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Common Fixed Point Results for Various Mappings in Fuzzy Metric Spaces with Application

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ABSTRACT: In this paper, first we discuss the variants of the weakly commuting and compatible mappings in the context of coupled fixed point theory of fuzzy metric spaces. Secondly, we investigate the existence and uniqueness of the common fixed point for pairs of weakly compatible mappings satisfying a new contraction condition in the setup of fuzzy metric spaces with Hadžić type t-norm . Further, we talk about some results for the variants of weakly commuting and compatible mappings. At the end, as an application, we obtain metrical version of the discussed results.

Key Words: Fuzzy metric spaces, Hadžić type t-norm, Weakly commuting mappings and variants, Compatible mappings and variants, Weak compatible mappings.

Contents

1. Introduction and Preliminaries

In 1965, Zadeh [28] introduced the notion of fuzzy sets that provides quick headways into different branches of mathematics and its areas of applications. In particular, the fuzzy version of the metric spaces has been given by various authors, resulting into different definitions of fuzzy metric spaces in numerous non-equivalent ways (see e.g., Deng [6], Erceg [8], George and Veeramani [9,10], Kaleva and Seikkala [19], Kramosil and Michalek [20]).

Grabiec [11] presented the fuzzy version of the famous Banach contraction principle, in the sense of Kramosil and Michalek (in short, KM) [20]. George and Veeramani (in short, GV) [9,10] modified the concept of fuzzy metric spaces due to Kramosil and Michalek [20]. Afterwards, various authors established several fixed point results in fuzzy metric spaces in the sense of GV. Some examples in this

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direction can be found in the works of Gregori and Sapena [12], Murthy et al. [22], Singh and Chauhan [24]. The present work deals with the definition of the fuzzy metric space as discussed by George and Veeramani [9,10].

Over the years, in metric fixed point theory, authors are continuously making an effort to extend and generalize the Banach's contraction mapping principle in different directions in different spaces. In order to extend this famous contraction mapping principle for the pair of mappings, researchers have successively introduced the notions of commutative mappings, compatible mappings, weakly compatible mappings, variants of weakly commuting and compatible mappings.

During the same course of time, coupled fixed point results were receiving much attention in metric fixed point theory. Although the concept of coupled fixed point was introduced by Guo and Lakshmikantham [13] but the line of research in this direction developed rapidly after the worth mentioning work of Bhaskar and Lakshmikantham [2]. In [2], the authors proved a contraction mapping theorem in partially ordered metric spaces in the context of coupled fixed point theory. Similar to ordinary fixed point theory, authors introduced the notions of commutative mappings, compatible mappings and weak compatible mappings in the context of coupled fixed theory. Various instances of such works can be found in [2,17,21].

In fuzzy metric spaces, coupled fixed point theorem for contraction mappings was first proved by Sedghi et al. [25]. Unfortunately, Zhu and Xiao [27] proved the falsity of the work presented by Sedghi et al. [25] and thereby they presented a correct modification of the results proved by Sedghi et al. [25]. On the other hand, Hu [14] presented a coupled common fixed point theorem for a pair of compatible mappings under a ϕ-contraction in fuzzy metric spaces, which was followed by the works of Choudhury et al. [4,5], Jain et al. [17,18], Hu et al. [15], etc. Subsequently, Abbas et al. [1] introduced the notion of w-compatible mappings as a generalization of compatible mappings. Recently, in order to obtain the existence and uniqueness of the coupled common fixed points for mappings in the setup of fuzzy metric spaces Jain et al. [18] introduced the notions of weakly commuting mappings and their variants, that is, R-commuting mappings, R-weakly commuting mappings of type (A_F), R-weakly commuting mappings of type (A_g), R-weakly commuting mappings of type (P). On the other hand, Sumitra and Masmali [7] studied the notions of variants of compatible mappings that includes compatible mappings of type (A), compatible mappings of type (B), compatible mappings of type (C), compatible mappings of type (P), compatible mappings of type (A_F), compatible mappings of type (A_g) in the context of coupled fixed point problems in fuzzy metric spaces.

The purpose of this paper is to present a discussion on the variants of weakly commuting and compatible mappings and to prove a common fixed point result for pairs of weakly compatible mappings satisfying a new contraction condition in the context of coupled fixed point theory of fuzzy metric spaces. Further, we give results for the variants of weakly commuting and compatible mappings. At the end, the metrical version of the notions and results discussed in different sections of the present manuscript are also established.

Here, we state some allied definitions and results which are required for the development of the present study.

Definition 1.1 ([28]). *A fuzzy set A in X is a function with domain X and values in $[0,1]$.*

Definition 1.2 ([26]). *A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:*

1. *$*$ is associative and commutative,*
2. *$*$ is continuous,*
3. *$a*1=a$ for all $a\in[0,1]$,*
4. *$a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.*

Some examples of the continuous t-norm are $a*_1 b=ab$ and $a*_2 b=\min\{a,b\}$ for all $a,b\in[0,1]$.

Definition 1.3 ([16]). *Let $\sup\limits_{0<t<1}\Delta(t,t)=1$. A t-norm Δ is said to be Hadžić type t-norm (in short, H-type t-norm), if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t=1$, where*

$$\Delta^1(t)=t,\ \Delta^{m+1}(t)=t\Delta(\Delta^m(t)),\ \text{for } t\in[0,1] \text{ and } m=1,2,\dots.$$

A t-norm Δ is a H-type t-norm iff for any $\lambda\in(0,1)$, there exists $\delta(\lambda)\in(0,1)$ such that $\Delta^m(t)>(1-\lambda)$ for all $m\in N$, when $t>(1-\delta)$.

Clearly, $*_2$ is an example of t-norm of H-type.

Definition 1.4 ([9]). *The 3-tuple $(X,M,*)$ is called a fuzzy metric space (in the sense of GV), if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^{\times}(0,\infty)$ satisfying the following conditions for each $x,y,z\in X$ and $t,s>0$:*

1. *$M(x,y,t)>0$,*
2. *$M(x,y,t)=1$ if and only if $x=y$,*
3. *$M(x,y,t)=M(y,x,t)$,*
4. *$M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,*
5. *$M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.*

Definition 1.5 ([9]). *Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be*

1. *Convergent to a point $x\in X$, if $\lim\limits_{n\to\infty} M(x_n,x,t)=1$, for all $t>0$;*
2. *Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists a positive integer n_0 such that $M(x_n,x_m,t)>1-\varepsilon$ for each $n,m\ge n_0$.*

A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.6 ([11]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.*

Lemma 1.7 ([23]). *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.8 ([25]). *Let $(X, M, *)$ be a fuzzy metric space. M is said to satisfy the n-property on $X^2 \times (0, \infty)$ if $\lim_{n\to\infty} [M(x, y, k^n t)]^{n^p} = 1$, whenever $x, y \in X$, $k > 1$ and $p > 0$.*

Definition 1.9 ([14]). *Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:*

1. *ϕ is non-decreasing;*
2. *ϕ is upper semicontinuous from the right;*
3. *$\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in N$.*

Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

The notion of coupled fixed points was initiated by Guo and Lakshmikantham [13]. Since then, the concept has been of interest to the researchers in metrical fixed point theory. On the other hand, Bhaskar and Lakshmikantham [2] introduced the notion of mixed monotone property, and thereby proved some coupled fixed point theorems for mappings satisfying this property in ordered metric spaces.

Definition 1.10 ([2,13]). *An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Lakshmikantham and Ćirić [21] extended the work of Bhaskar and Lakshmikantham [2] for a pair of commutative mappings.

Definition 1.11 ([21]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be commutative, if $F(gx, gy) = gF(x, y)$ for all $x, y \in X$.*

Definition 1.12 ([21]). *An element $(x, y) \in X \times X$, is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 1.13 ([18]). *An element $(x, y) \in X \times X$, is called a coupled common fixed point of the mappings $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$ and $T : X \to X$ if $B(a, b) = S(a) = a = T(a) = A(a, b)$ and $B(b, a) = S(b) = b = T(b) = A(b, a)$.*

Definition 1.14 ([18]). *An element $x \in X$, is called a common fixed point of the mappings $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$ and $T : X \to X$ if $A(a, a) = B(a, a) = S(a) = T(a) = a$.*

2. Discussion on Variants of Weakly Commuting and Compatible Mappings

In this section, we study the notions of the weakly commuting and compatible mappings, their variants and the weakly compatible mappings in the fuzzy metric spaces for problems concerning the computation of coupled coincidence and coupled fixed points.

Recently, Choudhury et al. [3] introduced the following notion of the compatible mappings to establish the existence of coupled coincidence points in ordered metric spaces:

Definition 2.1 ([3]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(g(x_n), g(y_n))) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, x_n), F(g(y_n), g(x_n))) = 0,$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n)$ for some $x, y \in X$.

Hu [14] defined the following notion as the fuzzy counterpart of the definition of compatibility, which was introduced in Choudhury et al. [3] for coupled fixed point problems in ordered metric spaces:

Definition 2.2 ([14]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1,$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$. $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

The following notions were given by Jain et al. [18], which extend the definitions of variants of weakly commuting mappings from ordinary fixed point theory to coupled fixed point theory in the setup of fuzzy metric spaces:

Definition 2.3 ([18]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be weakly commuting if*

$$M(F(gx, gy), gF(x, y), t) \geq M(F(x, y), gx, t),$$
$$M(F(gy, gx), gF(y, x), t) \geq M(F(y, x), gy, t) \text{ for all } x, y \text{ in } X \text{ and } t > 0.$$

Definition 2.4 ([18]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be*

1. *R-weakly commuting if there exists some $R > 0$ such that*

$$M(F(gx, gy), gF(x, y), t) \geq M(F(x, y), gx, t/R),$$
$$M(F(gy, gx), gF(y, x), t) \geq M(F(y, x), gy, t/R) \text{ for all } x, y \text{ in } X \text{ and } t > 0.$$

2. *R-weakly commuting maps of type (A_F) if there exists some $R > 0$ such that*

$$M(F(gx, gy), ggx, t) \geq M(F(x, y), gx, t/R),$$
$$M(F(gy, gx), ggy, t) \geq M(F(y, x), gy, t/R) \text{ for all } x, y \text{ in } X \text{ and } t > 0.$$

3. *R-weakly commuting maps of type (A_g) if there exists some $R > 0$ such that*

$$M(gF(x, y), F(F(x, y), F(y, x)), t) \geq M(F(x, y), gx, t/R),$$
$$M(gF(y, x), F(F(y, x), F(x, y)), t) \geq M(F(y, x), gy, t/R)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

4. *R-weakly commuting maps of type (P) if there exists some $R > 0$ such that*

$$M(F(F(x, y), F(y, x)), ggx, t) \geq M(F(x, y), gx, t/R),$$
$$M(F(F(y, x), F(x, y)), ggy, t) \geq M(F(y, x), gy, t/R)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

Now we present some illustrations and discuss the relations between these variants.

