathfrak{I}}}(A) = \left\{L, \emptyset, \underline{\theta_{\mathfrak{I}}}(A), \overline{\theta_{\mathfrak{I}}}(A), B_{\theta_{\mathfrak{I}}}(A)\right\}$ where $A \subseteq L$ and $\tau_{\theta_{\mathfrak{I}}}(A)$ satisfies the following axioms.

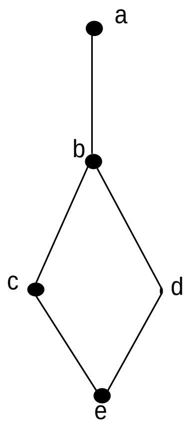
(i) L and $\emptyset \in \tau_{\theta_{\mathfrak{I}}}(A)$

(ii) The union of elements of any subcollection $\tau_{\theta_{\mathfrak{I}}}(A)$ is in $\tau_{\theta_{\mathfrak{I}}}(A)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_{\theta_{\mathfrak{I}}}(A)$ is in $\tau_{\theta_{\mathfrak{I}}}(A)$.

That is $\tau_{\theta_{\mathfrak{I}}}(A)$ forms a Topology on L called as the nano topology on Lattice.

Example 3.3: Let L $= \{a, b, c, d, e\}$ be a lattice, the Hasse diagram of L is given below in the figure. Let $\mathfrak{I} = \{c, e\}$ be an ideal of L,



Hasse diagram for Example 3.3

then the congruence classes with respect to $\mathcal{J}$, $\theta_{\mathcal{J}} = \{\{a\}, \{b, d\}, \{c, e\}\}$ and $A = \{a, b\} \subseteq L$. Thus $\underline{\theta_{\mathcal{J}}}(A) = \{a\}$, $\overline{\theta_{\mathcal{J}}}(A) = \{a, b, d\}$, $B_{\theta_{\mathcal{J}}}(A) = \{b, d\}$ and hence the nano topology on L $\tau_{\theta_{\mathcal{J}}}(A) = \{\{L, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}$.

Theorem 3.4: Let L be a lattice and $\mathcal{J}$ be an ideal of L and for very subset A⊆L,we have

(i) $\underline{\theta_{\mathcal{J}}}(A) \subseteq A \subseteq \overline{\theta_{\mathcal{J}}}(A)$

(ii) $\underline{\theta_{\mathcal{J}}}(\emptyset) = \emptyset = \overline{\theta_{\mathcal{J}}}(\emptyset)$

(iii) $\underline{\theta_{\mathcal{J}}}(L) = L = \overline{\theta_{\mathcal{J}}}(L)$

Theorem 3.5: Let L be a lattice and $\mathcal{J}$ be an ideal of L and for very subsets A , B ⊆L, If $A \subseteq B$ then

(i) $\underline{\theta_{\mathcal{J}}}(A) \subseteq \underline{\theta_{\mathcal{J}}}(B)$

(ii) $\overline{\theta_{\mathcal{J}}}(A) \subseteq \overline{\theta_{\mathcal{J}}}(B)$

(iii) $B_{\theta_{\mathcal{J}}}(A) \subseteq B_{\theta_{\mathcal{J}}}(B)$

Proof: (i). Let $x \in \underline{\theta_{\mathcal{J}}}(A)$. Then $[x]_{\theta_{\mathcal{J}}} \subseteq A$. This implies $[x]_{\theta_{\mathcal{J}}} \subseteq B$, since $A \subseteq B$. Hence x∈ $\underline{\theta_{\mathcal{J}}}(B)$. Thus $\underline{\theta_{\mathcal{J}}}(A) \subseteq \underline{\theta_{\mathcal{J}}}(B)$.
(ii). Let x∈ $\overline{\theta_{\mathcal{J}}}(A)$,then $[x]_{\theta_{\mathcal{J}}} \cap A \neq \emptyset$ and hence $[x]_{\theta_{\mathcal{J}}} \cap B \neq \emptyset$, which implies x∈ $\overline{\theta_{\mathcal{J}}}(A)$. Thus $\overline{\theta_{\mathcal{J}}}(A) \subseteq \overline{\theta_{\mathcal{J}}}(B)$.
(iii). Subtracting (ii) and (i) on both sides we get $\overline{\theta_{\mathcal{J}}}(A) - \underline{\theta_{\mathcal{J}}}(A) \subseteq \overline{\theta_{\mathcal{J}}}(B) - \underline{\theta_{\mathcal{J}}}(B)$, which implies $B_{\theta_{\mathcal{J}}}(A) \subseteq B_{\theta_{\mathcal{J}}}(B)$. □

