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Solvability of Bigeometric Differential Equations by Numerical Methods *

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ABSTRACT: The objective of this paper is to derive and analyze Bigeometric-Euler, Taylor's Bigeometric-series and Bigeometric-Runge-Kutta methods of different orders for the approximation of initial value problems of Bigeometric-differential equations.

Key Words: Bigeometric-calculus, geometric real numbers, geometric arithmetic.

Contents

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1. Introduction

Differential and integral calculi were created independently by Isaac Newton and Gottfried Wilhelm Leibniz in the second half of the 17th century. After that Leonhard Euler redirected calculus by giving a central place to the concept of function, and thus founded analysis. Differentiation and integration are the basic operations in calculus and analysis. Actually, they are the infinitesimal versions of the subtraction and addition on numbers, respectively. Often this calculus is referred as Newtonian calculus. The underlying feature of Newtonian calculus is that it studies functions by comparing them with linear functions.

In the period between 1967 to 1970 Michael Grossman and Robert Katz [12] indicated that Newtonian calculus can be realised in the forms of comparison of functions with nonlinear functions calling them non-Newtonian calculi. Multiplicative and bigeometric calculi are the two most popular non-Newtonian calculi. In fact, they are modification of each other. In these calculi the role of addition and subtraction are changed to multiplication and division.

Bigeometric calculus was prompted by Grossman [11]. We refer to Stanley [20], Jane Grossman [13], Grosssman et al. [14], Grossman et al. [15], Grossman [16], Campbell [9], Michael Coco [10], Córdova-Lepe [18], Bashirov et al. [3,4], Spivey [19], Bashirov and Rıza [2], Çakmak and Başar [8], Kadak and Özlük [17], Tekin and Başar [21], Türkmen and Başar [22] for different types of non-Newtonian calculi and applications. For numerical analysis we refer the book of Burden and Faires [7].

Bigeometric-calculus is an alternative to the usual calculus of Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in bigeometric-calculus. Generally, in growth related problems, price elasticity, numerical approximation problems bigeometric-calculus can be advocated instead of a traditional Newtonian one. Below in Section 2, a quick outlook on generators of arithmetic and geometric arithmetic is given. Then we establish relations between geometric arithmetic and classical arithmetic. In Section 3, we mention some previously established results which are essential to deduce new results in the present paper. In Section 4, we introduce numerical new methods, namely, bigeometric Euler method, Taylor's bigeometric series method and bigeometric Runge Kutta method for approximation of bigeometric initial value problems together with examples.

Throughout the article , instead of the phase "bigeometric calculus" term "*G*-calculus" will be used because, depending on the pioneering works of Grossman [11] and Grossman and Katz's [12], we are trying to develop their work with the help of geometric arithmetic system.

2. $\alpha-$generator and geometric real field

A generator is a one-to-one function whose domain is the set $\mathbb{R}$ of real numbers and range is a subset $B \subset \mathbb{R}$. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic and exponential function generates geometric arithmetic. As a generator, we choose the function α such that whose basic algebraic operations are defined as follows:

$$
\begin{array}{ll}
\alpha - \text{addition} & x\dot{+}y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)], \\
\alpha - \text{subtraction} & x\dot{-}y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)], \\
\alpha - \text{multiplication} & x\dot{\times}y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)], \\
\alpha - \text{division} & x\dot{/}y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)], \\
\alpha - \text{order} & x\dot{<}y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)
\end{array}
$$

for x and y from the range of the function α.

If we choose the exponential function $\alpha(x) = e^x$ for $x \in \mathbb{R}$ as an α-generator then $\alpha^{-1}(z) = \ln z$ and $\alpha - arithmetic$ turns out to be geometric arithmetic.

$$
\begin{array}{ll}
\text{geometric addition} & x \oplus y = e^{(\ln x + \ln y)} = x \cdot y, \\
\text{geometric subtraction} & x \ominus y = e^{(\ln x - \ln y)} = x/y, \\
\text{geometric multiplication} & x \odot y = e^{(\ln x \ln y)} = x^{\ln y} = y^{\ln x}, \\
\text{geometric division} & x \oslash y = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}}, y \neq 1.
\end{array}
$$

Since the logarithmic function is strictly increasing, that is $\ln(x) < \ln(y)$ for $0 < x < y$, without loss of generality, we use $x < y$ instead of the geometric order.

Türkmen and Başar [22] defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows:

$$
\begin{aligned}
\mathbb{Z}(G) &= \{e^x : x \in \mathbb{Z}\} \\
\mathbb{R}(G) &= \{e^x : x \in \mathbb{R}\} = (0, \infty) \\
\mathbb{C}(G) &= \{e^z : z \in \mathbb{C}\} = \mathbb{C}\backslash\{0\}.
\end{aligned}
$$

If $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real number line, then $\bar{\mathbb{R}}(G) = [0, \infty)$.

Remark 2.1. $(\mathbb{R}(G), \oplus, \odot)$ *is a field with the newtral elements of addition and multiplication to be 1 and* e, *respectively, since*

1. $(\mathbb{R}(G), \oplus)$ *is a geometric additive Abelian group with geometric zero* 1,
2. $(\mathbb{R}(G)\backslash 1, \odot)$ *is a geometric multiplicative Abelian group with geometric identity* e,
3. $\odot$ *is distributive over* $\oplus$.

But $(\mathbb{C}(G), \oplus, \odot)$ is not a field. This is a consequence of a multivalued nature of complex logarithm. Let us show that the geometric binary operation $\odot$ is not associative in $\mathbb{C}(G)$. Take $x = e^{1/4}$, $y = e^4$, and $z = e^{(1+i\pi/2)} = ie$. Then $(x \odot y) \odot z = e \odot z = z = ie$ while $x \odot (y \odot z) = x \odot e^4 = e$.