Example 2.1. *Let $X = (0, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $F : X \times X \to X$ as $F(x, y) = \frac{x+y}{2}$ for all x, y in X and $g : X \to X$ as $g(x) = \frac{x}{2}$ for all x in X. Then, clearly for all x, y in X and $t > 0$, we have*

$$M(F(gx, gy), gF(x, y), t) = 1 > \frac{2t}{2t + y} = M(F(x, y), gx, t),$$
$$M(F(gy, gx), gF(y, x), t) = 1 > \frac{2t}{2t + x} = M(F(y, x), gy, t),$$

which shows that the pair (F, g) is weakly commuting.

Moreover, for all x, y in X and $t > 0$, we have

$$M(F(gx, gy), gF(x, y), t) = 1 > \frac{2t}{2t + Ry} = M\left(F(x, y), gx, \frac{t}{R}\right),$$
$$M(F(gy, gx), gF(y, x), t) = 1 > \frac{2t}{2t + Rx} = M\left(F(y, x), gy, \frac{t}{R}\right), \text{ for each } R > 0,$$

which shows that the pair (F, g) is R-weakly commuting for each $R > 0$.

Further, we note the followings:

For $R \geq \frac{1}{2}$, the pair (F, g) are R-weakly commuting of type (A_F), since for all x, y in X and $t > 0$, we have

$$M(F(gx, gy), ggx, t) = \frac{4t}{4t + y} \geq \frac{2t}{2t + Ry} = M\left(F(x, y), gx, \frac{t}{R}\right),$$
$$M(F(gy, gx), ggy, t) = \frac{4t}{4t + x} \geq \frac{2t}{2t + Rx} = M\left(F(y, x), gy, \frac{t}{R}\right).$$

For $R \geq 1$ and $x = y$, the pair (F, g) satisfies the property of R-weakly commuting of type (A_g), since

$$\begin{aligned} M(gF(x, y), F(F(x, y), F(y, x)), t) &= \frac{4t}{4t + (x + y)} \geq \frac{2t}{2t + Ry} \\ &= M\left(F(x, y), gx, \frac{t}{R}\right), \\ M(gF(y, x), F(F(y, x), F(x, y)), t) &= \frac{4t}{4t + (x + y)} \geq \frac{2t}{2t + Rx} \\ &= M\left(F(y, x), gy, \frac{t}{R}\right), \text{ for each } t > 0, \end{aligned}$$

Finally, we proceed towards R-weakly commutativity of type (P).

For $R \geq \frac{3}{2}$ and $x = y$, the pair (F, g) satisfies the property of R-weakly commuting of type (P) since,

$$\begin{aligned} M(F(F(x, y), F(y, x)), ggx, t) &= \frac{4t}{4t + (x + 2y)} \geq \frac{2t}{2t + Ry} \\ &= M\left(F(x, y), gx, \frac{t}{R}\right), \\ M(F(F(y, x), F(x, y)), ggy, t) &= \frac{4t}{4t + (2x + y)} \geq \frac{2t}{2t + Rx} \\ &= M\left(F(y, x), gy, \frac{t}{R}\right), \text{ for } t > 0. \end{aligned}$$

Clearly, the pair (F, g) is R-weakly commuting for each $R > 0$ but R-weakly commuting of type (A_F) for $R \geq \frac{1}{2}$ and R-weakly commuting of type (A_g) for $R \geq 1$.

Remark 2.5. *Example 2.1 shows that R-weakly commuting pair of mappings of type (A_F) need not be R-weakly commuting of type (A_g) nor it can be R-weakly commuting of type (P).*

The following example illustrates that if the pair of mappings is R-weakly commuting for some value of $R > 0$, then that pair of mappings need not be weakly commuting, nor R weakly commuting of type (A_F), nor R-weakly commuting of type (A_g), nor weakly commuting of type (P) for the same value of R.

Example 2.2. *Let $X = [1, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $F : X \times X \to X$ as $F(x, y) = 2(x+y)+1$, for all x, y in X and $g : X \to X$ as $g(x) = 2x+2$ for all x in X. The mappings F and g are not commuting, since $F(gx, gy) = [4(x + y) + 9] \neq [4(x + y) + 4] = gF(x, y)$, for x, y in X.*

Also, for all x, y in X and $t > 0$, we have

$$\begin{aligned} M(F(gx, gy), ggx, t) &= \frac{t}{t + |4y + 3|} \geq \frac{t}{t + R|2y - 1|} \\ &= M\left(F(x, y), gx, \frac{t}{R}\right), \\ M(F(gy, gx), gF(y, x)), t) &= \frac{t}{t + |4x + 3|} \geq \frac{t}{t + R|2x - 1|} \\ &= M\left(F(y, x), gy, \frac{t}{R}\right), \text{ for each } R \geq 7, \end{aligned}$$

which shows that the pair (F, g) is R-weakly commuting of type (A_F) for $R \geq 7$.

Further, we note that the pair (F, g) is R-weakly commuting for each $R \geq 5$ but neither weakly commuting, nor R-weakly commuting of type (A_g), nor weakly commuting of type (P) for any $R > 0$.

Remark 2.6. *In Example 2.2, the pair (F, g) of the mappings is R-weakly commuting but not R-weakly commuting of type (A_F) for $R = 5$.*

We observe in general that, every pair of commuting mappings is always weakly commuting but converse need not be true. Further, the pair of R-weakly commuting mappings of type (A_g) need not be R-weakly commuting nor R-weakly commuting of type (A_F), nor R-weakly commuting of type (P) as shown in the following illustration:

Example 2.3. *Let $X = [1, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $F : X \times X \to X$ as $F(x, y) = \frac{x}{2}$ for all x, y in X and $g : X \to X$ as $g(x) = x^2$ for all x in X. The mappings F and g are not commuting, since $F(gx, gy) = F(x^2, y^2) = \frac{x^2}{2} \neq \frac{x^2}{4} = g\left(\frac{x}{2}\right) = gF(x, y)$ for x, y in X.*

Again, for all x, y in X and $t > 0$, we have

$$\begin{aligned} M(F(gx, gy), gF(x, y), t) &= \frac{t}{t + |\frac{x^2}{4}|} \geq \frac{t}{t + |x^2 - \frac{x}{2}|} = M(F(x, y), gx, t), \\ M(F(gy, gx), gF(y, x), t) &= \frac{t}{t + |\frac{y^2}{4}|} \geq \frac{t}{t + |y^2 - \frac{y}{2}|} = M(F(y, x), gy, t), \end{aligned}$$

which shows that the pair (F, g) is weakly commuting.

Moreover, for all x, y in X and $t > 0$,

$$M(F(gx, gy), gF(x, y), t) = \frac{t}{t + |\frac{x^2}{4}|} \geq \frac{t}{t + R|x^2 - \frac{x}{2}|} = M\left(F(x, y), gx, \frac{t}{R}\right),$$

$$M(F(gy, gx), gF(y, x), t) = \frac{t}{t + |\frac{y^2}{4}|} \geq \frac{t}{t + R|y^2 - \frac{y}{2}|} = M\left(F(y, x), gy, \frac{t}{R}\right),$$

$$\text{for each } R \geq \frac{1}{2},$$

which shows that the pair (F, g) is R-weakly commuting for each $R \geq \frac{1}{2}$.

Further, we note the followings:

- *The pair (F, g) is not R-weakly commuting of type (A_F) for any $R > 0$.*
- *The pair (F, g) is R-weakly commuting of type (A_g) for $R \geq \frac{1}{4}$.*
- *The pair (F, g) is not R-weakly commuting of type (P) for any $R > 0$.*

Clearly, for $R = \frac{1}{4}$, the pair (F, g) is R-weakly commuting of type (A_g) but not R-weakly commuting.

Example 2.4. *Let $X = [1, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $F : X \times X \to X$ as $F(x, y) = \frac{x+y+1}{2}$ for all x, y in X and $g : X \to X$ as $g(x) = \frac{x}{2}$ for all x in X. The mappings F and g are not commuting, since $F(gx, gy) = F\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{x+y+2}{4} \neq \frac{x+y+1}{4} = g\left(\frac{x+y+1}{2}\right) = gF(x, y)$ for x, y in X.*

Now, for all x, y in X and $t > 0$, we have

$$M(F(gx, gy), gF(x, y), t) = \frac{t}{t + |\frac{1}{4}|} \geq \frac{t}{t + R|\frac{y+1}{2}|} = M\left(F(x, y), gx, \frac{t}{R}\right),$$

$$M(F(gy, gx), gF(y, x), t) = \frac{t}{t + |\frac{1}{4}|} \geq \frac{t}{t + R|\frac{x+1}{2}|} = M\left(F(y, x), gy, \frac{t}{R}\right),$$

$$\text{for each } R \geq \frac{1}{4},$$

which shows that the pair (F, g) is R-weakly commuting for $R \geq \frac{1}{4}$. Also, since for $R = 1$, the R-weakly commuting property coincides with weakly commuting property of the mappings, therefore, the pair (F, g) is also weakly commuting.

Also, for all x, y in X and $t > 0$,

$$M(F(gx, gy), ggx, t) = \frac{t}{t + |\frac{x+y+2}{4} - \frac{x}{4}|} \geq \frac{t}{t + R|\frac{y+1}{2}|} = M\left(F(x, y), gx, \frac{t}{R}\right),$$

$$M(F(gy, gx), gF(y, x), t) = \frac{t}{t + |\frac{x+y+2}{4} - \frac{y}{4}|} \geq \frac{t}{t + R|\frac{x+1}{2}|} = M\left(F(y, x), gy, \frac{t}{R}\right),$$

for each $R \geq \dfrac{3}{4}$,

which shows that the pair (F, g) is R-weakly commuting of type (A_F) for each $R \geq \frac{3}{4}$.

Further, we note that the pair (F, g) is neither R-weakly commuting of type (A_g), nor R weakly commuting of type (P) for any $R > 0$.

Example 2.5. *Let $X = [1, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $F : X \times X \to X$ as $F(x, y) = 2x + 1$, for all x, y in X and $g : X \to X$ as $g(x) = x + 1$ for all x in X. Now, for x, y in X, we have $F(gx, gy) = 2x + 3$, $F(gy, gx) = 2y + 3$, $gF(x, y) = 2x+2$, $gF(y, x) = 2y+2$, $F(F(x, y), F(y, x)) = 4x+3$, $F(F(y, x), F(x, y)) = 4y+3$, $ggx = x + 2, ggy = y + 2$. Then, the pair (F, g) is not commuting; R-weakly commuting for each $R \geq 1$ (and hence weakly commuting); R-weakly commuting of type (A_F) for each $R \geq 2$; R-weakly commuting of type (A_g) for each $R \geq 3$; R-weakly commuting of type (P) for each $R \geq 4$.*

In the setup of fuzzy metric spaces, now, we study the following notions of variants of compatible mappings, which are due to Sumitra and Masmali [7]:

Definition 2.7 ([7]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible of type (A) if*

$$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1,$$
$$\lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t) = 1,$$

and

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x,$$

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y,$$

for some $x, y \in X$ and $t > 0$.

