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n-absorbing and Strongly n-absorbing Second Submodules

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ABSTRACT: In this paper, we introduce the concepts of n-absorbing and strongly n-absorbing second submodules as a dual notion of n-absorbing submodules of modules over a commutative ring and obtain some related results. In particular, we investigate some results concerning strongly 2-absorbing second submodules.

Key Words: Strongly n-absorbing second submodule, n-absorbing second submodule, Weakly strongly 2-absorbing second submodule.

Contents

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Let N be a submodule of an R-module M. For $r \in R$, $(N :_M r)$ will denote $(N :_M r) = \{m \in M : rm \in N\}$. Clearly, $(N :_M r)$ is a submodule of M containing N.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [17]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [26]. In this case $Ann_R(S)$ is a prime ideal of R. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [19].

The concept of 2-absorbing ideals was introduced in [11] and then extended to n-absorbing ideals in [1]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Let I be a proper ideal of R and n a positive integer. I is called an *n-absorbing ideal* of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose their product is in I.

The authors in [15] and [24], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if

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whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. A proper submodule N of M is said to be a *weakly 2-absorbing submodule* of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ [15]. A proper submodule N of M is called *n-absorbing submodule* of M if whenever $a_1 ... a_n m \in N$ for $a_1, ..., a_n \in R$ and $m \in M$, then either $a_1 ... a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose their product with m is in N [16]. Several authors investigated properties of 2-absorbing, and some generalization of 2-absorbing submodules, for example [15,16,22,23,24,25].

In [2], the authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of M and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of M* if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of M* if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$. Also, in [3,4], the authors introduced and investigated some generalization of 2-absorbing second and strongly 2-absorbing second submodules.

The purpose of this paper is to introduce the concepts of n-absorbing and strongly n-absorbing second submodules as dual notion of n-absorbing submodules of modules and provide some information concerning these new classes of modules. Furthermore, we study some properties of strongly 2-absorbing second submodules of an R-module M. Also, we introduce the concept of weakly strongly 2-absorbing second submodules of M as a dual notion of weakly 2-absorbing submodules and obtain some related results.

2. n-absorbing and strongly n-absorbing second submodules

Definition 2.1. *Let N be a non-zero submodule of an R-module M and n be a positive integer. We say that N is an* n-absorbing second submodule *of M if whenever $a_1 ... a_n N \subseteq L$ for $a_1, ..., a_n \in R$ and a completely irreducible submodule L of M, then either $a_1 ... a_n \in Ann_R(N)$ or there are $n-1$ of a_i's whose their product with N is a subset of L.*

Remark 2.2. *Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [9, 2.1].*

We recall that an R-module M is said to be a *cocyclic module* if $Soc_R(M)$ is a large and simple submodule of M [27]. (Here $Soc_R(M)$ denotes the sum of all minimal submodules of M.). A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [19, 12.1.1].

Proposition 2.3. *Let N be an n-absorbing second submodule of an R-module M. Then we have the following.*

(a) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $(L :_R N)$ is an n-absorbing ideal of R.

(b) If M is a cocyclic module, then $Ann_R(N)$ is an n-absorbing ideal of R.

(c) If $a \in R$, then $a^n N = a^{n+1} N$.

Proof. (a) Since $N \not\subseteq L$, we have $(L :_R N) \neq R$. Let $a_1, a_2, ..., a_n, a_{n+1} \in R$ and $a_1 a_2 ... a_{n+1} \in (L :_R N)$. Then $a_1 a_2 ... a_n N \subseteq (L :_M a_{n+1})$. Thus there are $n-1$ of a_i's whose their product with N is a subset of $(L :_M a_{n+1})$, where $1 \leq i \leq n$ or $a_1 a_2 ... a_n N = 0$ because by [10, 2.1], $(L :_M a_{n+1})$ is a completely irreducible submodule of M. Therefore, there are n of a_i's whose their product lies in $(L :_R N)$ for some $1 \leq i \leq n+1$ or $a_1 ... a_n \in (L :_R N)$ as needed.

(b) Since M is cocyclic, the zero submodule of M is a completely irreducible submodule of M. Thus the result follows from part (a).