Let us define geometric positive real numbers and geometric negative real numbers as follows:

$$\mathbb{R}^+(G) = \{x \in \mathbb{R}(G) : x > 1\}$$
$$\mathbb{R}^-(G) = \{x \in \mathbb{R}(G) : x < 1\}.$$

For $x \in \mathbb{R}(G)$, define its geometric absolute value by

$$|x|^G = \begin{cases} x, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } 0 < x < 1 \end{cases}$$

Then for all $x, y \in \mathbb{R}(G)$, the following relations hold:

- $x \oplus y = xy$.
- $x \ominus y = x/y$.
- $x \odot y = x^{\ln y} = y^{\ln x}$.
- $x \oslash y$ or $\frac{x}{y}_G = x^{\frac{1}{\ln y}}, y \neq 1$.
- $x^{2_G} = x \odot x = x^{\ln x}$.
- $x^{p_G} = x^{\ln^{p-1} x}$.
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$.
- $x^{-1_G} = e^{\frac{1}{\log x}}$.
- $x \odot e = x$ and $x \oplus 1 = x$.
- $e^n \odot x = x^n$.
- $|x|^G \geq 1$.
- $\sqrt{x^{2_G}}^G = |x|^G$
- $|e^y|^G = e^{|y|}$.
- $|x \odot y|^G = |x|^G \odot |y|^G$.
- $|x \oplus y|^G \leq |x|^G \oplus |y|^G$.
- $|x \oslash y|^G = |x|^G \oslash |y|^G$.
- $|x \ominus y|^G \geq |x|^G \ominus |y|^G$.
- $(0_G \ominus 1_G) \odot (x \ominus y) = y \ominus x$.

Briefly, the last relation can be written as $\ominus (x \ominus y) = y \ominus x$. Further, $e^{-x} = \ominus e^x$ holds for all $x \in \mathbb{Z}^+$. Thus the set of all geometric integers turns out to the following:

$$\mathbb{Z}(G) = \{\ldots, e^{-3}, e^{-2}, e^{-1}, e^0, e^1, e^2, e^3, \ldots\} = \{\ldots, \ominus e^3, \ominus e^2, \ominus e, 1, e, e^2, e^3, \ldots\}.$$

3. Basic definitions and results

The following definitions and results were introduced and studied by Boruah and Hazarika [5,6].

3.1. Geometric Binomial Formula

$$(i)\ (a \oplus b)^{2_G} = a^{2_G} \oplus e^2 \odot a \odot b \oplus b^{2_G}.$$
$$(ii)\ (a \oplus b)^{3_G} = a^{3_G} \oplus e^3 \odot a^{2_G} \odot b \oplus e^3 \odot a \odot b^{2_G} \oplus b^{3_G}.$$

In general

$$\begin{aligned}(iii)\ (a \oplus b)^{n_G} &= a^{n_G} \oplus e^{\binom{n}{1}} \odot a^{(n-1)_G} \odot b \\ &\oplus e^{\binom{n}{2}} \odot a^{(n-2)_G} \odot b^{2_G} \oplus .... \oplus b^{n_G} \\ &= {}_G\sum_{r=0}^{n} e^{\binom{n}{r}} \odot a^{(n-r)_G} \odot b^{r_G}.\end{aligned}$$

Similarly

$$(a \ominus b)^{n_G} = {}_G\sum_{r=0}^{n} (\ominus e)^{r_G} \odot e^{\binom{n}{r}} \odot a^{(n-r)_G} \odot b^{r_G}.$$

Note: $x \oplus x = x^2$. Also $e^2 \odot x = x^{\ln(e^2)} = x^2$. So, $e^2 \odot x = x^2 = x \oplus x$.

3.2. Geometric Factorial

In [5], we defined geometric factorial notation $!_G$ as

$$n!_G = e^n \odot e^{n-1} \odot e^{n-2} \odot \cdots \odot e^2 \odot e = e^{n!}.$$

3.3. Generalized Geometric Forward Difference Operator

Let

$$\begin{aligned}\Delta_G f(a) &= f(a \oplus h) \ominus f(a).\\ \Delta_G^2 f(a) &= \Delta_G f(a \oplus h) \ominus \Delta_G f(a)\\ &= f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a).\\ \Delta_G^3 f(a) &= \Delta_G^2 f(a \oplus h) \ominus \Delta_G^2 f(a)\\ &= f(a \oplus e^3 \odot h) \ominus e^3 \odot f(a \oplus e^2 \odot h) \oplus e^3 \odot f(a \oplus h) \ominus f(a).\end{aligned}$$

Thus n^{th} forward difference is

$$\Delta_G^n f(a) = {}_G\sum_{k=0}^{n} (\ominus e)^{k_G} \odot e^{\binom{n}{k}} \odot f(a \oplus e^{n-k} \odot h), \text{with } (\ominus e)^{0_G} = e.$$

3.4. Generalized Geometric Backward Difference Operator

Let

$$\begin{aligned}
\nabla_G f(a) &= f(a) \ominus f(a \ominus h).\\
\nabla_G^2 f(a) &= \nabla_G f(a) \ominus \nabla_G f(a \ominus h)\\
&= f(a) \ominus e^2 \odot f(a \ominus h) \oplus f(a \ominus e^2 \odot h).\\
\nabla_G^3 f(a) &= \nabla_G^2 f(a) \ominus \nabla_G^2 f(a - h)\\
&= f(a) \ominus e^3 \odot f(a \ominus h) \oplus e^3 \odot f(a \ominus e^2 \odot h) \ominus f(a \ominus e^3 \odot h).
\end{aligned}$$

Thus, n^{th} geometric backward difference is

$$\nabla_G^n f(a) = {}_G\sum_{k=0}^{n} (\ominus e)^{k_G} \odot e^{\binom{n}{k}} \odot f(a \ominus e^k \odot h).$$

3.5. Geometric Newton-Gregory formula for forward interpolation

In [5], we introduced the Geometric Newton-Gregory formula for forward interpolation as follows:

$$\begin{aligned}
f(a \oplus h \odot x) =& f(a) \oplus x \odot \Delta_G f(a) \oplus \frac{x \odot (x \ominus e)}{2!_G}{}_G \odot \Delta_G^2 f(a)\\
&\oplus \frac{x \odot (x \ominus e) \odot (x \ominus e^2)}{3!_G}{}_G \odot \Delta_G^3 f(a) \oplus \cdots \\
&\oplus \frac{x \odot (x \ominus e) \odot (x \ominus e^2) \odot \cdots \odot (x \ominus e^{n-1})}{n!_G}{}_G \odot \Delta_G^n f(a).
\end{aligned} \tag{3.1}$$

3.6. Geometric Newton-Gregory formula for backward interpolation

In [5] we introduced the Geometric Newton-Gregory formula for backward interpolation as follows:

$$\begin{aligned}
f(a \oplus e^n \odot h \oplus x \odot h) =& f(a \oplus e^n \odot h) \oplus x \odot \nabla_G f(a \oplus e^n \odot h)\\
&\oplus \frac{x \odot (x \oplus e)}{2!_G}{}_G \odot \nabla_G^2 f(a \oplus e^n \odot h)\\
&\oplus \frac{x \odot (x \oplus e) \odot (x \oplus e^2)}{3!_G}{}_G \odot \nabla_G^3 f(a \oplus e^n \odot h) \oplus \cdots \\
&\oplus \frac{x \odot (x \oplus e) \odot (x \oplus e^2) \odot \cdots \odot (x \oplus e^{n-1})}{n!_G}{}_G\\
&\odot \nabla_G^n f(a \oplus e^n \odot h).
\end{aligned} \tag{3.2}$$