Definition 2.8 ([7]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible of type (B) if*

$$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t)$$
$$\geq \frac{1}{2}\left\{\lim_{n\to\infty} M(F(gx_n, gy_n), F(x, y), t) + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n), t))\right\},$$
$$\lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t)$$
$$\geq \frac{1}{2}\left\{\lim_{n\to\infty} M(F(gy_n, gx_n), F(y, x), t) + \lim_{n\to\infty} M(F(y, x), F(F(y_n, x_n), F(x_n, y_n), t))\right\}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} M(gF(x_n,y_n),F(F(x_n,y_n),F(y_n,x_n)),t)\\
&\geq \frac{1}{2}\left\{\lim_{n\to\infty} M(gF(x_n,y_n),gx,t)+\lim_{n\to\infty} M(gx,g^2x_n,t)\right\},\\
&\lim_{n\to\infty} M(gF(y_n,x_n),F(F(y_n,x_n),F(x_n,y_n)),t)\\
&\geq \frac{1}{2}\left\{\lim_{n\to\infty} M(gF(y_n,x_n),gy,t)+\lim_{n\to\infty} M(gy,g^2x_n,t)\right\},
\end{aligned}$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\lim_{n\to\infty} F(x_n,y_n)=\lim_{n\to\infty} g(x_n)=x,$$

$$\lim_{n\to\infty} F(y_n,x_n)=\lim_{n\to\infty} g(y_n)=y,$$

for some $x,y\in X$ *and* $t>0$.

Definition 2.9 ([7])**.** *The mappings* $F: X\times X\to X$ *and* $g: X\to X$ *are said to be compatible of type (P) if*

$$\begin{aligned}
&\lim_{n\to\infty} M(F(F(x_n,y_n),F(y_n,x_n)),g^2x_n,t)=1,\\
&\lim_{n\to\infty} M(F(F(y_n,x_n),F(x_n,y_n)),g^2y_n,t)=1,
\end{aligned}$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n,y_n)=\lim_{n\to\infty} g(x_n)=x$, $\lim_{n\to\infty} F(y_n,x_n)=\lim_{n\to\infty} g(y_n)=y$ *for some* $x,y\in X$ *and* $t>0$.

Definition 2.10 ([7])**.** *The mappings* $F: X\times X\to X$ *and* $g: X\to X$ *are said to be compatible of type (C) if*

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gx_n,gy_n),g^2x_n,t)\\
&\qquad\geq \frac{1}{3}\{\lim_{n\to\infty} M(F(gx_n,gy_n),F(x,y),t)+\lim_{n\to\infty} M(F(x,y),g^2x_n,t)\\
&\qquad\quad+\lim_{n\to\infty} M(F(x,y),F(F(x_n,y_n),F(y_n,x_n)),t)\},\\
&\lim_{n\to\infty} M(F(gy_n,gx_n),g^2y_n,t)\\
&\qquad\geq \frac{1}{3}\{\lim_{n\to\infty} M(F(gy_n,gx_n),F(y,x),t)+\lim_{n\to\infty} M(F(y,x),g^2y_n,t)\\
&\qquad\quad+\lim_{n\to\infty} M(F(y,x),F(F(y_n,x_n),F(x_n,y_n)),t)\},
\end{aligned}$$

and

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t)$$
$$\geq \frac{1}{3}\{\lim_{n\to\infty} M(gF(x_n, y_n), gx, t) + \lim_{n\to\infty} M(gx, F(F(x_n, y_n), F(y_n, x_n)), t)$$
$$+ \lim_{n\to\infty} M(gx, g^2x_n, t)\}$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t)$$
$$\geq \frac{1}{3}\{\lim_{n\to\infty} M(gF(y_n, x_n), gy, t) + \lim_{n\to\infty} M(gy, F(F(x_n, y_n), F(x_n, y_n)), t)$$
$$+ \lim_{n\to\infty} M(gy, g^2y_n, t)\}$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = x$ *for some* $x, y \in X$ *and* $t > 0$.

Definition 2.11 ([7])**.** *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type* (A_F) *if*

$$\lim_{n\to\infty} M(F(gx_n, gy_n), ggx_n, t) = 1\,, \quad \lim_{n\to\infty} M(F(gy_n, gx_n), ggy_n, t) = 1$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, *for some* $x, y \in X$ *and* $t > 0$.

Definition 2.12 ([7])**.** *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type* (A_g) *if*

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t) = 1$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$ *and* $t > 0$.

We now discuss the relationship between these variants as follows:

In the following example, we show that compatible mappings need not be compatible of type (A), nor compatible of type (P), nor compatible of type (A_F), nor compatible of type (A_g).

Example 2.6. *Let* $X = \mathbb{R}$. *Define* $a * b = ab$ *and* $M(x, y, t) = \frac{t}{t+|x-y|}$, *for all* $x, y \in X$ *and* $t > 0$. *Then* $(X, M, *)$ *is a FM-space. Define the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *by*

$$F(x, y) = \begin{cases} \frac{1}{(xy)^3}, & xy \neq 0 \\ 3, & \textit{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 4, & x = 0, \end{cases} \text{ for } x, y \in X.$$

We claim that the pair (F, g) is compatible but not compatible of type (A), nor compatible of type (P), nor compatible of type (A_F), nor compatible of type (A_g).

For, let $\{x_n = n^2, n \geq 1\}$ and $\{y_n = 2n^2, n \geq 1\}$.

Then

$$\lim_{n\to\infty} F(x_n, y_n) = 0 = \lim_{n\to\infty} g(x_n)$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = 0 = \lim_{n\to\infty} g(y_n).$$

Also, since

$$\begin{aligned} &F(gx_n, gy_n) = 64n^{24}, \quad F(gy_n, gx_n) = 64n^{24}, \quad g^2x_n = n^8, \quad g^2y_n = 16n^8, \\ &F(F(x_n, y_n), F(y_n, x_n)) = (64n^{24})^3, F(F(y_n, x_n), F(x_n, y_n)) = (64n^{24})^3, \\ &gF(x_n, y_n) = 64n^{24}, \quad gF(y_n, x_n) = 64n^{24}, \end{aligned}$$

we have

$$\begin{aligned} &\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) \neq 1, \\ &\lim_{n\to\infty} M(F(F(x_n, y_n), F(y_n, x_n)), g^2x_n, t) \neq 1, \\ &\lim_{n\to\infty} M(F(gx_n, gy_n), ggx_n, t) \neq 1, \\ &\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), Fy_n, x_n)), t) \neq 1. \end{aligned}$$

Thus, the pair (F, g) is none of the following:

1. *compatible of type (A),*
2. *compatible of type (P),*
3. *compatible of type (A_F),*
4. *compatible of type (A_g).*

Also, for the sequences $\{x_n\}$ and $\{y_n\}$, with $\lim_{n\to\infty} F(x_n, y_n) = a = \lim_{n\to\infty} g(x_n)$ and $\lim_{n\to\infty} F(y_n, x_n) = b = \lim_{n\to\infty} g(y_n)$, for some $a, b \in X$, we have $gF(x_n, y_n) = (x_n y_n)^6 = F(g(x_n, gy_n))$ and $g(F(y_n, x_n)) = (x_n y_n)^6 = F(g(y_n), g(x_n)))$ so that, we have

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = \lim_{n\to\infty} \frac{t}{t + |gF(x_n, y_n) - F(g(x_n), g(y_n)))|} = 1.$$

Similarly, $\lim_{n\to\infty} M(gF(y_n, x_n), F(gy_n, gx_n), t) = 1$, so that the mappings F and g are compatible.

Next we illustrate that mappings of compatible of type (A) need not be compatible.

Example 2.7. *Let $X = [0,6]$. Define $a * b = ab$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define the mappings $F : X \times X \to X$ and $g : X \to X$ by*

$$F(x,y) = \begin{cases} \frac{x+y}{2}, & \textit{if both } x,y \in [0,3) \\ 6, & \textit{otherwise} \end{cases} \quad \textit{and} \quad g(x) = \begin{cases} 6-x, & \textit{if } x \in [0,3) \\ 6, & \textit{otherwise.} \end{cases}$$

Let $\{x_n = 3 - \frac{1}{n}, n \geq 1\}$ and $\{y_n = 3 - \frac{1}{2n}, n \geq 1\}$ be two sequences. Then, we obtain that

$$\lim_{n\to\infty} F(x_n, y_n) = 3 = \lim_{n\to\infty} g(x_n) \textit{ and } \lim_{n\to\infty} F(y_n, x_n) = 3 = \lim_{n\to\infty} g(y_n).$$

Now, we have $gF(x_n, y_n) = \left(3 + \frac{3}{4n}\right)$, $gF(y_n, x_n) = \left(3 + \frac{3}{4n}\right)$, $F(gx_n, gy_n) = 6$, $F(gy_n, gx_n) = 6$, $g^2x_n = 6$, $g^2y_n = 6$, $F(F(x_n, y_n), F(y_n, x_n)) = \left(3 - \frac{3}{4n}\right)$, $F(F(y_n, x_n), F(x_n, y_n)) = \left(3 - \frac{3}{4n}\right)$.

Now, the pair (F, g) is not compatible, since

$$\begin{aligned} \lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) &= \lim_{n\to\infty} M\left(3 + \frac{3}{4n}, 6, t\right) \\ &= \lim_{n\to\infty} \frac{t}{t + |3 + \frac{3}{4n} - 6|} \\ &\not\to 1. \end{aligned}$$

By routine calculation, it is easy to notice that the pair (F, g) is compatible of type (A).

Lemma 2.13. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that the pair (F, g) is compatible of type (A) and one of the mappings F and g is continuous, then the pair (F, g) is compatible.*

Proof. Without loss of generality, we assume that the mapping g is continuous. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$. Then

$$\begin{aligned} &M(F(g(x_n), g(y_n)), g(F(x_n, y_n), t) \\ &\quad \geq M(F(g(x_n), g(y_n), g^2x_n, t/2) * M(g^2x_n, gF(x_n, y_n), t/2), \end{aligned}$$

since the mappings F and g are compatible of type (A) and by continuity of g, on letting $n \to \infty$, it follows that $\lim_{n\to\infty} M(F(g(x_n), g(y_n)), gF(x_n, y_n), t) = 1$. Similarly, it is easy to obtain that $\lim_{n\to\infty} M(F(g(y_n), g(x_n)), gF(y_n, x_n), t) = 1$. Therefore, the mappings F and g are compatible. Analogously, it can be proved that if the mapping F is continuous and the pair (F, g) is compatible of type (A), then the pair (F, g) is also compatible. □

Lemma 2.14. *If the pair of mappings $F : X \times X \to X$ and $g : X \to X$ is compatible and both the mappings F and g are continuous, then the pair (F, g) is compatible of type (A).*

Lemma 2.15. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If the mapping g is continuous, then the pair (F, g) is compatible of type (A_F) iff the pair (F, g) is compatible.*

Proof. Let g be the continuous mapping. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Let the pair of mappings (F, g) be compatible of type (A_F), then

$$\begin{aligned} &M(F(g(x_n), g(y_n)), gF(x_n, y_n), t) \\ &\quad \geq M(F(g(x_n), g(y_n)), g^2x_n, t/2) * M(g^2(x_n, gF(x_n, y_n), t/2), \end{aligned}$$

on letting $n \to \infty$ and by the continuity of the mapping g, it follows that

$$\lim_{n\to\infty} M(F(g(x_n), g(y_n)), gF(x_n, y_n), t) = 1.$$

Similarly,

$$\lim_{n\to\infty} M(F(g(y_n), g(x_n)), gF(y_n, x_n), t) = 1.$$

Hence, the pair (F, g) is compatible.