(c) It is clear that $a^{n+1} N \subseteq a^n N$. Let L be a completely irreducible submodule of M such that $a^{n+1} N \subseteq L$. Then $a^n N \subseteq (L :_M a)$. Since N is n-absorbing second submodule and $(L :_M a)$ is a completely irreducible submodule of M by [10, 2.1], $a^{n-1} N \subseteq (L :_M a)$ or $a^n N = 0$. Therefore, $a^n N \subseteq L$. This implies that $a^n N \subseteq a^{n+1} N$ by Remark 2.2. □

Definition 2.4. *Let N be a non-zero submodule of an R-module M and n be a positive integer. We say that N is a* strongly n-absorbing second submodule *of M if whenever $a_1 ... a_n N \subseteq K$ for $a_1, ..., a_n \in R$ and a submodule K of M, then either $a_1 ... a_n \in Ann_R(N)$ or there are $n-1$ of a_i's whose their product with N is a subset of K.*

Clearly every strongly n-absorbing second submodule is an n-absorbing second submodule. It is natural to ask the following question:

Question 2.5. *Let M be an R-module. Is every n-absorbing second submodule of M a strongly n-absorbing second submodule of M?*

Note 1. *Let $a_1, a_2, ..., a_n \in R$. We denote by $\widehat{a_i}$ the element $a_1 ... a_{i-1} a_{i+1} ... a_n$. In this case, the definition of an n-absorbing (resp. a strongly n-absorbing) second submodule can be reformulated as: a non-zero submodule N of an R-module M is called* n-absorbing *(resp.* strongly n-absorbing*)* second *if whenever $a_1, ..., a_n \in R$ and L is a completely irreducible submodule (resp. K is a submodule) of M with $a_1 ... a_n N \subseteq L$ (resp. $a_1 ... a_n N \subseteq K$), then either $a_1 ... a_n \in Ann_R(N)$ or $\widehat{a_i} N \subseteq L$ (resp. $\widehat{a_i} N \subseteq K$) for some $1 \leq i \leq n$.*

Proposition 2.6. *Let M be an R-module and let $\{K_\lambda\}_{\lambda \in \Lambda}$ be a chain of n-absorbing second submodules of M. Then $\cup_{\lambda \in \Lambda} K_\lambda$ is an n-absorbing second submodule of M.*

Proof. Let $a_1, ..., a_n \in R$, L be a completely irreducible submodule of M, and $a_1 ... a_n (\cup_{\lambda \in \Lambda} K_\lambda) \subseteq L$. Assume that $\widehat{a_i} (\cup_{\lambda \in \Lambda} K_\lambda) \not\subseteq L$. Then for each $1 \leq i \leq n$ there is $\beta_i \in \Lambda$, where $\widehat{a_i} K_{\beta_i} \not\subseteq L$. Hence, for every $K_{\beta_i} \subseteq K_{\alpha_i}$ we have $\widehat{a_i} K_{\alpha_i} \not\subseteq L$. Therefore, for each submodule K_α such that $K_{\beta_i} \subseteq K_\alpha$ (for each $1 \leq i \leq n$), we have $\widehat{a_i} K_\alpha \not\subseteq L$ for each $1 \leq i \leq n$. Thus $a_1 ... a_n K_\alpha = 0$ as K_α is an n-absorbing second submodules of M. Let K_α be a submodule of M such that $K_{\beta_i} \subseteq K_\alpha$ for each $1 \le i \le n$. As $\{K_\lambda\}_{\lambda\in\Lambda}$ is a chain, we have

$$\cup_{\lambda\in\Lambda} K_\lambda = (\cup_{K_\lambda \subseteq K_\alpha} K_\lambda) \cup (\cup_{K_\alpha \subset K_\lambda} K_\lambda) = K_\alpha \cup (\cup_{K_\alpha \subset K_\lambda} K_\lambda).$$

Therefore $a_1...a_n(\cup_{\lambda\in\Lambda} K_\lambda) = 0$, as needed. □

Definition 2.7. *We say that an n-absorbing second submodule N of an R-module M is a* maximal n-absorbing second submodule *of a submodule K of M, if $N \subseteq K$ and there does not exist an n-absorbing second submodule H of M such that $N \subset H \subset K$.*

Lemma 2.8. *Let M be an R-module. Then every n-absorbing second submodule of M is contained in a maximal n-absorbing second submodule of M.*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.6. □