3.7. G-Derivative

[6] G-differentiation of a bi-positive function f is defined as

$$\frac{d^G f}{dx^G} = f^G(x) = {}_G\lim_{h \to 1} \frac{f(x \oplus h) \ominus f(x)}{h}{}_G \text{ for } h \in \mathbb{R}(\mathbb{G}). \tag{3.3}$$

Equivalently

$$\frac{d^G f}{dx} = {}_G\lim_{h\to 1} \frac{f(x\oplus h)\ominus f(x)}{h}{}_G$$
$$= \lim_{h\to 1}\left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}}.$$

The n^{th} G-derivative of $f(x)$ is denoted by $f^{[n]}(x)$. The relation between G-derivative and classical derivative is

$$f^G(x) \text{ or } f^{[1]}(x) = e^{x\frac{f'(x)}{f(x)}}. \tag{3.4}$$

3.8. Geometric Taylor's Series

If f possesses G-derivative of every order in $[x, xh]$ then its Taylor's expansion is

$$f(xh) = f(x).\left[f^{[1]}(x)\right]^{\ln h}.\left[f^{[2]}(x)\right]^{\frac{\ln^2 h}{2!}} \ldots \left[f^{[n]}(x)\right]^{\frac{\ln^n h}{n!}} \ldots = \prod_{n=0}^{\infty}\left[f^{[n]}(x)\right]^{\frac{\ln^n h}{n!}}. \tag{3.5}$$

which can be written in geometric form as follows:

$$f(x\oplus h) = f(x)\oplus h\odot f^{[1]}(x)\oplus \frac{h^{2_G}}{2!_G}{}_G\odot f^{[2]}(x)\oplus\ldots\oplus\frac{h^{n_G}}{n!_G}{}_G\odot f^{[n]}(x)\oplus\ldots$$
$$= {}_G\sum_{n=0}^{\infty}\frac{h^{n_G}}{n!_G}{}_G\odot f^{[n]}(x), \tag{3.6}$$

where $n!_G = e^{n!}$ and $h^{n_G} = h^{\ln^{(n-1)} h}$. The equivalent expressions (3.5) and (3.6) are called Taylor's product and Geometric Taylor's series, respectively.

4. Numerical Methods and Solution of G-differential Equations

An equation involving G-differential coefficient is called a G-differential equation. If the exact solution of a given G-differential equation can not be determined, numerical approximation methods can be adopted to determine the solution of the equation at different mesh points. Of course, to compare the exact value and the approximated value, we will take examples of initial value problems whose exact solutions are known. Here we deduce the Euler's, Taylor's and Runge-Kutta methods to compute numerical solutions of initial value problems of G-calculus. In [1] Aniszewska has been developing such methods for multiplicative calculi taking arguments as $a, a+h, a+2h, \ldots$ which form an arithmetic progression. But, all the methods going to be discussed here are based on geometric arithmetic and G-calculus [5]. So, values of the argument will be taken as $a, a\oplus h, a\oplus e^2\odot h, a\oplus e^2\odot h, \ldots$. i.e. in ordinary sense, values of the argument form a geometric progression $a, ah, ah^2, ah^3, \ldots$

4.1. G-Euler's Method

In classical calculus, Euler's method is the most elementary approximation technique for solving initial value problems. Here we deduce Euler's method for G-derivatives which we'll call as G-Euler's method. Though it gives weak approximation in practice, the simplicity of its derivation can be used to illustrate the techniques involved in the construction of more advanced techniques.

The objective of G-Euler's method is to obtain approximations to initial value problems having unique solution at given value of the argument. Let us consider the initial value problem

$$\frac{d^G y}{dx^G} = f(x,y), y(a) = y_0, \quad a \leq x \leq b \text{ and } a \geq 1. \tag{4.1}$$

We first make the stipulation that the mesh points form a geometric progression throughout the interval $[a,b]$. For, we choose a positive integer N and selecting the mesh points

$$x_i = ah^i, \text{ for each } i = 0,1,2,...,N.$$

The common geometric ratio $h = \left(\frac{b}{a}\right)^{\frac{1}{N}} = \frac{x_{i+1}}{x_i}$ will be called step size.

We will use Geometric Taylor's Theorem stated in [6] to derive G-Euler's method. Let $y(x)$ be the unique solution to (4.1) having two consecutive G-derivatives on $[a,b]$, so that for each $i = 0,1,2,...,N-1$,

$$y(x_{i+1}) = y(x_i).\left[y^{[1]}(x_i)\right]^{\ln\left(\frac{x_{i+1}}{x_i}\right)}.\left[y^{[2]}(\xi_i)\right]^{\frac{\ln^2\left(\frac{x_{i+1}}{x_i}\right)}{2!}}$$

for some $\xi \in (x_i, x_{i+1})$. Putting $h = \frac{x_{i+1}}{x_i}$, we get

$$y(x_{i+1}) = y(x_i).\left[y^{[1]}(x_i)\right]^{\ln h}.\left[y^{[2]}(\xi_i)\right]^{\frac{\ln^2 h}{2}}.$$

As $y(x)$ satisfies the differential equation (4.1),

$$y(x_{i+1}) = y(x_i).\left[f\left(x_i, y(x_i)\right)\right]^{\ln h}.\left[y^{[2]}(\xi_i)\right]^{\frac{\ln^2 h}{2}}. \tag{4.2}$$

Neglecting second and higher order G-derivatives, let ω_i be the approximation to $y(x_i)$, i.e. $\omega_i \approx y(x_i)$, for $i = 0,1,2,3,\ldots,N-1$. Thus the G-Euler's method is

$$\begin{aligned} &\omega_0 = y_0, \\ &\omega_{i+1} = \omega_i.\left[f(x_i,\omega_i)\right]^{\ln h}, \text{ for } i = 0,1,2,3,\ldots,N-1. \end{aligned} \tag{4.3}$$

In terms of geometric arithmetic, G-Euler's method can written as

$$\begin{aligned} &\omega_0 = y_0, \\ &\omega_{i+1} = \omega_i \oplus h \odot f(x_i,\omega_i), \text{ for } i = 0,1,2,3,...,N-1. \end{aligned} \tag{4.4}$$

Equation (4.4) will be called G-difference equation associated to G-Euler's Method.