We conclude the proof by showing that the pair (F, g) is compatible of type (A_F), if the pair (F, g) is compatible.

For,

$$\begin{aligned} &M(F(g(x_n), g(y_n)), g^2x_n, t) \\ &\quad \geq M(F(g(x_n), g(y_n)), gF(x_n, y_n), t/2) * M(gF(x_n, y_n), g^2x_n, t/2), \end{aligned}$$

then, by continuity of g, on letting $n \to \infty$, it follows that
$\lim_{n\to\infty} M(F(g(x_n), g(y_n)), g^2x_n, t) = 1$.
Similarly, $\lim_{n\to\infty} M(F(g(y_n), g(x_n)), g^2y_n, t) = 1$. Thus, the pair (F, g) is compatible of type (A_F). This completes the proof. □

The following lemma establishes the relationship between the pair of compatible mappings and the pair of compatible mappings of type (A_g):

Lemma 2.16. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If the mapping F is continuous, then the pair (F, g) is compatible of type (A_g) iff the pair (F, g) is compatible.*

Proof. The result can be proved analogously as Lemma 2.3. □

Lemma 2.17. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If the pair (F, g) is compatible of type (A), then the pair (F, g) is*

1. *compatible of type (B),*
2. *compatible of type (P),*
3. *compatible of type (A_F),*
4. *compatible of type (A_g).*

Proof. By using the definitions of variants of compatible mappings, the proof holds trivially. □

Remark 2.18. *Using Lemma 2.5, the Example 2.7 illustrates the fact that "the pair of the mappings that are compatible of type (B) or compatible of type (P) or compatible of type (A_F) or compatible of type (A_g) need not be compatible".*

Lemma 2.19. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two continuous mappings. Then the pair (F, g) is compatible of type (B) (or compatible of type (C) or compatible of type (P)) iff the pair (F, g) is compatible.*

Proof. First, assume that the pair (F, g) of the mappings is compatible of type (B). We shall show that the pair (F, g) of the mappings is compatible. For, let $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$. Then on using the continuity hypotheses of the mappings F and g in the definition of the compatible mappings of type (B), by the condition

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t)\\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(F(gx_n, gy_n), F(x, y), t)\\
&\qquad + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\},
\end{aligned}$$

we have that $M(F(x, y), gx, t) \geq 1$, that is $F(x, y) = gx$. Similarly, it can be obtained that $F(y, x) = g(y)$. Now, for $t > 0$

$$\begin{aligned}
&M(gF(x_n, y_n), F(g(x_n), g(y_n), t)\\
&\quad \geq M(gF(x_n, y_n), gx, (t/2)) * M(gx, F(g(x_n), g(y_n), (t/2)),
\end{aligned}$$

then, on letting $n \to \infty$, and using the continuity conditions of the mappings F and g in the last inequality, we obtain that

$$\begin{aligned}
\lim_{n\to\infty} M(g(F(x_n, y_n), F(g(x_n), g(y_n)), t) &\geq M(gx, gx, (t/2)) * M(gx, F(x, y), (t/2))\\
&= 1 * 1 = 1,
\end{aligned}$$

that is, $\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = 1$. Similarly, we obtain that $\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1$. Hence the pair (F, g) of the mappings is compatible. Interestingly, since the mappings F and g are continuous, so the conditions

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1$$

implies that $gx = F(x, y)$ and $gy = F(y, x)$, respectively, which has already been noted.

Conversely, assume that the pair (F, g) of the mappings be compatible. To show that it is compatible of type (B). For, let $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$. Now, we have

$$\begin{aligned} &M(F(g(x_n), g(y_n)), g^2x_n, t) \\ &\qquad \geq M(F(g(x_n), g(y_n), gF(x_n, y_n), (t/2)) * M(gF(x_n, y_n), g^2x_n, t/2), \end{aligned}$$

on letting $n \to \infty$, and using the compatibility of the mappings F and g along with the continuity of the mapping g, we obtain that $\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) \geq 1$, that is, $\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1$. Also, on using the continuity hypothesis of the mapping F, we obtain that

$$\frac{1}{2}\{\lim_{n\to\infty} M(F(gx_n, gy_n), F(x, y), t) + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\} = 1.$$

Hence, we can conclude that

$$\begin{aligned} &\lim_{n\to\infty} M(F(g(x_n)), g(y_n), g^2x_n, t) \\ &\qquad \geq \frac{1}{2}\{\lim_{n\to\infty} M(F(g(x_n), g(y_n)), F(x, y), t) \\ &\qquad + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\}. \end{aligned}$$

Similarly, we can show that if the pair (F, g) of the mappings is compatible and the mappings F and g are continuous, then all the other conditions for the mappings F and g to be the compatible of type (B) holds.

Analogously, it can be easily proved that if the mappings F and g are both continuous, then the pair (F, g) is compatible of type (C) (or, compatible of type (P)) iff the pair (F, g) is compatible. □

Next example illustrates that compatible mappings of type (B) need not be compatible, nor compatible of type (A), nor compatible of type (C), nor compatible of type (P).

Example 2.8. *Let $X = [0,2]$. Define $a * b = ab$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define the mappings $F : X \times X \to X$ and $g : X \to X$ by*

$$F(x,y) = \begin{cases} \frac{1}{2} + x, & \text{if } x,y \in [0, \frac{1}{2}) \\ 2, & \text{if } x = y = \frac{1}{2} \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{2} - x, & \text{if } x \in [0, \frac{1}{2}) \\ x - \frac{1}{2}, & \text{if } x \in (\frac{1}{2}, 1) \\ 1, & \text{otherwise.} \end{cases}$$

The pair (F,g) is not compatible but compatible of type (B).

For, let $\left\{x_n = \frac{1}{n}, n \geq 3\right\}$ and $\left\{y_n = \frac{1}{2n}, n \geq 3\right\}$. Then

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = \frac{1}{2} = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = \frac{1}{2} = y.$$

Also for $n \geq 3$, $gF(x_n, y_n) = \frac{1}{n}$, $gF(y_n, x_n) = \frac{1}{2n}$, $F(gx_n, gy_n) = 1 - \frac{1}{n}$, $F(gy_n, gx_n) = 1 - \frac{1}{2n}$, $g^2 x_n = \frac{1}{n}$, $g^2 y_n = \frac{1}{2n}$, $F(F(x_n, y_n), F(y_n, x_n)) = 1$, $F(F(y_n, x_n), F(x_n, y_n)) = 1$, $F(x,y) = 2$, $F(y,x) = 2$, $gx = 1$, $gy = 1$.

Since,

$$\begin{aligned} \lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) &= \lim_{n\to\infty} M\left(\frac{1}{n}, \left(1 - \frac{1}{n}\right), t\right) \\ &= \lim_{n\to\infty} \frac{t}{t + |\frac{1}{n} - (1 - \frac{1}{n})|} \not\to 1, \end{aligned}$$

hence, the pair (F,g) is not compatible.

But the pair is compatible of type (B), since

$$\begin{aligned} \lim_{n\to\infty} M(F(gx_n, gy_n), g^2 x_n, t) &= \lim_{n\to\infty} M\left(\left(1 - \frac{1}{n}\right), \frac{1}{n}, t)\right) \\ &= \lim_{n\to\infty} \frac{t}{t + |(1 - \frac{1}{n}) - \frac{1}{n}|} \\ &= \frac{t}{t+1}. \end{aligned}$$

Also,

$$\begin{aligned} &\frac{1}{2}\{\lim_{n\to\infty} M(F(g(x_n), g(y_n)), F(x,y), t) + \lim_{n\to\infty} M(F(x,y), F(F(x_n, y_n), F(y_n, x_n)), t)\} \\ &= \frac{1}{2}\left\{\lim_{n\to\infty} M\left(\left(1 - \frac{1}{n}\right), 2, t\right) + \lim_{n\to\infty} M(2, 1, t)\right\} \\ &= \frac{1}{2}\left\{\lim_{n\to\infty} \frac{t}{t + |(1 - \frac{1}{n}) - 2|} + \lim_{n\to\infty} \frac{t}{t + |2 - 1|}\right\} \\ &= \frac{1}{2}\left\{\frac{t}{t+1} + \frac{t}{t+1}\right\} \\ &= \frac{t}{t+1}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t)\\
&\quad = \frac{1}{2}\{\lim_{n\to\infty} M(F(g(x_n), g(y_n)), F(x, y), t)\\
&\qquad + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\}.
\end{aligned}$$

Similarly, it can be easily checked that

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t)\\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(F(g(y_n), g(x_n)), F(y, x), t)\\
&\qquad + \lim_{n\to\infty} M(F(y, x), F(F(y_n, x_n), F(x_n, y_n)), t)\}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t)\\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(gF(x_n, y_n), gx, t) + \lim_{n\to\infty} M(gx, g^2x_n, t)\},\\
&\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t)\\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(gF(y_n, x_n), gy, t) + \lim_{n\to\infty} M(gy, g^2y_n, t)\}.
\end{aligned}$$

Thus, the pair (F, g) is compatible of type (B). Also, we note that the pair (F, g) is not compatible of type (A), since

$$\begin{aligned}
\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) &= \lim_{n\to\infty} M\left(\left(1 - \frac{1}{n}\right), \frac{1}{n}, t\right)\\
&= \lim_{n\to\infty} \frac{t}{t + |(1 - \frac{1}{n}) - \frac{1}{n}|}\\
&= \frac{t}{t+1} \neq 1.
\end{aligned}$$

Further, the pair (F, g) is not compatible of type (P), since

$$\begin{aligned}
\lim_{n\to\infty} M(F(F(x_n, y_n), F(y_n, x_n), g^2x_n, t) &= \lim_{n\to\infty} M\left(1, \frac{1}{n}, t\right)\\
&= \lim_{n\to\infty} \frac{t}{t + |1 - \frac{1}{n}|}\\
&= \frac{t}{t+1}\\
&\neq 1.
\end{aligned}$$

Further, simple calculation shows that the pair (F, g) is not compatible of type (C).

Lemma 2.20. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If the pair (F, g) is compatible of type (B) (or compatible of type (C)) and both the mappings F, g are continuous, then the pair (F, g) is compatible of type (A).*

Proof. First, let us assume that the pair (F, g) of the mappings is compatible of type (B) and both the mappings F, g are continuous, then the pair (F, g) is compatible of type (A). For, let $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$. Since the pair (F, g) is compatible of type (B), we have

$$\begin{aligned}&\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t)\\ &\quad\geq \frac{1}{2}\{\lim_{n\to\infty} M(F(gx_n, gy_n), F(x, y), t)\\ &\qquad+ \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\},\end{aligned}$$

then, on using the continuity hypothesis of the mapping F on the right side of the above inequality, we obtain that $\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) \geq 1$, that is,

$$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1.$$

Similarly, we can obtain that

$$\lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t) = 1.$$

We now show that

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1.$$

Since the pair (F, g) is compatible of type (B), we have

$$\begin{aligned}&\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t)\\ &\quad\geq \frac{1}{2}\{\lim_{n\to\infty} M(gF(x_n, y_n), gx, t) + \lim_{n\to\infty} M(gx, g^2x_n, t)\},\end{aligned}$$

then, on using the continuity hypothesis of the mapping g on the right side of the above inequality, we obtain that

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) \geq 1,$$

that is,

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1.$$

Similarly, we can obtain that $\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x, y_n)), t) = 1$. Hence, the pair (F, g) is compatible of type (A).