Theorem 2.9. *Every Artinian R-module M has only a finite number of maximal n-absorbing second submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of non-zero submodules N of M such that N has an infinite number of maximal n-absorbing second submodules. The collection Σ is non-empty because $M \in \Sigma$ and hence has a minimal member, S say. Then S is not n-absorbing second submodule. Thus there exist $a_1, ..., a_n \in R$, and a completely irreducible submodule L of M such that $a_1...a_n S \subseteq L$ but $\widehat{a_i} S \not\subseteq L$ (for each $1 \le i \le n$) and $a_1...a_n S \neq 0$. Let V be a maximal n-absorbing second submodule of M contained in S. Then $\widehat{a_i} V \subseteq L$ for some $1 \le i \le n$ or $a_1...a_n V = 0$. Thus $V \subseteq (L :_M \widehat{a_i})$ or $V \subseteq (0 :_M a_1...a_n)$. Therefore, $V \subseteq (L :_S \widehat{a_i})$ or $V \subseteq (0 :_S a_1...a_n)$. Hence every maximal n-absorbing second submodule of S is a maximal n-absorbing second submodule of $(L :_S \widehat{a_i})$ or $(0 :_S a_1...a_n)$. By the choice of S, the modules $(L :_S \widehat{a_i})$ and $(0 :_S a_1...a_n)$ have only finitely many maximal n-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S which is a contradiction. □

Definition 2.10. *We say that a strongly n-absorbing second submodule N of an R-module M is a* maximal strongly n-absorbing second submodule *of a submodule K of M, if $N \subseteq K$ and there does not exist a strongly n-absorbing second submodule H of M such that $N \subset H \subset K$.*

Remark 2.11. *One can check that, by using the same technique, that the results in Proposition 2.6, Lemma 2.8, and Theorem 2.9 about n-absorbing second submodules is also true for strongly n-absorbing second submodules.*

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [5].

A proper ideal I is a *strongly n-absorbing ideal* of R if whenever $I_1...I_{n+1} \subseteq I$ for ideals I_1,...,I_{n+1} of R then there are n of the I_i's whose their product is in I [1]. Clearly a strongly n-absorbing ideal of R is also an n-absorbing ideal of R. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n-absorbing if and only if I is an n-absorbing ideal of R [1, Corollary 6.9].

Theorem 2.12. *Let N be a submodule of an R-module M. Then we have the following.*

(a) If N is a strongly n-absorbing second submodule of M, then $Ann_R(N)$ is an n-absorbing ideal of R.

(b) If M is a comultiplication R-module and $Ann_R(N)$ is a strongly n-absorbing ideal of R, then N is a strongly n-absorbing second submodule of M.

Proof. (a) Let N be a strongly n-absorbing second submodule of M. Assume that $a_1, ..., a_{n+1} \in R$ with $a_1...a_{n+1} \in Ann_R(N)$. For each $1 \leq i \leq n$, let $\widehat{a_i}$ be the element of R which is obtained by eliminating a_i from $a_1...a_n$. Then $a_1...a_nN \subseteq a_1...a_nN$ implies that $\widehat{a_i}N \subseteq a_1...a_nN$ for some $1 \leq i \leq n$ because N is strongly n-absorbing second. Thus $\widehat{a_i}a_{n+1}N = 0$ that is, $Ann_R(N)$ is n-absorbing.

(b) Assume that $Ann_R(N)$ is a strongly n-absorbing ideal of R. Let $a_1, ..., a_n \in R$ and K be a submodule of M such that $a_1...a_nN \subseteq K$ and $a_1...a_nN \neq 0$. Then $a_1...a_nAnn_R(K)N = 0$. Now as $Ann_R(N)$ is a strongly n-absorbing ideal of R, $\widehat{a_i}Ann_R(K) \subseteq Ann_R(N)$ since $a_1...a_n \notin Ann_R(N)$. Thus $Ann_R(K) \subseteq Ann_R(\widehat{a_i}N)$. It follows that $\widehat{a_i}N \subseteq K$ since M is a comultiplication R-module that is, N is strongly n-absorbing second submodule of M. □

Theorem 2.13. *Let N be a strongly n-absorbing second submodule of an R-module M. Then rN is a strongly n-absorbing second submodule of M for all $r \in R \setminus Ann_R(N)$.*