4. Characterization based on nano approximation

In this section, we have given some characterisations based on nano approximations on the nano topology induced by lattices.
Theorem 4.1: Let $\mathfrak{I}$ be an ideal of L, then the nano topology on L is $\tau_{\theta_{\mathfrak{I}}}(\mathfrak{I}) = \{L, \emptyset, \underline{\theta_{\mathfrak{I}}}(\mathfrak{I})\}$.

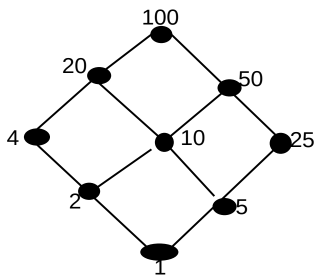
Proof: It's enough if we prove that, $\underline{\theta_{\mathfrak{I}}}(\mathfrak{I})\} = \mathfrak{I} = \overline{\theta_{\mathfrak{I}}}(\mathfrak{I})\}$. From Theorem 3.4(i). we have, $\underline{\theta_{\mathfrak{I}}}(\mathfrak{I}) \subseteq \mathfrak{I} \subseteq \overline{\theta_{\mathfrak{I}}}(\mathfrak{I})$. On the other hand, let x$\in \overline{\theta_{\mathfrak{I}}}(\mathfrak{I})$,then $[x]_{\theta_{\mathfrak{I}}} \cap \mathfrak{I} \neq \emptyset$, then there exist $a \in \mathfrak{I}$ *and* $d \in \mathfrak{I}$ such that x v d = a v d. Since $\mathfrak{I}$ is an ideal, we have x v d$\in \mathfrak{I}$ and thus $x \in \mathfrak{I}$. This means $\overline{\theta_{\mathfrak{I}}}(\mathfrak{I}) \subseteq \mathfrak{I}$.
Moreover, let $x \in \mathfrak{I}$ and $a \in [x]_{\theta_{\mathfrak{I}}}$. Then there exists $d \in \mathfrak{I}$ such that $a \vee d = x \vee d$. Then we have $a \vee d \in \mathfrak{I}$, and thus $a \in \mathfrak{I}$. So, $[x]_{\theta_{\mathfrak{I}}} \subseteq \mathfrak{I}$, which implies $x \in \underline{\theta_{\mathfrak{I}}}(\mathfrak{I}$. Therefore $\mathfrak{I} \subseteq \underline{\theta_{\mathfrak{I}}}(\mathfrak{I})\}$.
From the above, we have $\underline{\theta_{\mathfrak{I}}}(\mathfrak{I})\} = \mathfrak{I} = \overline{\theta_{\mathfrak{I}}}(\mathfrak{I})\}$. Hence $\tau_{\theta_{\mathfrak{I}}}(\mathfrak{I}) = \{L, \emptyset, \underline{\theta_{\mathfrak{I}}}(\mathfrak{I})\}$. □

Theorem 4.2: Let L be a lattice and $\mathfrak{I}$ and $\mathfrak{J}$be ideals of L, then $\overline{\theta_{\mathfrak{I}}}(\mathfrak{I} \cap \mathfrak{J}) = \mathfrak{I}$.