Analogously, it can be easily proved that if both the mappings F, g are continuous and the pair (F, g) is compatible of type (C), then it is compatible of type (A). □

Remark 2.21. *In view of the above discussion, various relations between the variants of compatible mappings could be easily established under certain conditions. For example, we can easily observe that "If the mappings F and g are both continuous, then the pair (F, g) is compatible of type (B) iff the pair (F, g) is compatible of type (C)".*

Recently, Abbas et al. [1], introduced the concept of w-compatible mappings, following which, some authors established coupled common fixed point results for the similar notion of weakly compatible mappings. Works noted in [15, 17, 18] are some examples in this direction.

Definition 2.22 ([1])**.** *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be w-compatible if $gF(x, y) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.*

Definition 2.23 ([15,17,18])**.** *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be weakly compatible if $gF(x, y) = F(gx, gy)$ and $gF(y, x) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.*

Interestingly, the concepts of w-compatible mappings and weakly compatible mappings are equivalent.

Lemma 2.24. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If F and g are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), or compatible of type (A_F), or compatible of type (A_g), then they are weakly compatible (or, w-compatible).*

Proof. First, we shall show that if the pair (F, g) of the mappings be compatible, then it is also weakly compatible. For, if the pair (F, g) of the mappings be compatible, then by definition of compatible mappings, we have

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n), t) = 1 \quad \text{and}$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1,$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x,$$

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y,$$

for some $x, y \in X$. Taking $x_n = a$ and $y_n = b$, we obtain that $ga = F(a, b)$ and $gb = F(b, a)$ implies that $gF(a, b) = F(ga, gb)$ and $gF(b, a) = F(gb, ga)$. Hence every pair of compatible mappings is always weakly compatible (or, we can say w-compatible).

Next, we shall show that if the pair (F, g) of the mappings be compatible of type (A), then it is also a weakly compatible pair. For, if the pair (F, g) of the mappings be compatible of type (A), then by definition of compatible mappings of type (A), we have

$$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1, \quad \lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t) = 1$$

and

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$. Taking $x_n = a$ and $y_n = b$, we obtain that $ga = F(a, b) = x$ and $gb = F(b, a) = y$. And the condition $\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1$ becomes $M(F(ga, gb), g^2a, t) = 1$, that is, $M(F(ga, gb), gga, t) = 1$, that is, $M(F(ga, gb), gFa, b, t) = 1$ which implies that $F(ga, gb) = gF(a, b)$. Similarly, we can obtain that $F(gb, ga) = gF(b, a)$. Therefore, $ga = F(a, b)$ and $gb = F(b, a)$ implies that $F(ga, gb) = gF(a, b)$ and $F(gb, ga) = gF(b, a)$. Hence, we can conclude that every pair of compatible mappings of type (A) is always weakly compatible (or, we can say w-compatible).

Now, if the pair (F, g) of the mappings be compatible of type (B), then it is also a weakly compatible pair. For, if the pair (F, g) of the mappings be compatible of type (B), by taking $x_n = a$ and $y_n = b$ in the definition of compatible mappings of type (B), we obtain that $ga = F(a, b) = x$ and $gb = F(b, a) = y$. Then, the condition

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) \\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(F(gx_n, gy_n), F(x, y), t) \\
&\qquad + \lim_{n\to\infty} M(F(x, y), F(F(x_n, y_n), F(y_n, x_n)), t)\}
\end{aligned}$$

in the definition of compatible mappings of type (B) becomes

$$\begin{aligned}
M(F(ga, gb), g^2a, t) \geq &\frac{1}{2}\left\{M(F(ga, gb), F(x, y), t)\right. \\
&\left.+M(F(x, y), F(F(a, b), F(b, a)), t)\right\},
\end{aligned}$$

that is,

$$M(F(ga, gb), gF(a, b), t) \geq \frac{1}{2}\{M(F(ga, gb), F(x, y), t) + M(F(x, y), F(x, y), t)\},$$

that is,

$$M(F(ga, gb), gF(a, b), t) \geq \frac{1}{2}\{M(F(ga, gb), F(ga, gb), t) + M(F(x, y), F(x, y), t)\},$$

that is, $M(F(ga,gb),gF(a,b),t) \geq 1$, that is, $M(F(ga,gb),gF(a,b),t) = 1$, hence, $F(ga,gb) = F(ga,gb)$. Similarly, we can obtain that $F(gb,ga) = gF(b,a)$. Therefore, $ga = F(a,b)$ and $gb = F(b,a)$ implies that $F(ga,gb) = gF(a,b)$ and $F(gb,ga) = gF(b,a)$. Hence, we can conclude that every pair of compatible mappings of type (B) is always weakly compatible (or, we can say w-compatible).

Similarly, we can prove that if the pair of the mappings (F,g) is compatible of type (P), or compatible of type (C), or compatible of type (A_F), or compatible of type (A_g), then it is weakly compatible (or, w-compatible). □

The following example illustrates that weakly compatible mappings need not be compatible nor compatible of type (A), nor compatible of type (B), nor compatible of type (P), nor compatible of type (C), nor compatible of type (A_F).

Example 2.9. *Let* $X = [1,20]$ *and* $*$ *being any continuous* t*-norm. Define*

$$M(x,y,t) = e^{-|x-y|/t},$$

for all $x,y \in X$ *and* $t > 0$. *Then* $(X,M,*)$ *is a FM-space. Define the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *respectively by*

$$F(x,y) = \begin{cases} 1, & \text{if } x = 1, \text{ or } x > 4, y \in X \\ 5, & \text{if } 1 < x \leq 4, y \in X \end{cases} \quad \text{and } g(x) = \begin{cases} 1, & \text{if } x = 1 \\ 12, & \text{if } 1 < x \leq 4 \\ x-3, & \text{if } x > 4. \end{cases}$$

Then the only coupled coincidence point for the pair (F,g) *is* $(1,1)$. *The mappings* F *and* g *are not compatible, since for the sequences* $\{x_n\}$ *and* $\{y_n\}$ *with* $x_n := 4+\frac{1}{2n}$ *and* $y_n := 4 + \frac{1}{2n+1}$ *for* $n \geq 1$, *we have* $F(x_n,y_n) = 1$, $g(x_n) \to 1$, $F(y_n,x_n) = 1$, $g(y_n) \to 1$, $M(gF(x_n,y_n),F(g(x_n),g(y_n)),t) = e^{-4/t} \not\to 1$ *as* $n \to \infty$.

Also, for the above defined sequences $\{x_n\}$ *and* $\{y_n\}$, *we have*

$$M(F(g(x_n),g(y_n)),g^2x_n,t) = e^{-7/t} \not\to 1$$

as $n \to \infty$, *so that the functions* F *and* g *are not compatible of type (A) and not compatible of type* (A_F). *We next show that the mappings* F *and* g *are also not compatible of type (B). On the contrary, assume that the mappings* F *and* g *are compatible of type (B), then, we must have*

$$\begin{aligned}
&\lim_{n\to\infty} M(F(gx_n,gy_n),g^2x_n,t) \\
&\quad \geq \frac{1}{2}\{\lim_{n\to\infty} M(F(gx_n,gy_n),F(x,y),t) \\
&\qquad + \lim_{n\to\infty} M(F(x,y),F(F(x_n,y_n),F(y_n,x_n)),t)\},
\end{aligned}$$

iff $e^{-7/t} \geq \frac{1}{2}(1+e^{-4/t})$ *iff* $2 \geq e^{7/t}+e^{3/t}$, *which is not possible for* $t > 0$. *Hence, the mappings* F *and* g *are not compatible of type (B). In a similar way, we can easily show that the mappings* F *and* g *are neither compatible of type (C) nor compatible of type (P). But the mappings* F *and* g *are weakly compatible, since they commute at their coupled coincidence point* $(1,1)$.

Remark 2.25. *Since every pair of compatible mappings is weakly compatible, so that the mappings F and g defined in the Example 2.6 being compatible are also weakly compatible.*

Hence, Example 2.6 illustrates the fact that weakly compatible mappings need not be compatible of type (A_g).

Remark 2.26. *Similarly, we can obtain that if $(X, M, *)$ is a fuzzy metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings, and if the pair (F, g) is commuting, or weakly commuting, or R-weakly commuting, or R-weakly commuting of type (A_F), or R-weakly commuting of type (A_g), or R-weakly commuting of type (P), then the pair (F, g) is also weakly compatible (or, w-compatible). However, in general, the converse need not be true.*

In the following section, we establish the existence and uniqueness of common fixed points for the mappings satisfying a new contraction condition.

3. Main results

Let us define by $\mathcal{W}$ the class of all continuous, non-decreasing functions $\omega : [0, 1] \to [0, 1]$ with the property that $\omega(t) = 1$ iff $t = 1$. Also, define by $\mathcal{V}$ the class of all continuous functions $\gamma : [0, 1] \to [0, 1]$.

Lemma 3.1. *Let $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$. Assume that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$. Then $\gamma(1) = 1$.*

Proof. Let $\{a_n\} \subseteq (0, 1)$ be a non-decreasing sequence with $\lim_{n\to\infty} a_n = 1$. By hypothesis we have $\gamma(a_n) \geq \omega(a_n)$, $n \in \mathbb{N}$. Using the properties of γ and ω, we can obtain that $\gamma(1) \geq \omega(1) = 1$, which implies that $\gamma(1) = 1$. This completes the proof. □

Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm of H-type and $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying the following conditions:

1. *$A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$,*
2. *there exists $\phi \in \Phi$ such that*

$$\begin{aligned}&\omega(M(A(x,y), B(u,v), \phi(t)) * M(A(y,x), B(v,u), \phi(t)))\\&\quad \geq \gamma(M(Sx, Tu, t) * M(Sy, Tv, t)),\end{aligned}$$

   for all x, y, u, v in X and $t > 0$, where $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$ such that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$.

Then for the arbitrary points x_0, y_0 in X, by (3.1), we can choose x_1, y_1 in X such that $T(x_1) = A(x_0, y_0)$, $T(y_1) = A(y_0, x_0)$.

Again, by (3.1), we can choose x_2, y_2 in X such that $S(x_2) = B(x_1, y_1)$ and $S(y_2) = B(y_1, x_1)$.

Continuing in this way, we can construct two sequences $\{z_n\}$ and $\{z'_n\}$ in X such that

3. $z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1})$, $z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2})$ *and*

4. $z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1})$, $z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2})$, *for all* $n \geq 0$.

In order to prove our main result, we first give the following lemma:

Lemma 3.2. *The sequences* $\{z_n\}$ *and* $\{z'_n\}$ *defined by (3.3) and (3.4) respectively are Cauchy in* X.