Proof. Let $a_1...a_nrN \subseteq K$ for some $a_1, ..., a_n \in R$ and a submodule K of M. Then $a_1a_2...a_nN \subseteq (K :_M r)$. So either $a_1...a_n \in Ann_R(N)$ or there are $n-1$ of a_i's whose their product with N is a subset of $(K :_M r)$. If $a_1...a_n \in Ann_R(N)$, since $Ann_R(N) \subseteq Ann_R(rN)$ we are done. In other case, there are $n-1$ of a_i's whose their product with N is a subset of $(K :_M r)$ implies that there is a product of $n-1$ of the a_i's with rN is a subset of K. Thus rN is a strongly n-absorbing second submodule of M. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [12].

Corollary 2.14. *Let R be a principal ideal domain and M be a multiplication strongly n-absorbing second R-module. Then every submodule of M is a strongly n-absorbing second submodule of M.*

Proof. This follows from Theorem 2.13. □

Proposition 2.15. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a strongly n-absorbing second submodule of M, then $f(N)$ is a strongly n-absorbing second submodule of $\acute{M}$.*

(b) *If $\acute{N}$ is a strongly n-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a strongly n-absorbing second submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a_1, a_2, ..., a_n \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $a_1a_2...a_nf(N) \subseteq \acute{K}$. Then $a_1a_2...a_nN \subseteq f^{-1}(\acute{K})$. As N is strongly n-absorbing second submodule, $\widehat{a_i}N \subseteq f^{-1}(\acute{K})$ for some $1 \leq i \leq n$ or $a_1a_2...a_nN = 0$. Therefore,

$$\widehat{a_i}f(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $a_1a_2...a_nf(N) = 0$, as needed.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = f(f^{-1}(\acute{N})) = f(0) = 0$. By assumption, $\acute{N} \subseteq f(M)$. Therefore $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a_1, a_2, ..., a_n \in R$, K be a submodule of M, and $a_1a_2...a_nf^{-1}(\acute{N}) \subseteq K$. Then

$$a_1a_2...a_n\acute{N} = a_1a_2...a_n(f(M) \cap \acute{N}) = a_1a_2...a_nf(f^{-1}(\acute{N})) \subseteq f(K).$$

Thus as $\acute{N}$ is strongly n-absorbing second submodule, $\widehat{a_i}\acute{N} \subseteq f(K)$ for some $1 \leq i \leq n$ or $a_1a_2...a_n\acute{N} = 0$. Therefore, $\widehat{a_i}f^{-1}(\acute{N}) \subseteq f^{-1}(f(K)) = K$ or $a_1a_2...a_nf^{-1}(\acute{N}) = 0$, as desired. □

Theorem 2.16. *Let M be an R-module. If N_i is a strongly n_i-absorbing second submodule of M for each $1 \leq i \leq k$, then $N_1 + \ldots + N_k$ is a strongly n-absorbing second submodule of M for $n = n_1 + \ldots + n_k$. In particular, if $N_1, \ldots, N_n$ are second submodules of M, then $N_1 + \ldots + N_n$ is a strongly n-absorbing second submodule of M.*

Proof. Let $a_1, ..., a_n \in R$ and K be a submodule of M with $a_1...a_n(N_1 + ... + N_k) \subseteq K$ such that $\widehat{a_i}(N_1 + ... + N_k) \not\subseteq K$ for each $1 \leq i \leq n$. As $a_1...a_n(N_1 + ... + N_k) \subseteq K$, we have $a_1...a_nN_j \subseteq K$ for every $1 \leq j \leq k$. Therefore, $a_1....a_n \in Ann_R(N_j)$ for every $1 \leq j \leq k$ since N_j is a strongly n_j-absorbing second submodule of M and $n_j \leq n$. Therefore $a_1 + ... + a_n \in Ann_R(N_1) \cap ... \cap Ann_R(N_k) = Ann_R(N_1 + ... + N_k)$, that is, $N_1 + ... + N_k$ is strongly n-absorbing second. The "In particular" statement follows from the fact that every second submodule is a strongly n-absorbing second submodule. □

Let N be a non-zero submodule of an R-module M. It is clear that if N is an n-absorbing (resp. a strongly n-absorbing) second submodule, then it is an m-absorbing (resp. a strongly m-absorbing) second submodule of M for every integer