Proof: Let $x \in \mathfrak{I}$, we have x$\wedge y \in \mathfrak{I} \cap \mathfrak{J}$ for every $y \in \mathfrak{J}$. On the other hand,
x $\vee x = (x \wedge y) \vee x$ which means that $x \wedge y \equiv x(mod\theta_{\mathfrak{I}})$. Hence, $[x]_{\theta_{\mathfrak{I}}} \cap (\mathfrak{I} \cap \mathfrak{J}) \neq \emptyset$, so $x \in \overline{\theta_{\mathfrak{I}}}(\mathfrak{I} \cap \mathfrak{J})$. Conversely, by Theorem 3.5.(ii) and Theorem 4.1 we have $\overline{\theta_{\mathfrak{I}}}(\mathfrak{I} \cap \mathfrak{J}) \subseteq \overline{\theta_{\mathfrak{I}}}(\mathfrak{I}) = \mathfrak{I}$. □

Remark 4.3: The above theorem can also revealed by the following example.

Example 4.4: Let L $= \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$, the factors of 100 which forms a lattice under divisibility. Its Hasse Diagram is given below.



Hasse Diagram for factors of 100

Let $\mathfrak{I} = \{1, 5, 25\}$ and $\mathfrak{J} = \{1, 2, 5, 10\}$ be ideals of L, then $\mathfrak{I} \cap \mathfrak{J} = \{1, 5\}$. The congruence classes with respect to $\mathfrak{I}$ and $\mathfrak{J}$ are $\theta_{\mathfrak{I}} = \{\{1, 5, 25\}, \{2, 10, 50\}, \{4, 20, 100\}$ $\theta_{\mathfrak{J}} = \{\{1, 2, 5, 10\}, \{4, 20\}, \{25, 50\}, \{100\}\}$. Thus $\overline{\theta_{\mathfrak{I}}}(\mathfrak{I} \cap \mathfrak{J}) = \{1, 5, 25\} = \mathfrak{I}$ and also $\overline{\theta_{\mathfrak{J}}}(\mathfrak{I} \cap \mathfrak{J}) = \{1, 2, 5, 10\} = \mathfrak{J}$. Hence the Nano toplogy $\tau_{\theta_{\mathfrak{I}}}(\mathfrak{I} \cap \mathfrak{J}) = \{L, \emptyset, \{1, 5, 25\}\}$. $\tau_{\theta_{\mathfrak{J}}}(\mathfrak{I} \cap \mathfrak{J}) = \{L, \emptyset, \{1, 2, 5, 10\}\}$.

Theorem 4.5: Let L be a lattice and $\mathfrak{I}$ be an ideal of L and for very subset A $\subseteq$L, then

(i) $\underline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A)) = \underline{\theta_{\mathcal{I}}}(A)$

(ii) $\overline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A)) = \overline{\theta_{\mathcal{I}}}(A)$

(iii) $\overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A)) = \underline{\theta_{\mathcal{I}}}(A)$

(iv) $\underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A)) = \overline{\theta_{\mathcal{I}}}(A)$