Proof. Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

1. $\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq 1 - \varepsilon$, for all $p \in \mathbb{N}$.

Since $\lim\limits_{t \to \infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

1. $M(Sx_0, Tx_1, t_0) \geq (1 - \delta)$ and $M(Sy_0, Ty_1, t_0) \geq (1 - \delta)$.

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

1. $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$.

Using condition (3.2), we have

$$\begin{aligned}
&\omega(M(z_1, z_2, \phi(t_0)) * M(z'_1, z'_2, \phi(t_0))) \\
&\quad = \omega(M(A(x_0, y_0), B(x_1, y_1), \phi(t_0)) * M(A(y_0, x_0), B(y_1, x_1), \phi(t_0)) \\
&\quad \geq \gamma(M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0)) \\
&\quad \geq \omega(M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0)),
\end{aligned}$$

which implies by the monotonic property of function ω, that

$$M(z_1, z_2, \phi(t_0)) * M(z'_1, z'_2, \phi(t_0)) \geq M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0).$$

Again using the condition (3.2), we can get

$$\begin{aligned}
&\omega(M(z_2, z_3, \phi^2(t_0)) * M(z'_2, z'_3, \phi^2(t_0))) \\
&\quad = \omega(M(B(x_1, y_1), A(x_2, y_2), \phi^2(t_0)) * M(B(y_1, x_1), A(y_2, x_2), \phi^2(t_0))) \\
&\quad \geq \gamma(M(Sx_2, Tx_1, \phi(t_0)) * M(Sy_2, Ty_1, \phi(t_0))) \\
&\quad \geq \omega(M(Sx_2, Tx_1, \phi(t_0)) * M(Sy_2, Ty_1, \phi(t_0))) \\
&\quad = \omega(M(z_2, z_1, \phi(t_0)) * M(z'_2, z'_1, \phi(t_0)))
\end{aligned}$$

which implies by the monotonic property of function ω, that

$$M(z_2, z_3, \phi^2(t_0)) * M(z'_2, z'_3, \phi^2(t_0)) \geq M(z_1, z_2, \phi(t_0)) * M(z'_1, z'_2, \phi(t_0)).$$

Similarly, we can obtain that

$$M(z_3, z_4, \phi^3(t_0)) * M(z'_3, z'_4, \phi^3(t_0)) \geq M(z_2, z_3, \phi^2(t_0)) * M(z'_2, z'_3, \phi^2(t_0)).$$

Continuing in this way, for all $n > 0$, we can obtain that

$$\begin{aligned} &M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * M(z'_{n+1}, z'_{n+2}, \phi^{n+1}(t_0)) \\ &\quad \geq M(z_n, z_{n+1}, \phi^n(t_0)) * M(z'_n, z'_{n+1}, \phi^n(t_0)), \end{aligned}$$

which implies that

$$\begin{aligned} &M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0) * M(z'_{n+1}, z'_{n+2}, \phi^{n+1}(t_0)) \\ &\quad \geq M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0). \end{aligned}$$

Using (3.5)-(3.7), for $m > n \geq n_0$, we have

$$\begin{aligned} &M(z_n, z_m, t) * M(z'_n, z'_m, t) \\ &\geq M\left(z_n, z_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) * M\left(z'_n, z'_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(z_n, z_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) * M\left(z'_n, z'_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\ &\geq [M(z_n, z_{n+1}, \phi^n(t_0)) * M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * \ldots * M(z_{m-1}, z_m, \phi^{m-1}(t_0))] \\ &\quad * [M(z'_n, z'_{n+1}, \phi^n(t_0) * Mz'_{n+1}, z'_{n+2}, \phi^{n+1}(t_0) * \ldots * Mz'_{m-1}, z'_m, \phi^{m-1}(t_0))] \\ &= [M(z_n, z_{n+1}, \phi^n(t_0)) * M(z'_n, z'_{n+1}, \phi^n(t_0))] \\ &\quad * [M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * M(z'_{n+1}, z'_{n+2}, \phi^{n+1}(t_0))] \\ &\vdots \\ &\quad * [M(z_{m-1}, z_m, \phi^{m-1}(t_0)) * M(z'_{m-1}, z'_m, \phi^{m-1}(t_0))] \\ &\geq [M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0)] * [M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0)] \cdots \\ &\qquad * [M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0)] \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{2(m-n)} \\ &\geq 1 - \varepsilon, \end{aligned}$$

which implies that $M(z_n, z_m, t) * M(z'_n, z'_m, t) \geq 1 - \varepsilon$, for all $m, n \in \mathbb{N}$ with $m > n > n_0$ and $t > 0$. So that $\{z_n\}$ and $\{z'_n\}$ both are Cauchy sequences in X.□

We are now ready to give our main result as follows:

Theorem 3.3. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type and $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying (3.1), (3.2) and the following conditions:*

8. *the pairs (A,S) and (B,T) are weakly compatible,*

9. *one of the subspaces $A(X\times X)$ or $T(X)$ and one of $B(X\times X)$ or $S(X)$ are complete.*

Then there exists a unique point α in X such that

$$A(\alpha,\alpha)=S(\alpha)=\alpha=T(\alpha)=B(\alpha,\alpha).$$

Proof. By Lemma 3.2, the sequences $\{z_n\}$ and $\{z'_n\}$ defined respectively by (3.3) and (3.4) are both Cauchy sequences. We shall divide the proof in to four steps as follows:
Step 1: In this step we shall show the existence of some elements $\alpha,\beta\in X$ such that $T(\alpha)=B(\alpha,\beta)$, $T(\beta)=B(\beta,\alpha)$ and $S(\alpha)=A(\alpha,\beta)$, $S(\beta)=A(\beta,\alpha)$.

Without loss of generality, we assume that the subspaces $T(X)$ and $S(X)$ are complete. Since $\{z_{2n+1}\}$, $\{z_{2n+2}\}$ and $\{z'_{2n+1}\}$, $\{z'_{2n+2}\}$ are the sub-sequences of the Cauchy sequences $\{z_n\}$ and $\{z'_n\}$ respectively, so they are also Cauchy sequences. By completeness of $T(X)$, there exists α,β in $T(X)\subseteq X$ such that $\{z_{2n+1}\}\to\alpha$ and $\{z'_{2n+1}\}\to\beta$. By the convergence of the sub-sequences $\{z_{2n+1}\}$ and $\{z'_{2n+1}\}$, it is easy to establish the convergence of the original Cauchy sequences $\{z_n\}$ and $\{z'_n\}$ respectively, so that $\{z_n\}\to\alpha$ and $\{z'_n\}\to\beta$.

Consequently, it follows that the sequences $\{z_{2n+1}\}$, $\{z_{2n+2}\}$, $\{z_n\}$ converges to α and $\{z'_{2n+1}\}$, $\{z'_{2n+2}\}$, $\{z'_n\}$ converges to β. Since $\alpha,\beta\in T(X)$, there exist some $p,q\in X$ such that $T(p)=\alpha$, $T(q)=\beta$, so that we have

$$\begin{aligned}\lim_{n\to\infty} z_{2n+1}&=\lim_{n\to\infty} A(x_{2n},y_{2n})=\lim_{n\to\infty} T(x_{2n+1})=\alpha=T(p),\\ \lim_{n\to\infty} z_{2n+2}&=\lim_{n\to\infty} B(x_{2n+1},y_{2n+1})=\lim_{n\to\infty} S(x_{2n+2})=\alpha=T(p)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} z'_{2n+1}&=\lim_{n\to\infty} A(y_{2n},x_{2n})=\lim_{n\to\infty} T(y_{2n+1})=\beta=T(q),\\ \lim_{n\to\infty} z'_{2n+2}&=\lim_{n\to\infty} B(y_{2n+1},x_{2n+1})=\lim_{n\to\infty} S(y_{2n+2})=\beta=T(q).\end{aligned}$$

By condition (3.2), we obtain that

$$\begin{aligned}&\omega(M(A(x_{2n},y_{2n}),B(p,q),\phi(t))*M(A(y_{2n},x_{2n}),B(q,p),\phi(t))\\ &\quad\geq\gamma(M(Sx_{2n},T(p),t)*M(Sy_{2n},T(q),t))\\ &\quad\geq\omega(M(Sx_{2n},T(p),t)*M(Sy_{2n},T(q),t)),\end{aligned}$$

then on using the monotonic property of ω, we obtain that

$$\begin{aligned}&M(A(x_{2n},y_{2n}),B(p,q),\phi(t)*M(A(y_{2n},x_{2n}),B(q,p),\phi(t))\\ &\quad\geq M(Sx_{2n},T(p),t)*M(Sy_{2n},T(q),t),\end{aligned}$$

on letting $n\to\infty$, we obtain that $M(T(p),B(p,q),\phi(t)*M(T(q),B(q,p),\phi(t))\geq 1$, which implies that $T(p)=B(p,q)=\alpha$ and $T(q)=B(q,p)=\beta$. As the pair (B,T)

is weakly compatible, so that $T(p) = B(p,q) = \alpha$ implies that $T(a) = B(\alpha, \beta)$. Similarly, we can obtain that $T(\beta) = B(\beta, \alpha)$. Again, since the subspace $S(X)$ is complete, so that $\alpha, \beta \in S(X)$, which implies the existence of r, s in X so that $S(r) = \alpha$, $S(s) = \beta$.

Again using the condition (3.2), we obtain that

$$\begin{aligned}
&\omega(M(A(r,s), B(x_{2n+1}, y_{2n+1}), \phi(t)) * M(A(s,r), B(y_{2n+1}, x_{2n+1}), \phi(t))) \\
&\geq \gamma(M(S(s), Tx_{2n+1}, t) * M(S(s), Ty_{2n+1}, t)),
\end{aligned}$$

on letting $n \to \infty$ and using the continuity of ω, γ we can obtain that

$$\omega(M(A(r,s), \alpha, \phi(t)) * M(A(s,r), \beta, \phi(t))) \geq \gamma(1) = 1,$$

which implies that $A(r,s) = \alpha = S(r)$ and $A(s,r) = \beta = S(s)$. Since the pair (A, S) is weakly compatible, it follows that $A(\alpha, \beta) = S(\alpha)$ and $A(\beta, \alpha) = S(\beta)$.
Step 2: Next we show that $S(\alpha) = T(\alpha)$ and $S(\beta) = T(\beta)$.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq 1 - \varepsilon, \quad \text{for all } p \in \mathbb{N}.$$

Since $\lim\limits_{t\to\infty} M(x,y,t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(S(\alpha), T(\alpha), (t_0)) \geq (1-\delta) \text{ and } M(S(\beta), T(\beta), (t_0)) \geq (1-\delta).$$

Also, by condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0)\infty < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$.