$m \geq n$. If N is a strongly n-absorbing second submodule of M for some positive integer n, then $\omega^c_M(N) = min\{n : N\ is\ an\ n-absorbing\ second\ submodule of\ M\}$; otherwise, set $\omega^c_M(N) = \infty$. Moreover, we define $\omega^c_M(0) = 0$. Therefore, for any submodule N of M, we have $\omega^c_M(N) \in \mathbb{N} \cup \{0, \infty\}$, with $\omega^c_M(N) = 1$ if and only if N is a second submodule of M and $\omega^c_M(N) = 0$ if and only if $N = 0$. Then $\omega^c_M(N)$ measures, in some sense, how far N is from being a second submodule of M. One can ask how $\omega^c_M(N)$ and $\omega^c_R(Ann_R(N))$ compare.

Corollary 2.17. *Let M be an R-module. Then we have the following.*

(a) If $N_1, ..., N_k$ are strongly n-absorbing second submodules of M, then $\omega^c_M(N_1 + ... + N_k) \leq \omega^c_M(N_1) + ... + \omega^c_M(N_k)$.

(b) If $N_1, ..., N_n$ are second submodules of M, then $\omega^c_M(N_1 + ... + N_n) \leq n$.

Proof. This follows from Theorem 2.16. □

Theorem 2.18. *Let M be an R-module and N be a P-secondary submodule of M such that $P^n \subseteq Ann_R(N)$. Then N is a strongly n-absorbing second submodule of M. Moreover, $\omega^c_M(N) \leq n$. In particular, if $(0 :_M P^n)$ is a P-secondary submodule of M, then $(0 :_M P^n)$ is a strongly n-absorbing second submodule of M. Moreover, $\omega^c_M((0 :_M P^n)) \leq n$.*

Proof. Assume that $a_1, ..., a_n \in R$ and K be a submodule of M with $a_1...a_nN \subseteq K$ such that $\widehat{a_i}N \not\subseteq K$ for each $1 \leq i \leq n$. For every $1 \leq i \leq n$, as $\widehat{a_i}a_iN \subseteq K$ with $\widehat{a_i}N \not\subseteq K$ and N is a P-secondary submodule of M, we have $a_i \in P$. Consequently, $a_1...a_n \in P^n \subseteq Ann_R(N)$, that is, N is a strongly n-absorbing second submodule of M. The "In particular" statement follows from the fact that $P^n \subseteq Ann_R((0 :_M P^n))$. □

Theorem 2.19. *Let R be a Noetherian ring and let M be a finitely cogenerated R-module. Then every non-zero proper submodule of M is a strongly n-absorbing second submodule of M for some positive integer n.*

Proof. Let N be a P-secondary submodule of M. So $Ann_R(N)$ is a P-primary ideal of R. Since R is a Noetherian ring, there exists a positive integer m for which $P^m \subseteq Ann_R(N)$. Thus N is a strongly m-absorbing second submodule of M by Theorem 2.18. Now assume that K is a non-zero submodule of M. Since M is an Artinian R-module, K has a secondary representation by [20, 6.11]. Let $K = N_1 + ... + N_k$ be a secondary representation of K, where each N_i is a P_i-secondary submodule of M for any $1 \leq i \leq n$. By the first part, each N_i $(1 \leq i \leq n)$ is a strongly m_i-absorbing second submodule of M for some positive integer m_i. Now K is a strongly n-absorbing second submodule in which $n = m_1 + ... + m_k$ by Theorem 2.16. Therefore the result follows. □

Theorem 2.20. *Let N be a strongly n-absorbing second submodule of an R-module M with $n \geq 2$ and $Ann_R(N) \subset \sqrt{Ann_R(N)}$. Suppose that $r \in \sqrt{Ann_R(N)} \setminus Ann_R(N)$ and let $t(\geq 2)$ be the least positive integer such that $r^t \in Ann_R(N)$. Then $r^{t-1}N$ is a strongly $(n-t+1)$-absorbing second submodule of M.*