Example 4.1. *We consider an G-initial value problem as*

$$\frac{d^G y}{dx^G} = e^{\left(\frac{x-1}{y}\right)}, 1 \leq x \leq 4, y(1) = 1 \tag{4.5}$$

whose exact solution is

$$y(x) = x - \ln x \tag{4.6}$$

and corresponding ordinary initial value problem is

$$\frac{dy}{dx} = 1 - \frac{1}{x}, 1 \leq x \leq 4, y(1) = 1. \tag{4.7}$$

To approximate (4.5) with the help of G-Euler's method, we consider $N = 6$. Then

$$h = \left[\frac{b}{a}\right]^{1/N} = \left[\frac{4}{1}\right]^{1/6} = 1.25992105$$

$$x_i = a.h^i = (1.25992105)^i, \qquad \omega_0 = 1, \text{ and}$$

$$\omega_{i+1} = \omega_i.\left[f(x_i, \omega_i)\right]^{\ln h} = \omega_i.\left[e^{\left(\frac{x_i-1}{\omega_i}\right)}\right]^{\frac{\ln 4}{6}}, \text{ for } i = 0, 1, 2, ..., 6.$$

Therefore

$$\omega_1 = \omega_0.\left[e^{\left(\frac{x_0-1}{\omega_0}\right)}\right]^{\frac{\ln 4}{6}} = 1.\left[e^{\left(\frac{1-1}{1}\right)}\right]^{\frac{\ln 4}{6}} = 1,$$

$$\omega_2 = \omega_1.\left[e^{\left(\frac{x_1-1}{\omega_1}\right)}\right]^{\frac{\ln 4}{6}} = 1.\left[e^{\left(\frac{1.25992105-1}{1}\right)}\right]^{\frac{\ln 4}{6}}$$

$$= e^{(0.25992105 \times 0.23104906)} = 1.061894433,$$

and so on.

On the other hand, corresponding actual values $y_0, y_1, y_2, ..., y_6$ are given by

$$y_i = y(x_i) = x_i - \ln(x_i), \text{ for } i = 0, 1, 2, ..., 6.$$

TABLE 1 shows the comparison between approximated values ω_i and corresponding actual values y_i at x_i. Also we have shown the error $y_i - x_i$.

Table 1: G-Euler's Approximation at $x = 4$

i	x_i	ω_i	Exact value= $y_i = y(x_i)$	Error= $\lvert y_i - \omega_i \rvert$
0	1.000000000	1.000000000	1.000000000	0.000000000
1	1.259921050	1.000000000	1.028871990	0.028871990
2	1.587401052	1.061894433	1.125302932	0.063408498
3	2.000000000	1.206667442	1.306852819	0.100185377
4	2.519842100	1.461318892	1.595645859	0.134326967
5	3.174802104	1.858261811	2.019556803	0.161294992
6	4.000000000	2.435246520	2.613705639	0.178459118

From the TABLE 1, it is seen that error grows rapidly as the value x_i increases. Though errors can be minimized by increasing the number of steps, i.e. N, still they are not negligible errors. That is why G-Euler's method will be less admissible for better approximation. Of course G-Euler method gives better accuracy than the ordinary Euler's method with same number of steps. For, we consider the corresponding ordinary initial value problem given in the above example by equation (4.7). In this case

$$h = \frac{b-a}{N} = \frac{4-1}{6} = 0.5$$
$$x_i = a + ih = 1 + 0.5i, \quad \omega_0 = 1, \text{ and difference equations are given by}$$
$$\omega_{i+1} = \omega_i + h.f(x_i, \omega_i) = \omega_i + 0.5(1 - \frac{1}{x_i}), \text{ for } i = 0, 1, 2, ..., 6.$$

TABLE 2 gives the approximate value of the function at $x = 4$ with respect to the ordinary initial value problem.

Table 2: Ordinary Euler's Approximation at $x = 4$

i	x_i	ω_i	Exact value= $y_i = y(x_i)$	Error= $\lvert y_i - \omega_i \rvert$
0	1.0	1.000000000	1.000000000	0.000000000
1	1.5	1.000000000	1.094534892	0.094534892
2	2.0	1.166666667	1.306852819	0.140186153
3	2.5	1.416666667	1.583709268	0.167042601
4	3.0	1.716666667	1.901387711	0.184721045
5	3.5	2.050000000	2.247237032	0.197237032
6	4.0	2.407142857	2.613705639	0.206562782

From TABLE 1, $\omega_4 = 2.435246520$ and from Table 2, $\omega_4 = 2.407142857$ whereas the exact value of the function at $x = 4$ is $y(4) = 2.613705639$. Thus, it is clear that G-Euler's method gives better approximation.

4.2. Taylor's G-Series Method

We consider an initial value problem having first order G-derivative as

$$\frac{d^G y}{dx^G} = f(x, y), y(a) = y_0, \quad a \leq x \leq b \text{ and } a \geq 1. \tag{4.8}$$

We take

$$h = \left(\frac{b}{a}\right)^{\frac{1}{N}},$$

where $N \in \mathbb{N}$ will be called number of steps and h, the step-size. We denote, $x_0 = a, x_1 = ah, x_2 = ah^2, ..., x_N = ah^N = b$. For fixed a and b, values of h decreases as N increases. But N should be so chosen that $h > 1$ and hence $x_0, x_1, x_2, ...$ will be in increasing order.