Proof: (i): By definition, $\underline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A)) = \{x \in L : [x]_{\theta_{\mathcal{I}}} \subseteq \underline{\theta_{\mathcal{I}}}(A)\} = \{x \in L : [x]_{\theta_{\mathcal{I}}} \subseteq A\}$, since $\underline{\theta_{\mathcal{I}}}(A) \subseteq A = \underline{\theta_{\mathcal{I}}}(A)$.
(ii): $\overline{\theta_{\mathcal{I}}}(A)$. $= \{x \in L : [x]_{\theta_{\mathcal{I}}} \cap A \neq \emptyset\} = \{x \in L : [x]_{\theta_{\mathcal{I}}} \cap \overline{\theta_{\mathcal{I}}}(A) \neq \emptyset\}$, since $A \subseteq \overline{\theta_{\mathcal{I}}}(A) = \overline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A))$.
(iii): Let $a \in \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A)) \Rightarrow [a]_{\theta_{\mathcal{I}}} \cap \underline{\theta_{\mathcal{I}}}(A) \neq \emptyset \Rightarrow x \in [a]_{\theta_{\mathcal{I}}}$ and $x \in \underline{\theta_{\mathcal{I}}}(A)$ $\Rightarrow [x]_{\theta_{\mathcal{I}}} \subseteq A \Rightarrow [a]_{\theta_{\mathcal{I}}} \subseteq A \Rightarrow a \in \underline{\theta_{\mathcal{I}}}(A)$. Thus $\overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A)) \subseteq \underline{\theta_{\mathcal{I}}}(A))$.
Suppose if possible $\underline{\theta_{\mathcal{I}}}(A)) \nsubseteq \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A))$. Then there exist $a \in \underline{\theta_{\mathcal{I}}}(A))$ such that $a \notin \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A))$. Now $a \notin \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A))$ implies $[a]_{\mathcal{I}} \cap \underline{\theta_{\mathcal{I}}}(A)) = \emptyset$ and hence $a \notin \underline{\theta_{\mathcal{I}}}(A))$, which is a contradiction. Hence $\underline{\theta_{\mathcal{I}}}(A)) \subseteq \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A))$. Thus $\underline{\theta_{\mathcal{I}}}(A)) = \overline{\theta_{\mathcal{I}}}(\underline{\theta_{\mathcal{I}}}(A))$.
(iv). Let $a \in \underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A))$. Then $[a]_{\mathcal{I}} \subseteq \overline{\theta_{\mathcal{I}}}(A)$. This implies $a \in \overline{\theta_{\mathcal{I}}}(A)$. Thus $\underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A)) \subseteq \overline{\theta_{\mathcal{I}}}(A)$.
If possible $\overline{\theta_{\mathcal{I}}}(A) \nsubseteq \underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A))$. Then there exist $a \in \overline{\theta_{\mathcal{I}}}(A)$ such that $a \notin \underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A))$. Hence there exist $x \in [a]_{\mathcal{I}}$ such that $x \notin \overline{\theta_{\mathcal{I}}}(A)$, This implies $[x]_{\theta_{\mathcal{I}}} \cap A = \emptyset$. Thus $[a]_{\theta_{\mathcal{I}}} \cap A = \emptyset$. (Since $x \in [a]_{\theta_{\mathcal{I}}} \Rightarrow [x]_{\theta_{\mathcal{I}}} = [a]_{\theta_{\mathcal{I}}}$), which is a contradiction. Hence $\overline{\theta_{\mathcal{I}}}(A) \subseteq \underline{\theta_{\mathcal{I}}}(\overline{\theta_{\mathcal{I}}}(A))$. □

Theorem 4.6: Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of L and $\mathcal{I} \subseteq \mathcal{J}$. If A is a non-empty subset of L, then $\underline{\theta_{\mathcal{J}}}(A) \subseteq \underline{\theta_{\mathcal{I}}}(A)$ and $\overline{\theta_{\mathcal{I}}}(A) \subseteq \overline{\theta_{\mathcal{J}}}(A)$.

Remark 4.7: In the above theorem, if we choose A as an ideal of L then the $\overline{\theta_{\mathcal{I}}}(\mathcal{J})$, $\underline{\theta_{\mathcal{I}}}(\mathcal{J})$ will be an ideal of L which can be shown by the following theorem.

Theorem 4.8: Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of L and $\mathcal{I} \subseteq \mathcal{J}$ then the following statements are equivalent:

(i) $\mathcal{I} \subseteq \mathcal{J}$

(i) $\mathcal{J} = \overline{\theta_{\mathcal{I}}}(\mathcal{J})$

(ii) $\mathcal{J} = \underline{\theta_{\mathcal{I}}}(\mathcal{J})$

Proof: $(i) \Leftrightarrow (ii)$: If $\mathcal{I} \subseteq \mathcal{J}$, let $x \in \overline{\theta_{\mathcal{I}}}(\mathcal{J})$, then there exist $y \in \mathcal{J}$ and $d \in \mathcal{I} \subseteq \mathcal{J}$ such that x V d = y V d. Since $\mathcal{J}$ is an ideal we have x V d = y V d $\in \mathcal{J}$, then $x \in \mathcal{J}$ which implies $\overline{\theta_{\mathcal{I}}}(\mathcal{J}) \subseteq \mathcal{J}$, and since $\mathcal{J} \subseteq \overline{\theta_{\mathcal{I}}}(\mathcal{J})$ thus we obtain $\mathcal{J} = \overline{\theta_{\mathcal{I}}}(\mathcal{J})$.
$(ii) \Leftrightarrow (i)$ If $\mathcal{J} = \overline{\theta_{\mathcal{I}}}(\mathcal{J})$. By Theorem 3.5 (ii)and Theorem 4.2, $\mathcal{I} = \overline{\theta_{\mathcal{I} \cap \mathcal{J}}} \subseteq \overline{\theta_{\mathcal{I}}}(\mathcal{J}) = \mathcal{J}$.
$(ii) \Leftrightarrow (iii)$: If $\mathcal{J} = \overline{\theta_{\mathcal{I}}}(\mathcal{J})$, let $x \in \mathcal{J}$ and $y \equiv x(mod\theta_{\mathcal{I}})$. Assume that$y \notin \mathcal{J}$,then $y \notin \overline{\theta_{\mathcal{I}}}(\mathcal{J})$. Hence, $[x]_{\theta_{\mathcal{I}}} \cap \mathcal{J} = [y]_{\theta_{\mathcal{I}}} \cap \mathcal{J} = \emptyset$ which implies that $x \notin \overline{\theta_{\mathcal{I}}}(\mathcal{J}) = \mathcal{J}$. It contradicts with $x \in \mathcal{J}$, so $y \in \mathcal{J}$. Thus $[x]_{\theta_{\mathcal{I}}} \subseteq \mathcal{J}$ this means that $x \in \underline{\theta_{\mathcal{I}}}(\mathcal{J})$ and by

Theorem 3.5(i) we have $\mathcal{J} = \underline{\theta_{\mathcal{I}}}(\mathcal{J})$.
$(iii) \Leftrightarrow (ii)$: Now suppose that $\mathcal{J} = \underline{\theta_{\mathcal{I}}}(\mathcal{J})$. Let $x \in \overline{\theta_{\mathcal{I}}}(J)$, there exist $y \in \mathcal{J}$ such that $y \equiv x(mod\theta_{\mathcal{I}})$. Since $\mathcal{J} = \underline{\theta_{\mathcal{I}}}(\mathcal{J})$. we have $[x]_{\theta_{\mathcal{I}}} = [y]_{\theta_{\mathcal{I}}} = \mathcal{J}$ which means $x \in \underline{\theta_{\mathcal{I}}}(\mathcal{J}) = \mathcal{J}$. Thus by Theorem 2.5(i) we have $\overline{\theta_{\mathcal{I}}}(\mathcal{J}) = \mathcal{J}$. □

Theorem 4.9: Suppose $\mathcal{I}$ and $\mathcal{J}$ be ideals of L and $\mathcal{I} \subseteq \mathcal{J}$, then the nano topology $\tau_{\theta_{\mathcal{I}}}(\mathcal{J}) = \{L, \emptyset, \mathcal{J}\}$.

Proof: By the above theorem we have if $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{J} = \overline{\theta_{\mathcal{I}}}(\mathcal{J}) = \underline{\theta_{\mathcal{I}}}(\mathcal{J})$ which implies that,$B_{\theta_{\mathcal{I}}}(\mathcal{J}) = \emptyset$. Thus $\tau_{\theta_{\mathcal{I}}}(\mathcal{J}) = \{L, \emptyset, \mathcal{J}\}$. □

Remark 4.10: The following example reveals the above theorem.

Example 4.11: Consider the lattice L in Example 4.4 and let $\mathcal{I} = \{1, 5\}$ and $\mathcal{J} = \{1, 5, 25\}$ be ideals of L such that $\mathcal{I} \subseteq \mathcal{J}$ and the congruence class with respect to the ideal $\mathcal{I}$ relation $\theta_{\mathcal{I}} = \{\{1, 5\}, \{2, 10\}, \{4, 20\}, \{25\}, \{50\}, \{100\}$ then the nano topology $\tau_{\theta_{\mathcal{I}}}(\mathcal{J}) = \{L, \emptyset, \{1, 5, 25\}\}$.

5. Conclusion

In this paper the universe of objects is endowed with a lattice structure and a join congruence relation is defined with respect to an ideal. This universal set can also be further extended over quotient ideals to give a new structure.

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