Proof. Choose $2 \leq t \leq n$. Then $n-t+1 \geq 1$. Let $a_1...a_{n-t+1}r^{t-1}N \subseteq K$ for some $a_1, ..., a_{n-t+1} \in R$ and a submodule K of M. Since $r^{t-1}a_1...a_{n-t+1}N \subseteq K$ and N is a strongly n-absorbing second submodule of M, therefore either $r^{t-1}\widehat{a_i}N \subseteq K$ or $r^{t-2}a_1...a_{n-t+1}N \subseteq K$ or $a_1...a_{n-t+1} \in Ann_R(r^{n-1}N)$. If $r^{t-1}\widehat{a_i}N \subseteq K$ or $a_1...a_{n-t+1} \in Ann_R(r^{n-1}N)$, then we are done. Hence assume that $r^{t-1}\widehat{a_i}N \not\subseteq K$ and $a_1...a_{n-t+1} \notin Ann_R(r^{n-1}N)$. Since N is a strongly n-absorbing second submodule of M, therefore $r^{t-2}a_1...a_{n-t+1}N \subseteq K$. Now $r^t \in Ann_R(N)$ and $r^{t-1}a_1...a_{n-t+1}N \subseteq K$ imply $rr^{t-2}a_1...a_{n-t}(a_{n-t+1}+r)N \subseteq K$. Again, since N is a strongly n-absorbing second and $r^{t-1}\widehat{a_i}N \not\subseteq K$ for any $1 \leq i \leq (n-t)$ and $rr^{t-2}a_1...a_{n-t}(a_{n-t+1}+r) \notin Ann_R(N)$ (as $r^t \in Ann_R(N)$), we must have $r^{t-2}a_1...a_{n-t}(a_{n-t+1}+r)N = r^{t-2}a_1...a_{n-t+1}N + r^{t-1}a_1...a_{n-t}N \subseteq K$. As $r^{t-2}a_1...a_{n-t+1}N \subseteq K$, we have $r^{t-1}a_1...a_{n-t}N \subseteq K$, a contradiction, since we assumed that the product of r^{t-1} with any $n-t$ of the a_i's with N is not a subset of K. Thus $r^{t-1}\widehat{a_i}N \subseteq K$ or $a_1...a_{n-t+1} \in Ann_R(r^{t-1}N)$, and hence $r^{t-1}N$ is a strongly $(n-t+1)$-absorbing second submodule of M. □

Remark 2.21. *One can see, by using the same technique, that the results in Theorems 2.16, 2.13, and Corollary 2.14 about strongly n-absorbing second submodules in this section is also true for n-absorbing second submodules.*

3. Strongly and weakly strongly 2-absorbing second submodules

Recall that an R-module M is said to be *sum-irreducible* precisely when it is nonzero and cannot be expressed as the sum of two proper submodules of itself [13, Definition and Exercise 7.2.8].

Theorem 3.1. *Let N be a strongly 2-absorbing second submodule of an R-module M. Then $aN = a^2N$ for all $a \in R \setminus \sqrt{Ann_R(N)}$. The converse holds, if N is a sum-irreducible submodule of M.*

Proof. Let $a \in R \setminus \sqrt{Ann_R(N)}$. Then $a^2 \in R \setminus Ann_R(N)$. Thus $aN = a^2N$ by [2, 3.3]. Conversely, let N be a sum-irreducible submodule of M and $abN \subseteq K$ for some $a, b \in R$ and a submodule K of M. Assume that, $ab \in R \setminus \sqrt{Ann_R(N)}$. We show that $aN \subseteq K$ or $bN \subseteq K$. As $ab \in R \setminus \sqrt{Ann_R(N)}$, we have $a, b \in R \setminus \sqrt{Ann_R(N)}$. Thus $aN = a^2N$ by assumption. Let $x \in N$. Then $ax \in aN = a^2N$. Hence $ax = a^2y$ for some $y \in N$. This implies that $x - ay \in (0 :_N a) \subseteq (K :_N a)$. Thus $x = x - ay + ay \in (K :_N a) + (K :_N b)$. Therefore, $N \subseteq (K :_N a) + (K :_N b)$. Clearly, $(K :_N a) + (K :_N b) \subseteq N$. Thus as N is sum-irreducible, $(K :_N a) = N$ or $(K :_N b) = N$ as needed. □

Proposition 3.2. *Let N be a submodule of an R-module M. Then we have the following.*