Let $y(x) = F(x)$ be the solution of (4.8) such that $F(a) \neq 1$ and $y(x)$ has $(n+1)$ continuous G-derivatives. Then from [6], if we expand the solution, $y(x)$, in terms of n^{th} Taylor product about a point $x_i = ah^i$ and evaluate at $x_{i+1} = ah^{i+1}$, we obtain

$$y(x_{i+1}) = y(x_i).\left[y^{[1]}(x_i)\right]^{\ln h}.\left[y^{[2]}(x_i)\right]^{\frac{\ln^2 h}{2!}}...\left[y^{[n]}(x_i)\right]^{\frac{\ln^n h}{n!}}.\left[y^{[n+1]}(\xi_i)\right]^{\frac{\ln^{(n+1)} h}{(n+1)!}} \tag{4.9}$$

for some $\xi_i \in (x_i, x_{i+1})$.
But from (4.8), we have

$$y^{[1]} = f(x,y) = F_1(x,y) \text{ (say)}. \tag{4.10}$$

Then its successive G-differentiation gives

$$y^{[2]} = f^{[1]}(x,y) = \frac{\partial^G F_1}{\partial x^G} \oplus \frac{\partial^G F_1}{\partial y^G} \odot y^{[1]} = F_2\left(x, y, y^{[1]}\right), \text{ (say)},$$

$$y^{[3]} = f^{[2]}(x,y) = \frac{\partial^G F_2}{\partial x^G} \oplus \frac{\partial^G F_2}{\partial y^G} \odot y^{[1]} \oplus \frac{\partial^G F_2}{\partial y^{[1]G}} \odot y^{[2]} = F_3\left(x, y, y^{[1]}, y^{[2]}\right), \text{ (say)},$$

$$\begin{aligned} y^{[4]} = f^{[3]}(x,y) &= \frac{\partial^G F_3}{\partial x^G} \oplus \frac{\partial^G F_3}{\partial y^G} \odot y^{[1]G} \oplus \frac{\partial^G F_3}{\partial y^{[1]G}} \odot y^{[2]} \oplus \frac{\partial^G F_3}{\partial y^{[2]G}} \odot y^{[3]} \\ &= F_4\left(x, y, y^{[1]}, y^{[2]}, y^{[3]}\right), \text{ (say)}. \end{aligned}$$

Proceeding in this way, we have

$$\begin{aligned} y^{[n]} = f^{[n-1]}(x,y) &= \frac{\partial^G F_{n-1}}{\partial x^G} \oplus \frac{\partial^G F_{n-1}}{\partial y^G} \odot y^{[1]G} \\ &\oplus \frac{\partial^G F_{n-1}}{\partial y^{[1]G}} \odot y^{[2]} \oplus \cdots \oplus \frac{\partial^G F_{n-1}}{\partial y^{[n-2]G}} \odot y^{[n-1]} \\ &= F_n\left(x, y, y^{[1]}, y^{[2]}, ..., y^{[n-1]}\right), \text{ (say)}. \end{aligned}$$

Putting $x = x_i, y_i = y(x_i)$ in above equations, and substituting these values into (4.9) we get

$$\begin{aligned} y(x_{i+1}) = y(x_i).\left[f(x_i,y_i)\right]^{\ln h}.\left[f^{[1]}(x_i,y_i)\right]^{\frac{\ln^2 h}{2!}}.\left[f^{[2]}(x_i,y_i)\right]^{\frac{\ln^3 h}{3!}}... \\ ...\left[f^{[n-1]}(x_i,y_i)\right]^{\frac{\ln^n h}{n!}}.\left[f^{[n]}(\xi_i,y_i)\right]^{\frac{\ln^{(n+1)} h}{(n+1)!}}. \end{aligned} \tag{4.11}$$

Neglecting the remainder term involving ξ_i, we get the approximation equations(or difference equations) for **Taylor G-series method of order** n as

$$\begin{aligned} \omega_0 &= y_0, \\ \omega_{i+1} &= y(x_i).\left[T^{[n]}(x_i,\omega_i)\right]^{\ln h}, \end{aligned} \tag{4.12}$$

where

$$T^{[n]}(x_i,\omega_i) = f(x_i,y_i).\left[f^{[1]}(x_i,y_i)\right]^{\frac{\ln h}{2!}} \ldots \left[f^{[n-1]}(x_i,y_i)\right]^{\frac{\ln^{n-1} h}{n!}}, i = 1,2,3,...,N. \tag{4.13}$$

It is observed that G-Euler method is Taylor G-series method of order one.

Definition 4.2. *If the difference method*

$$\omega_0 = y_0,$$
$$\omega_{i+1} = \omega_i.\left[\phi(x_i,\omega_i)\right]^{\ln h}, \text{ for } i = 0,1,2,\ldots,N-1$$

approximates an initial value problem, we define its ***local truncation error*** *as*

$$\tau_{i+1}(h) = \left[\frac{y_{i+1}}{y_i.\left\{\phi(x_i,y_i)\right\}^{\ln h}}\right]^{\frac{1}{\ln h}},$$
$$= \left[\frac{y_{i+1}}{y_i}\right]^{\frac{1}{\ln h}}.\frac{1}{\phi(x_i,y_i)}.$$

for each $i = 0,1,2,\ldots,N-1$, *where* y_i *and* y_{i+1} *denote the solution at* x_i *and* x_{i+1}, *respectively.*

Theorem 4.3. *Local truncation error in G-Euler's method is* $O(\ln h)$.

Proof: From (4.3), difference equations for G-Euler's method are

$$\omega_0 = y_0,$$
$$\omega_{i+1} = \omega_i.\left[f(x_i,\omega_i)\right]^{\ln h}, \text{ for } i = 0,1,2,3,\ldots,N-1.$$

Therefore local truncation error is

$$\tau_{i+1}(h) = \left[\frac{y_{i+1}}{y_i}\right]^{\frac{1}{\ln h}}.\frac{1}{f(x_i,y_i)}.$$

But, from equation (4.2),

$$\tau_{i+1}(h) = \left[y^{[2]}(\xi_i)\right]^{\frac{\ln h}{2}}$$

If $y^{[2]}(x_i)$ is bounded by a constant, say M on $[a,b]$, then

$$|\tau_{i+1}(h)| \leq M^{\frac{\ln h}{2}}.$$

So, the local truncation error in G-Euler's method is $O(\ln h)$. □

Example 4.4. *Apply Taylor's G-series method of order two with $N = 6$ to approximate the solution of the G-initial value problem*

$$\frac{d^G y}{dx^G} = x^2, \quad 1 < x < 4, \ y(1) = 1. \tag{4.14}$$

Solution: To compare the solutions, it is to be noted that exact solution of the equation (4.14) is

$$y = x^{\ln x},$$

and its corresponding ordinary initial value problem is

$$\frac{dy}{dx} = \frac{2y \ln x}{x}, \quad 1 < x < 4, \ y(1) = 1. \tag{4.15}$$

Since, $N = 6$, we have

$$\begin{aligned} h &= \left[\frac{b}{a}\right]^{1/N} = \left[\frac{4}{1}\right]^{1/6} = 1.25992105 \\ x_i &= a.h^i = h^i, \\ \omega_0 &= 1, \qquad \text{and} \\ \omega_{i+1} &= \omega_i. \left[T^{[2]}(x_i, \omega_i)\right]^{\ln h} \qquad \text{for } i = 0, 1, 2, ..., 6. \end{aligned}$$