By condition (3.2), we obtain that

$$\begin{aligned}
&\omega(M(S(\alpha), T(\alpha), \phi(t_0)) * M(S(\beta), T(\beta), \phi(t_0))) \\
&= \omega(M(A(\alpha,\beta), B(\alpha,\beta), \phi(t_0)) * M(A(\beta,\alpha), B(\beta,\alpha), \phi(t_0))) \\
&\geq \gamma(M(S(\alpha), T(\alpha), t_0) * M(S(\beta), T(\beta), t_0)).
\end{aligned}$$

Since $\gamma(t) \geq \omega(t)$ for $t \in [0,1]$, we can easily obtain from above inequality that

$$\begin{aligned}
&\omega(M(S(\alpha), T(\alpha), \phi(t_0)) * M(S(\beta), T(\beta), \phi(t_0))) \\
&\geq \omega(M(S(\alpha), T(\alpha), t_0) * M(S(\beta), T(\beta), t_0)).
\end{aligned}$$

By the monotonic property of ω, we obtain that

$$\begin{aligned}
&M(S(\alpha), T(\alpha), \phi(t_0) * M(S(\beta), T(\beta), \phi(t_0))) \\
&\geq M(S(\alpha), T(\alpha), t_0) * M(S(\beta), T(\beta), t_0).
\end{aligned}$$

Reasoning as above, we can obtain in general for all $n \geq 1$, that

$$\begin{aligned}
&M(S(\alpha), T(\alpha), \phi^n(t_0)) * M(S(\beta), T(\beta), \phi^n(t_0)) \\
&\geq M(S(\alpha), T(\alpha), t_0) * M(S(\beta), T(\beta), t_0).
\end{aligned}$$

Thus, for $t > 0$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&M(S(\alpha), T(\alpha), t) * M(S(\beta), T(\beta), t) \\
&\quad \geq M\left(S(\alpha), T(\alpha), \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) * M\left(S(\beta), T(\beta), \sum_{k=n_0}^{\infty} \phi^k(t_0))\right) \\
&\quad \geq M(S(\alpha), T(\alpha), \phi^{n_0} t_0) * M(S(\beta), T(\beta), \phi^{n_0} t_0) \\
&\quad \geq M(S(\alpha), T(\alpha), t_0) * M(S(\beta), T(\beta), t_0) \\
&\quad \geq (1-\delta) * (1-\delta) \geq (1-\varepsilon).
\end{aligned}$$

Hence, $S(\alpha) = T(\alpha)$ and $S(\beta) = T(\beta)$.
Therefore, $S(\alpha) = A(\alpha, \beta) = B(\alpha, \beta) = T(\alpha)$ and $S(\beta) = A(\beta, \alpha) = B(\beta, \alpha) = T(\beta)$.
Step 3: We next show that $S(\alpha) = \alpha$ and $S(\beta) = \beta$.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\varepsilon), \quad \text{for all } p \in \mathbb{N}.$$

Since $\lim_{t \to \infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(\alpha, S(\alpha), t_0) \geq (1-\delta) \quad \text{and} \quad M(\beta, S(\beta), t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{n=n_0}^{\infty} \phi^k(t_0)$.

Form condition (3.2), we have

$$\begin{aligned}
&\omega(M(S(\alpha), \alpha, \phi(t_0) * M(S(\beta), \beta, \phi(t_0))) \\
&\quad = \omega(M(A(\alpha, \beta), B(p, q), \phi(t_0)) * M(A(\beta, \alpha), B(q, p), \phi(t_0))) \\
&\quad \geq \gamma(M(S(\alpha), T(p), \phi(t_0)) * M(S(\beta), T(q), \phi(t_0))) \\
&\quad = \gamma(M(S(\alpha, \alpha), t_0) * M(S(\beta, \beta), t_0)),
\end{aligned}$$

using the fact that $\gamma(t) \geq \omega(t)$ for $t \in [0, 1]$ and the monotone property of ω, we obtain that

$$M(S(\alpha), \alpha, \phi(t_0)) * M(S(\beta), \beta, \phi(t_0)) \geq M(S(\alpha), \alpha, t_0) * M(S(\beta), \beta, t_0).$$

In general, we can obtain that

$$M(S(\alpha), \alpha, \phi^n(t_0) * M(S(\beta), \beta, \phi^n(t_0)) \geq M(S(\alpha), \alpha, t_0) * M(S(\beta), \beta, t_0)),$$

for all $n \geq 1$. Now, for all $t > 0$ and for any $\varepsilon > 0$, we have

$$\begin{aligned}
&M(S(\alpha), \alpha, t) * M(S(\beta), \beta, t) \geq M\left(S(\alpha), \alpha, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) * M\left(S(\beta), \beta, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
&\geq M(S(\alpha), \alpha, \phi^{n_0}(t_0)) * M(S(\beta), \beta, \phi^{n_0}(t_0)) \\
&\geq M(S(\alpha), \alpha, t_0) * M(S(\beta), \beta, t_0) \\
&\geq (1-\delta) * (1-\delta) \geq (1-\varepsilon).
\end{aligned}$$

Therefore, $S(\alpha) = \alpha$ and $S(\beta) = \beta$.

Thus, we have $B(\alpha, \beta) = S(\alpha) = \alpha = T(\alpha) = A(\alpha, \beta)$ and $B(\beta, \alpha) = S(\beta) = \beta = T(\beta) = A(\beta, \alpha)$.

Step 4: We now show that $\alpha = \beta$.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\varepsilon), \quad \text{for all } p \in \mathbb{N}.$$

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(\alpha, \beta, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$.

Using condition (3.2), we have

$$\begin{aligned}
&\omega(M(\alpha, \beta, \phi(t_0)) * M(\beta, \alpha, \phi(t_0))) \\
&\quad = \omega(M(A(p, q), B(q, p), \phi(t_0) * M(A(q, p), B(p, q), \phi(t_0)) \\
&\quad \geq \gamma(M(S(p), T(q), (t_0) * M(S(q), T(p), (t_0)) \\
&\quad = \gamma(M(\alpha, \beta, t_0) * M(\beta, \alpha, t_0),
\end{aligned}$$

using the fact that $\gamma(t) \geq \omega(t)$ for $t \in [0, 1]$ and the monotone property of ω, we obtain that

$$M(\alpha, \beta, \phi(t_0)) * M(\beta, \alpha, \phi(t_0))) \geq M(\alpha, \beta, t_0) * M(\beta, \alpha, t_0).$$

In general, we obtain that

$$M(\alpha, \beta, \phi^n(t_0)) * M(\beta, \alpha, \phi^n(t_0)) \geq M(\alpha, \beta, t_0) * M(\beta, \alpha, t_0),$$

for all $n \geq 1$. Then,

$$\begin{aligned}
M(\alpha, \beta, t) * M(\beta, \alpha, t) \geq & M(\alpha, \beta, \sum_{k=n_0}^{\infty} \phi^k(t_0)) * M\left(\beta, \alpha, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
\geq & M(\alpha, \beta, \phi^{n_0}(t_0)) * M(\beta, \alpha, \phi^{n_0}(t_0)) \\
\geq & M(\alpha, \beta, t_0) * M(\beta, \alpha, t_0) \geq (1-\delta) * (1-\delta) \\
\geq & (1-\varepsilon)
\end{aligned}$$

for all $t > 0$ and $\varepsilon > 0$, which implies that $\alpha = \beta$. Hence, there exists some point α in X such that $A(\alpha, \alpha) = T(\alpha) = \alpha = S(\alpha) = B(\alpha, \alpha)$. Uniqueness of the point α follows immediately by using (3.2). $\square$

Theorem 3.4. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type such that M has n-property on $X^2 \times (0, \infty)$. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X, T : X \to X$ be four mappings satisfying (3.1), (3.2), (3.8) and (3.9). Then there exists a unique point α in X such that $A(\alpha, \alpha) = S(\alpha) = \alpha = T(\alpha) = B(\alpha, \alpha)$.*

Proof. Since M has n-property on $X^2 \times (0, \infty)$, we have that $\lim_{n\to\infty} [M(x, y, k^n t)]^{n^p} = 1$, whenever $x, y \in X$, $k > 1$ and $p > 0$. We claim that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. If not, then using the fact that $M(x, y, \cdot) \in [0, 1]$ and the non-decreasing property of $M(x, y, \cdot)$, we can conclude the existence of some $a, b \in X$ such that $\lim_{t\to+\infty} M(a, b, t) = \gamma < 1$, then for any $t > 0$ and $k > 1$, we have that $k^n t \to +\infty$ as $n \to \infty$ and hence we obtain that $\lim_{n\to\infty} [M(x, y, k^n t)]^{n^p} = 0$ for $p > 0$, which is a contradiction. Now, the proof follows immediately by applying Theorem 3.1. □

On taking $\phi(t) = kt$, for $t > 0$, where $k \in (0, 1)$ and taking ω, γ to be the identity mapping on their respective domains, we obtain the following result:

Theorem 3.5. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type and $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying (3.1), (3.8), (3.9) and the following condition:*

1. *there exists $k \in (0, 1)$ such that*

$$M(A(x, y), B(u, v), kt) * M(A(y, x), B(v, u), kt) \geq M(Sx, Tu, t) * M(Sy, Tv, t),$$

for all x, y, u, v in X and $t > 0$. Then there exists a unique point α in X such that $A(\alpha, \alpha) = S(\alpha) = \alpha = T(\alpha) = B(\alpha, \alpha)$.

Taking $A = B = F$ and $S = T = g$ in Theorem 3.1, we have the following result:

Corollary 3.6. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type and $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$$\begin{aligned} &\omega(M(F(x, y), F(u, v), \phi(t)) * M(F(y, x), F(v, u), \phi(t)) \\ &\quad \geq \gamma M(gx, gu, t) * M(gy, gv, t), \end{aligned}$$

for all x, y, u, v in X and $t > 0$, where $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$ such that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$.

Suppose that $F(X \times X) \subseteq g(X)$ and F and g are weakly compatible. If one of the range spaces of F or g is complete, then there exits a unique α in X such that $\alpha = g(\alpha) = F(\alpha, \alpha)$.

Theorem 3.7. *Theorem 3.1 (and Theorems 3.2, 3.3) remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypotheses) of the following properties:*

1. *compatibility;*
2. *compatibility of type (A);*
3. *compatibility of type (P);*
4. *compatibility of type (B);*
5. *compatibility of type (C);*
6. *compatibility of type* (A_F)*;*
7. *compatibility of type* (A_g)*.*

Proof. On using Lemma 2.8, the proof follows immediately. □

Theorem 3.8. *Theorem 3.1 (and Theorems 3.2, 3.3) remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypothesis) of the following properties:*

1. *commuting;*
2. *weakly commuting;*
3. *R-weakly commuting;*
4. *R-weakly commuting of type* (A_F)*;*
5. *R-weakly commuting of type* (A_g)*;*
6. *R-weakly commuting of type (P).*

Proof. On using Remark 2.6, the proof follows immediately. □

4. Applications in metric spaces

In this section, we first give the metrical version of the definitions of variants of weakly commuting and compatible mappings, which were respectively given by Jain et al. [18] and Sumitra and Masmali [7] in the setup of fuzzy metric spaces. Then, as application of the results proved in the earlier sections of this paper, we obtain some common fixed point results in the framework of metric spaces.