(a) *If $(0:_M Ann_R(N)^3)$ is a strongly 2-absorbing second submodule of M, then $(0:_M Ann_R(N)^2) = (0:_M Ann_R(N)^3)$.*

(b) *If K is a strongly 2-absorbing second submodule of M such that $K \not\subseteq N$, then $(K+N)/N$ is a strongly 2-absorbing second of M/N and $\sqrt{(N:_R K+N)} \setminus (N:_R K) = \sqrt{Ann_R(K)} \setminus (N:_R K)$.*

Proof. (a) Clearly, $(0:_M Ann_R(N)^2) \subseteq (0:_M Ann_R(N)^3)$. As $(0:_M Ann_R(N)^3)$ is a strongly 2-absorbing second submodule of M and $Ann_R(N)^2(0:_M Ann_R(N)^3) \subseteq (0:_M Ann_R(N))$, we have $Ann_R(N)(0:_M Ann_R(N)^3) \subseteq (0:_M Ann_R(N))$ or $Ann_R(N)^2(0:_M Ann_R(N)^3) = 0$. So in any case, $Ann_R(N)^2(0:_M Ann_R(N)^3) = 0$. This implies that $(0:_M Ann_R(N)^3) \subseteq (0:_M Ann_R(N)^2)$.

(b) As $K \not\subseteq N$, we have $(K+N)/N \neq 0$. Let $ab((K+N)/N) \subseteq H/N$ for some $a, b \in R$ and a submodule H/N of M/N. Then $ab(K+N)+N \subseteq H$. This implies that $abK \subseteq H$. Now as K is a strongly 2-absorbing second submodule of M, we have either $aK \subseteq H$ or $bK \subseteq H$ or $abK = 0$. Therefore, either $a((K+N)/N) \subseteq H/N$ or $b((K+N)/N) \subseteq H/N$ or $ab((K+N)/N) = 0$ as needed. To see the second assertion, let $a \in \sqrt{(N:_R K+N)} \setminus (N:_R K)$. Then $a^nK \subseteq N$ for some positive integer n. Now as K is a strongly 2-absorbing second submodule of M and $a \notin (N:_R K)$, we have $a \in \sqrt{Ann_R(K)}$. Hence $\sqrt{(N:_R K+N)} \setminus (N:_R K) \subseteq \sqrt{Ann_R(K)} \setminus (N:_R K)$. The reverse inclusion is clear. □

For a submodule N of an R-module M the *second radical* (or second socle) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [14], [8]).

Theorem 3.3. *Let N be a strongly 2-absorbing second submodule of an R-module M. Then we have the following.*

(a) $\sqrt{Ann_R(N)}^2 \subseteq Ann_R(N)$.

(b) *If M is a finitely generated comultiplication R-module, then $N \subseteq (0:_M Ann^2_R(sec(N)))$.*

(c) *If $\sqrt{Ann_R(N)} \neq Ann_R(N)$, then for each $a \in \sqrt{Ann_R(N)} \setminus Ann_R(N)$, aN is a second R-module with $\sqrt{Ann_R(N)} \subseteq Ann_R(aN)$. Furthermore, we have $\{Ann_R(aN)\}_{a\in\sqrt{Ann_R(N)}\setminus Ann_R(N)}$ is a chain of prime ideals of R.*

Proof. (a) By [2, 3.5], $Ann_R(N)$ is a 2-absorbing ideal of R. Thus the result follows from [11, 2.4].

(b)By [7, 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Thus

$$Ann_R(sec(N))^2 \subseteq Ann_R(N),$$

by part (a). Hence $N \subseteq (0 :_M Ann^2_R(sec(N))$.

(c) Let $a \in \sqrt{Ann_R(N)} \setminus Ann_R(N)$. Then $aN \neq 0$ and there exists a positive integer t such that $a^t N = 0$ but $a^{t-1}N \neq 0$. Now let $b \in R$ such that $abN \neq 0$. We show that $abN = aN$. As N is a strongly 2-absorbing second submodule of M, $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$. If $aN \subseteq abN$, then we are done. If $bN \subseteq abN$, then $a^{t-1}bN \subseteq a^t bN = 0$. By [2, 3.5], $Ann_R(N)$ is a 2-absorbing ideal of R. Hence $a^{t-2}bN = 0$. Continuing in this way we obtain, $abN = 0$ which is a contradiction.