For the method of order two, we need the first G-derivative of $f(x, y) = x^2$ with respect to the variable x.

$$f^{[1]}(x, y) = e^2,$$

so

$$\begin{aligned} T^{[2]}(x_i, \omega_i) &= f(x_i, \omega_i). \left[f^{[1]}(x_i, \omega_i)\right]^{\frac{\ln h}{2}} \\ &= x_i^2.(e^2)^{\frac{\ln h}{2}} \\ &= (h^i)^2.h = h^{2i+1}. \end{aligned}$$

Now

$$\begin{aligned} \omega_{i+1} &= \omega_i. \left[T^{[2]}(x_i, \omega_i)\right]^{\ln h} \\ &= \omega_i.h^{(2i+1)\ln h} \\ &= \omega_i. \left(4^{1/6}\right)^{\frac{(2i+1)\ln 4}{6}} \\ &= \omega_i. \left(2^{\frac{\ln 2}{9}}\right)^{(2i+1)} \\ &= \omega_i.(1.054834274)^{2i+1}. \end{aligned}$$

Therefore the first step gives the approximation

$$y(1.2599210499) \approx \omega_1 = 1 \times (1.054834274)^1 = 1.054834274.$$

Second approximation is given by

$$y(1.5874010520) \approx \omega_2 = 1.054834274 \times (1.054834274)^3 = 1.238046424.$$

and so on. From the table, it is seen that in the first four steps, approximated values are same as the exact value of the function at corresponding mesh points. In the table, though it seems that in fifth and sixth steps, approximated values and exact values are equal too. But, if we extend the decimal places, actually they are differed by a very very small value after 14^{th} places of decimal. Here, we are using just the Taylor's G-series method of order two. If we use fourth order method, definitely it will give more accurate values. Therefore Taylor G-series approximation gives extremely reliable approximation.

Table 3: Taylor's G-series approximation of order 2 at $x = 4$

i	x_i	ω_i	Exact value $y(x_i)$	Error $\lvert y(x_i) - \omega_i \rvert$
0	1.0000000000	1	1	0
1	1.2599210499	1.054834274	1.054834274	0
2	1.5874010520	1.238046424	1.238046424	0
3	2.0000000000	1.616806672	1.616806672	0
4	2.5198420998	2.349349994	2.349349994	0
5	3.1748021039	3.798444745	3.798444745	4.44089×10^{-15}
6	4.0000000000	6.833329631	6.833329631	9.76996×10^{-15}

To compare the values with approximation of ordinary initial value problem, we consider the corresponding initial value problem (4.15) with same number of steps, i.e. $N = 6$. In this case,

$$y' = f(x, y) = \frac{2y \ln x}{x},$$

Differentiating with respect to x, we get

$$\begin{aligned}
f'(x,y) &= \frac{2y}{x^2}(2\ln^2 x - \ln x + 1).\\
h &= \frac{4-1}{6} = 0.5\\
T^{(2)}(x_i,\omega_i) &= f(x_i,\omega_i) + \frac{h}{2} f'(x_i,\omega_i)\\
&= \frac{2\omega_i \ln(x_i)}{x_i} + \frac{h}{2}\left(2\ln^2(x_i) - \ln(x_i) + 1\right)\\
&= \frac{\omega_i}{x_i^2}\left[2x_i \ln(x_i) + h(2\ln^2(x_i) - \ln(x_i) + 1)\right].\omega_{i+1}\\
&= \omega_i + hT^{(2)}(x_i,\omega_i)\\
&= \omega_i + \frac{h\omega_i}{x_i^2}\left[2x_i \ln(x_i) + h(2\ln^2(x_i) - \ln(x_i) + 1)\right].
\end{aligned}$$

TABLE 4 shows the approximation with the help of ordinary Taylor's series method. From the table, it is clear that ordinary Taylor's series method of order two gives poor approximation than Taylor's G-series method of order two.

Table 4: Ordinary Taylor series approximation of order 2 at $x = 4$

i	x_i	ω_i	Exact value $y(x_i)$	Error $\lvert y(x_i) - \omega_i \rvert$
0	1.0	1	1	0
1	1.5	1.250000000	1.054834274	0.195165726
2	2.0	1.716129090	1.238046424	0.478082666
3	2.5	2.446871475	1.616806672	0.830064803
4	3.0	3.516232022	2.349349994	1.166882028
5	3.5	5.030031744	3.798444745	1.231586999
6	4.0	7.126707955	6.833329631	0.293378324

4.3. G-Runge-Kutta Method:

Objective of a numerical techniques is to determine accurate approximations with minimal effort. Since, local truncation error in G-Euler's method is $O(\ln h)$. So, G-Euler's method is not applicable in practice. Higher order Taylor's G-series methods discussed above have higher order truncation error, but disadvantage of higher order Taylor's G-series methods is that they require the computation and evaluation of higher order G-derivatives of $f(x,y)$, which is complicated in practice. So, Taylor's methods are rarely used. Here we discuss about G-Runge-Kutta method, which has higher-order local truncation error as like Taylor's G-series methods but no need to compute higher order G-derivatives of $f(x,y)$.

4.3.1. G-Runge-Kutta Method of order four: To approximate the solution of the initial value problem

$$\frac{d^G y}{dx^G} = f(x,y), y(a) = y_0, a \leq x \leq b \text{ and } a \geq 1 \tag{4.16}$$

in the interval $[a,b]$, we take

$$\begin{aligned} h &= \left(\frac{b}{a}\right)^{\frac{1}{N}}; \\ x_0 &= a, x_i = ah^i, \text{ for } i = 1,2,...,N; \\ \omega_0 &= y_0. \end{aligned}$$