Let (X, d) be a metric space, then we define the following notions in X:

Definition 4.1. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be weakly commuting if*

$$\begin{aligned} &d(F(gx, gy), gF(x, y)) \leq d(F(x, y), gx) \text{ and} \\ &d(F(gy, gx), gF(y, x)) \leq d(F(y, x), gy), \quad \text{for all } x, y \text{ in } X. \end{aligned}$$

Definition 4.2. *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be*

1. *R-weakly commuting if there exists some $R > 0$ such that*

$$d(F(gx, gy), gF(x, y)) \leq Rd(F(x, y), gx) \text{ and}$$
$$d(F(gy, gx), gF(y, x)) \leq Rd(F(y, x), gy), \quad \text{for all } x, y \text{ in } X.$$

2. *R-weakly commuting maps of type (A_F) if there exists some $R > 0$ such that*

$$d(F(gx, gy), ggx) \leq Rd(F(x, y), gx) \text{ and}$$
$$d(F(gy, gx), ggy) \leq Rd(F(y, x), gy), \quad \text{for all } x, y \text{ in } X.$$

3. *R-weakly commuting maps of type (A_g) if there exists some $R > 0$ such that*

$$d(gF(x, y), F(F(x, y), F(y, x))) \leq Rd(F(x, y), gx) \text{ and}$$
$$d(gF(y, x), F(F(y, x), F(x, y))) \leq Rd(F(y, x), gy) \quad \text{for all } x, y \text{ in } X.$$

4. *R-weakly commuting maps of type (P) if there exists some $R > 0$ such that*

$$d(F(F(x, y), F(y, x)), ggx) \leq Rd(F(x, y), gx) \text{ and}$$
$$d(F(F(y, x), F(x, y)), ggy) \leq Rd(F(y, x), gy) \quad \text{for all } x, y \text{ in } X.$$

Definition 4.3. *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible of type (A) if*

$$\lim_{n\to\infty} d(F(gx_n, gy_n), g^2x_n) = 0, \quad \lim_{n\to\infty} d(F(gy_n, gx_n), g^2y_n) = 0$$

and

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n))) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Definition 4.4. *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible of type (B) if*

$$\lim_{n\to\infty} d(F(gx_n, gy_n), g^2x_n)$$
$$\leq \frac{1}{2}\{\lim_{n\to\infty} d(F(gx_n, gy_n), F(x, y)) + \lim_{n\to\infty} d(F(x, y), F(F(x_n, y_n), F(y_n, x_n)))\},$$
$$\lim_{n\to\infty} d(F(gy_n, gx_n), g^2y_n)$$
$$\leq \frac{1}{2}\{\lim_{n\to\infty} d(F(gy_n, gx_n), F(y, x)) + \lim_{n\to\infty} d(F(y, x), F(F(y_n, x_n), F(x_n, y_n)))\}$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} d(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n))) \\
&\quad \le \frac{1}{2}\{\lim_{n\to\infty} d(gF(x_n, y_n), gx) + \lim_{n\to\infty} d(gx, g^2x_n)\}, \\
&\lim_{n\to\infty} d(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t) \\
&\quad \le \frac{1}{2}\{\lim_{n\to\infty} d(gF(y_n, x_n), gy) + \lim_{n\to\infty} d(gy, g^2y_n)\}
\end{aligned}
$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$.

Definition 4.5. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type (P) if*

$$
\begin{aligned}
&\lim_{n\to\infty} d(F(F(x_n, y_n), F(y_n, x_n)), g^2x_n) = 0, \\
&\lim_{n\to\infty} d(F(F(y_n, x_n), F(x_n, y_n)), g^2y_n) = 0
\end{aligned}
$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$.

Definition 4.6. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type (C) if*

$$
\begin{aligned}
&\lim_{n\to\infty} d(F(gx_n, gy_n), g^2x_n) \\
&\quad \le \frac{1}{3}\Big\{\lim_{n\to\infty} d(F(gx_n, gy_n), F(x, y)) + \lim_{n\to\infty} d(F(x, y), g^2x_n) \\
&\qquad + \lim_{n\to\infty} d(F(x, y), F(F(x_n, y_n), F(y_n, x_n)))\Big\}, \\
&\lim_{n\to\infty} d(F(gy_n, gx_n), g^2y_n) \\
&\quad \le \frac{1}{3}\Big\{\lim_{n\to\infty} d(F(gy_n, gx_n), F(y, x)) + \lim_{n\to\infty} d(F(y, x), g^2y_n) \\
&\qquad + \lim_{n\to\infty} d(F(y, x), F(F(y_n, x_n), F(x_n, y_n)))\Big\}
\end{aligned}
$$

and

$$\begin{aligned}
&\lim_{n\to\infty} d(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n))) \\
&\quad \le \frac{1}{3}\Big\{\lim_{n\to\infty} d(gF(x_n, y_n), gx) + \lim_{n\to\infty} d(gx, F(F(x_n, y_n), F(y_n, x_n))) \\
&\qquad + \lim_{n\to\infty} d(gx, g^2x_n)\Big\}, \\
&\lim_{n\to\infty} d(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n))) \\
&\quad \le \frac{1}{3}\{\lim_{n\to\infty} d(gF(y_n, x_n), gy) + \lim_{n\to\infty} d(gy, F(F(y_n, x_n), F(x_n, y_n))) \\
&\qquad + \lim_{n\to\infty} d(gy, g^2y_n)\Big\}
\end{aligned}$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$.

Definition 4.7. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type* (A_F) *if*

$$\lim_{n\to\infty} d(F(gx_n, gy_n), ggx_n) = 0, \quad \lim_{n\to\infty} d(F(gy_n, gx_n), ggy_n) = 0$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$.

Definition 4.8. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are said to be compatible of type* (A_g) *if*

$$\begin{aligned}
&\lim_{n\to\infty} d(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n))) = 1, \\
&\lim_{n\to\infty} d(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n))) = 1
\end{aligned}$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ *for some* $x, y \in X$.

Remark 4.9. *Interestingly, the comparison and relation between various mappings in the setup of fuzzy metric spaces established earlier in the Section 2 of the present manuscript also holds among the metrical versions of those mappings.*

Theorem 4.10. *Let* (X, d) *be a metric space and suppose that* $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ *be four mappings satisfying the condition that there exists some* $k \in (0, 1)$ *such that*

1. $\max\{d(A(x, y), B(u, v)), d(A(y, x), B(v, u))\} \le \frac{k}{2}[d(Sx, Tu) + d(Sy, Tv)]$,

for all $x, y, u, v \in X$. *Also, suppose that* $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$, *the pairs* (A, S) *and* (B, T) *are weakly compatible, one of the subspaces* $A(X \times X)$ *or* $T(X)$ *and one of* $B(X \times X)$ *or* $S(X)$ *are complete. Then there exists a unique point* α *in* X *such that* $A(\alpha, \alpha) = S(\alpha) = \alpha = T(\alpha) = B(\alpha, \alpha)$.

Proof. For all $x, y \in X$ and $t > 0$, define $M(x,y,t) = \frac{t}{t+d(x,y)}$ and $a*b = \min\{a,b\}$. Then, $(X, M, *)$ is a fuzzy metric space and $*$ being the Hadžić type t-norm. Further, it is easy to see that $M(x,y,t) = \frac{t}{t+d(x,y)} \to 1$ as $t \to \infty$, for all $x, y \in X$. We next show that the inequality (4.1) implies (3.10). If otherwise, from (3.10), for some $t > 0$ and $x, y, u, v \in X$, we have

$$\min\left\{\frac{t}{t+\frac{1}{k}d(A(x,y),B(u,v))}, \frac{t}{t+\frac{1}{k}d(A(y,x),B(v,u))}\right\} < \min\left\{\frac{t}{t+d(Sx,Tu)}, \frac{t}{t+d(Sy,Tv)}\right\},$$

then, we have
either

1. $\dfrac{t}{t+\frac{1}{k}d(A(x,y),B(u,v))} < \min\left\{\dfrac{t}{t+d(Sx,Tu)}, \frac{t}{t+d(Sy,Tv)}\right\}$,

or,

1. $\dfrac{t}{t+\frac{1}{k}d(A(y,x),B(v,u)} < \min\left\{\dfrac{t}{t+d(Sx,Tu)}, \frac{t}{t+d(Sy,Tv)}\right\}$.

From (4.2), we obtain that

4. $t + \dfrac{1}{k}d(A(x,y),B(u,v)) > t + d(Sx,Tu)$,

5. $t + \dfrac{1}{k}d(A(x,y),B(u,v)) > t + d(Sy,Tv)$.

Combining (4.4) and (4.5), we obtain that

1. $d(A(x,y),B(u,v)) > \frac{k}{2}[d(Sx,Tu) + d(Sy,Tv)]$.

Similarly, from (4.3), we obtain that

1. $d(A(y,x),B(v,u)) > \frac{k}{2}[d(Sx,Tu) + d(Sy,Tv)]$.

Using (4.6) and (4.7), we have that

$$\max\{d(A(x,y),B(u,v),d(A(y,x),B(v,u)))\} > \frac{k}{2}[d(Sx,Tu) + d(Sy,Tv)],$$

which is a contradiction to (4.1). Then, the result holds immediately by applying Theorem 3.3. □

Theorem 4.11. *Let (X,d) be a metric space and suppose that $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be the four mappings satisfying the condition that there exists some $k \in (0,1)$ such that*

1. $d(A(x,y),B(u,v)) + d(A(y,x),B(v,u)) \leq k[d(Sx,Tu) + d(Sy,Tv)]$,

for all $x, y, u, v \in X$. Also, suppose that $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$, the pairs (A, S) and (B, T) are weakly compatible, one of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete. Then there exists a unique point α in X such that $A(\alpha, \alpha) = S(\alpha) = \alpha = T(\alpha) = B(\alpha, \alpha)$.

Proof. We know that $\frac{a+b}{2} \leq \max\{a, b\}$ for $a, b \in R^+$, thus, the proof follows immediately by applying Theorem 4.1. □

Taking into account the relation between the weakly compatible mappings and variants of compatible mappings, the following result holds immediately:

Theorem 4.12. *Theorem 4.1 (and Theorem 4.2) remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypotheses) of the following properties:*

1. *compatibility;*
2. *compatibility of type (A);*
3. *compatibility of type (P);*
4. *compatibility of type (B);*
5. *compatibility of type (C);*
6. *compatibility of type (A_F);*
7. *compatibility of type (A_g).*

In the similar way, the following result holds immediately:

Theorem 4.13. *Theorem 4.1 (and Theorem 4.2) remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypotheses) of the following properties:*

1. *commuting;*
2. *weakly commuting;*
3. *R-weakly commuting;*
4. *R-weakly commuting of type (A_F);*
5. *R-weakly commuting of type (A_g);*
6. *R-weakly commuting of type (P).*

5. Conclusion

In coupled fixed point theory of fuzzy metric spaces, we have discussed the relation among various variants of weakly commuting mappings and also among the variants of compatible mappings. The obtained main result for two pair of weakly compatible mappings has been extended for the variants of weakly commuting and compatible mappings. The corresponding notions of these variants have been discussed in the coupled fixed point theory of metric spaces. As application of the results proved in the setup of fuzzy metric spaces, the analogous results have been established in metric spaces. Further, due to the assumption of the new contraction condition, the proof of the main result of this paper is quite shorter and simpler than the proof of the results already present in the literature.

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