Then for $x_1 = ah$, the first increment in y is computed from the formulae

$$\begin{aligned} k_1 &= [f(x_0,y_0)]^{\ln h}, \\ k_2 &= \left[f(x_0\sqrt{h}, y_0\sqrt{k_1})\right]^{\ln h}, \\ k_3 &= \left[f(x_0\sqrt{h}, y_0\sqrt{k_2})\right]^{\ln h}, \\ k_4 &= [f(x_0 h, y_0 k_3)]^{\ln h}, \\ \omega_1 &= \omega_0.\left(k_1.k_2^2.k_3^2.k_4\right)^{\frac{1}{6}}, \end{aligned}$$

taken in the given order. In similar manner, the second and successive approximation ω_i to y_i are computed by means of the formulae

$$\begin{aligned} k_1 &= [f(x_i,y_i)]^{\ln h}, \\ k_2 &= \left[f(x_i\sqrt{h}, y_i\sqrt{k_1})\right]^{\ln h}, \\ k_3 &= \left[f(x_i\sqrt{h}, y_i\sqrt{k_2})\right]^{\ln h}, \\ k_4 &= [f(x_i h, y_i k_3)]^{\ln h}, \\ \omega_{i+1} &= \omega_i.\left(k_1.k_2^2.k_3^2.k_4\right)^{\frac{1}{6}} \end{aligned}$$

which can be written in terms of of geometric arithmetic as follows:

$$\begin{aligned} k_1 &= h \odot f(x_i,y_i), \\ k_2 &= h \odot f(x_i \oplus \frac{h}{2}, y_i \oplus \frac{k_1}{2}), \\ k_3 &= h \odot f(x_i \oplus \frac{h}{2}, y_i \oplus \frac{k_2}{2}), \\ k_4 &= h \odot f(x_i \oplus h, y_i \oplus k_3), \\ \omega_{i+1} &= \omega_i \oplus \left(\frac{k_1 \oplus e^2 \odot k_2 \oplus e^2 \odot k_3 \oplus k_4}{e^6}{}_G\right). \end{aligned}$$

Example 4.5. *We take the same initial value problem mentioned in Example 4.1 with $N = 6$.*

Then, $\omega_0 = 1, x_0 = a = 1, h = 1.25992105, x_i = (1.25992105)^i$. Now

$$k_1 = [f(x_0, y_0)]^{\ln h} = \left[e^{\left(\frac{x_0-1}{\omega_0}\right)}\right]^{\frac{\ln 4}{6}} = \left[e^{\left(\frac{1-1}{1}\right)}\right]^{\frac{\ln 4}{6}} = 1,$$

$$k_2 = \left[f(x_0\sqrt{h}, y_0\sqrt{k_1})\right]^{\ln h} = \left[e^{\left(\frac{x_0\sqrt{h}-1}{\omega_0\sqrt{k_1}}\right)}\right]^{\frac{\ln 4}{6}}$$

$$= \left[e^{\left(\frac{1.\sqrt{1.25992105}-1}{1\sqrt{1}}\right)}\right]^{\frac{\ln 4}{6}} = e^{(0.122462048 \times 0.23104906)} = 1.02869884,$$

$$k_3 = \left[f(x_0\sqrt{h}, y_0\sqrt{k_2})\right]^{\ln h} = \left[e^{\left(\frac{x_0\sqrt{h}-1}{\omega_0\sqrt{k_2}}\right)}\right]^{\frac{\ln 4}{6}}$$

$$= \left[e^{\left(\frac{1.\sqrt{1.25992105}-1}{1\sqrt{1.02869884}}\right)}\right]^{\frac{\ln 4}{6}} = e^{\left(\frac{0.122462048 \times 0.23104906}{1.014247918}\right)} = 1.028290036,$$

$$k_4 = [f(x_0 h, y_0 k_3)]^{\ln h} = \left[e^{\left(\frac{x_0 h-1}{\omega_0 k_3}\right)}\right]^{\frac{\ln 4}{6}}$$

$$= \left[e^{\left(\frac{1.25992105-1}{1.028290036}\right)}\right]^{\frac{\ln 4}{6}} = e^{\left(\frac{0.25992105 \times 0.23104906}{1.028290036}\right)} = 1.060141416,$$

$$\therefore \omega_1 = \omega_0. \left(k_1.k_2^2.k_3^2.k_4\right)^{\frac{1}{6}}$$

$$= 1.\left[1 \times (1.02869884)^2 \times (1.028290036)^2 \times 1.060141416\right]^{\frac{1}{6}} = 1.028873369.$$

Similarly we can compute successive approximations. Results and their errors computed with the help of Excel are shown in the TABLE 5. In the table, some decimal places are reduced to limit the size of the table for convenience.

Table 5: Approximation by G-Runge-Kutta Method at $x = 4$

i	x_i	ω_i	Exact Value $y_i = y(x_i)$	Error $\lvert y_i - \omega_i \rvert$	k_1	k_2	k_3	k_4
1	1.0000	1.0000000	1.0000000	0.000000	1.0000000	1.0286988	1.0282900	1.0601414
2	1.2599	1.0288734	1.0288720	0.000001	1.0601063	1.0945491	1.0929819	1.1282728
3	1.5874	1.1253093	1.1253029	0.000006	1.1281797	1.1631425	1.1604835	1.1935436
4	2.0000	1.3068688	1.3068528	0.000016	1.1933875	1.2232075	1.2201887	1.2463427
5	2.5198	1.5956740	1.5956459	0.000028	1.2461625	1.2676501	1.2650937	1.2826374
6	3.1748	2.0195966	2.0195568	0.000040	1.2824921	1.2956043	1.2939032	1.3037598
7	4.0000	2.6137542	2.6137056	0.000049	1.3036815	1.3102060	1.3093238	1.3135512

From the table it is seen that though error grows as values of x increases, still errors in respective steps are not huge enough. We can compare the approximation given by ordinary Runge-Kutta Method of order 4. For, we consider the corresponding ordinary initial value problem as stated in equation (4.7) as

$$\frac{dy}{dx} = 1 - \frac{1}{x}, 1 \leq x \leq 4, y(1) = 1.$$

If we consider $N = 6$, then $h = \frac{b-a}{N} = \frac{4-1}{6} = 0.5$ and the ordinary Runge-Kutta method gives the approximation equations as follows:

$$\begin{aligned}
k_1 &= hf(x_i, y_i),\\
k_2 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}),\\
k_3 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}),\\
k_4 &= hf(x_i h, y_i k_3),\\
\omega_{i+1} &= \omega_i + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right).
\end{aligned}$$

Table 6: Approximation by Ordinary Runge-Kutta Method at $x = 4$

i	x_i	ω_i	Exact Value $y_i = y(x_i)$	Error $\lvert y_i - \omega_i \rvert$	k_1	k_2	k_3	k_4
1	1.0	1.0000000	1.0000000	0.000000	0.0000000	0.1000000	0.1000000	0.1666667
2	1.5	1.0944444	1.0945349	0.000090	0.0431472	0.2142857	0.2142857	0.2500000
3	2.0	1.2861595	1.3068528	0.020693	0.1112457	0.2777778	0.2777778	0.3000000
4	2.5	1.5398856	1.5837093	0.043824	0.1753006	0.3181818	0.3181818	0.3333333
5	3.0	1.8367791	1.9013877	0.064609	0.2277844	0.3461538	0.3461538	0.3571429
6	3.5	2.1650362	2.2472370	0.082201	0.2690570	0.3666667	0.3666667	0.3750000
7	4.0	2.5168235	2.6137056	0.096882	0.3013369	0.3823529	0.3823529	0.3888889

From the both tables, if we observe the approximation at $x = 4$, it is obvious that G-Runge-Kutta method gives much better approximation than ordinary Runge-Kutta method. Of course, here, in the ordinary initial value problem, $f(x, y) = 1 - \frac{1}{x} = f(x)$, i.e. it is free from y. So, k_1, k_2, k_3, k_4 have less effect on the approximations in this particular example. That is also a reason for giving worse approximation.

In the G-Runge-Kutta method, if value of N is increased, error will be decreased. If we take $N = 10$, following table shows the approximation at $x = 4$ giving lesser error than for $N = 6$.

Table 7: Approximation by G-Runge-Kutta Method at $x = 4$ with $N = 10$

i	x_i	ω_i	Exact Value $y_i = y(x_i)$	Error $\lvert y_i - \omega_i \rvert$	k_1	k_2	k_3	k_4
1	1.00000	1.000000	1.000000	0.0000000	1.000000	1.010000	1.009950	1.020621
2	1.14870	1.010069	1.010069	0.0000001	1.020618	1.031900	1.031723	1.043420
3	1.31951	1.042249	1.042249	0.0000003	1.043414	1.055417	1.055093	1.067174
4	1.51572	1.099829	1.099828	0.0000006	1.067164	1.079178	1.078719	1.090456
5	1.74110	1.186585	1.186583	0.0000012	1.090442	1.101746	1.101197	1.111927
6	2.00000	1.306855	1.306853	0.0000020	1.111909	1.121918	1.121341	1.130583
7	2.29740	1.465623	1.465620	0.0000028	1.130565	1.138922	1.138378	1.145894
8	2.63902	1.668613	1.668610	0.0000037	1.145877	1.152471	1.152003	1.157779
9	3.03143	1.922402	1.922398	0.0000045	1.157766	1.162680	1.162309	1.166493
10	3.48220	2.234542	2.234537	0.0000051	1.166484	1.169924	1.169653	1.172481
11	4.00000	2.613711	2.613706	0.0000057	1.172476	1.174699	1.174520	1.176254

Also from TABLE 1 and TABLE 5, it is clear that fourth order G-Runge-Kutta method gives much better approximation than G-Euler's method.

5. Conclusion

From the whole discussion, it has come to light that numerical methods deduced for G-initial value problems are much reliable than ordinary numerical approximation methods. It is observed that if local truncation error in an ordinary method is $O(h^p)$, then local truncation error in the corresponding G-initial value problem is $O(\ln^p h)$. So, converting the ordinary initial value problems to G-initial value problem, we can get better approximated value. Of course, there are some demerits in the methods for G-initial value problems too. G-initial value problems with $x_0 \leq 0$ can not be approximated with the help of above methods. We hope, these barriers will be removed in near future with suitable approximation methods.

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References

1. D. Aniszewska, *Multiplicative Runge-Kutta methods*, Nonlinear Dyn. 50, 265?272, (2007).
2. A. E. Bashirov, M. Rıza, *On Complex multiplicative differentiation*, TWMS J. App. Eng. Math. 1(1), 75-85, (2011).
3. A. E. Bashirov, E. Mısırlı, Y. Tandoğdu and A. Özyapıcı, *On modeling with multiplicative differential equations*, Appl. Math. J. Chinese Univ. 26(4), 425-438, (2011).
4. A. E. Bashirov, E. M. Kurpınar and A. Özyapici, *Multiplicative Calculus and its applications*, J. Math. Anal. Appl. 337, 36-48, (2008).
5. K. Boruah, B. Hazarika, *Application of Geometric Calculus in Numerical Analysis and Difference Sequence Spaces*, J. Math. Anal. Appl. 449(2), 1265-1285, (2017).
6. K. Boruah, B. Hazarika, *G-Calculus*, TWMS J. Appl. Eng. 8(1), pp. 94-105, 2018.
7. R. L. Burden, J. D. Faires, *Numerical Analysis*, Ninth Edition, Youngstown State University.
8. A. F. Çakmak, F. Başar, *On Classical sequence spaces and non-Newtonian calculus*, J. Inequal. Appl. 2012, Art. ID 932734, 12pp, (2012).
9. D. Campbell, *Multiplicative Calculus and Student Projects*, Department of Mathematical Sciences, United States Military Academy, West Point, NY,10996, USA.

10. M. Coco, *Multiplicative Calculus*, Lynchburg College.
11. M. Grossman, *Bigeometric Calculus: A System with a scale-Free Derivative*, Archimedes Foundation, Massachusetts, 1983.
12. M. Grossman, R. Katz, *Non-Newtonian Calculus*, Lee Press, Piegon Cove, Massachusetts, 1972.
13. J. Grossman, M. Grossman, R. Katz, *The First Systems of Weighted Differential and Integral Calculus*, University of Michigan, 1980.
14. J. Grossman, *Meta-Calculus: Differential and Integral*, University of Michigan, 1981.
15. J. Grossman, R. Katz, *Averages, A new Approach*, University of Michigan, 1983.
16. M. Grossman, *The First Nonlinear System of Differential and Integral Calculus* University of California, 1979.
17. U. Kadak, M. Özlük, *Generalized Runge-Kutta method with respect to non-Newtonian calculus*, Abst. Appl. Anal., Vol. 2015 , Article ID 594685, 10 pages, (2015).
18. F. Córdova-Lepe, *The multiplicative derivative as a measure of elasticity in economics*, TMAT Revista Latinoamericana de Ciencias e Ingeneríria, 2(3), 8 pages, (2006).
19. M. Z. Spivey, *A Product Calculus*, University of Puget Sound, Tacoma, Washington 98416-1043.
20. D. Stanley, *A multiplicative calculus, Primus* IX 4 (1999) 310-326.
21. S. Tekin, F. Başar, *Certain Sequence spaces over the non-Newtonian complex field*, Abst. Appl. Anal. 2013. Article ID 739319, 11 pages, (2013).
22. C. Türkmen, F. Başar, *Some Basic Results on the sets of Sequences with Geometric Calculus*, Commun. Fac. Fci. Univ. Ank. Series A1. Vol G1. No 2, 17-34, (2012).

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