By part (a), $a\sqrt{Ann_R(N)} \subseteq \sqrt{Ann_R(N)}^2 \subseteq Ann_R(N)$. Thus $\sqrt{Ann_R(N)} \subseteq (Ann_R(N) :_R a) = Ann_R(aN)$.

As $Ann_R(N)$ is a 2-absorbing ideal of R, $\{Ann_R(aN)\}_{a \in \sqrt{Ann_R(N)} \setminus Ann_R(N)}$ is a chain of prime ideals of R by [11, 2.5], which completes the proof. □

Proposition 3.4. *Let N be a P-secondary submodule of an R-module M. Then N is a strongly 2-absorbing second submodule of M if and only if $P^2 \subseteq Ann_R(N)$.*

Proof. This follows from Theorem 3.3 (a) and Theorem 2.18. □

Definition 3.5. *Let N be a non-zero submodule of an R-module M. We say that N is a* weakly strongly 2-absorbing second submodule of M *if whenever $a, b \in R$, K is a submodule of M, $abM \not\subseteq K$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.*

Example 3.6. *Let M be an R-module. Clearly every strongly 2-absorbing second submodule of M is a weakly strongly 2-absorbing second submodule of M. Also, evidently M is a weakly strongly 2-absorbing second submodule of itself. In particular, $M = \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$ is not strongly 2-absorbing second $\mathbb{Z}$-module but M is a weakly strongly 2-absorbing second $\mathbb{Z}$-submodule of M.*

Theorem 3.7. *Let N be a weakly strongly 2-absorbing second submodule of an R-module M which is not a strongly 2-absorbing second submodule. Then $Ann^2_R(N) \subseteq (N :_R M)$.*

Proof. Assume on the contrary that $Ann^2_R(N) \not\subseteq (N :_R M)$. We show that N is a strongly 2-absorbing second submodule of M. Let $a, b \in R$ and K be a submodule of M such that $abN \subseteq K$. If $abM \not\subseteq K$, then we are done because N is a weakly strongly 2-absorbing second submodule of M. Thus suppose that $abM \subseteq K$. If $abM \not\subseteq N$, then $abM \not\subseteq N \cap K$. Hence $abN \subseteq N \cap K$ implies that $aN \subseteq N \cap K \subseteq K$ or $bN \subseteq N \cap K \subseteq K$ or $abN = 0$ as needed. So let $abM \subseteq N$. If $aAnn_R(N)M \not\subseteq K$, then $a(b + Ann_R(N))M \not\subseteq K$. Thus $a(b + Ann_R(N))N \subseteq K$ implies that $aN \subseteq K$ or $bN = (b + Ann_R(N))N \subseteq K$ or $abN = a(b + Ann_R(N))N = 0$, as required. So let $aAnn_R(N)M \subseteq K$. Similarly, we can assume that $bAnn_R(N)M \subseteq K$. Since $Ann_R(N)^2 \not\subseteq (N :_R M)$, there exist $a_1, b_1 \in Ann_R(N)$ such that $a_1b_1M \not\subseteq N$. Thus there exists a completely irreducible submodule L of M such that $N \subseteq L$ and $a_1b_1M \not\subseteq L$ by Remark 2.2. If $ab_1M \not\subseteq L$, then $a(b + b_1)M \not\subseteq L \cap K$. Thus $a(b + b_1)N \subseteq L \cap K$ implies that $aN \subseteq L \cap K \subseteq K$ or $bN = (b + b_1)N \subseteq$

$L \cap K \subseteq K$ or $abN = a(b + b_1)N = 0$ as needed. So let $ab_1M \subseteq L$. Similarly, we can assume that $a_1bM \subseteq L$. Therefore, $(a + a_1)(b + b_1)M \not\subseteq L \cap K$. Hence, $(a+a_1)(b+b_1)N \subseteq L\cap K$ implies that $aN = (a+a_1)N \subseteq K$ or $bN = (b+b_1)N \subseteq K$ or $abN = (a + a_1)(b + b_1)N = 0$, as desired. □

Let M be an R-module. A submodule N of M is said to be *idempotent* (resp. *coidempotent*) if $N = (N :_R M)^2M$ (resp. $N = (0 :_M Ann_R(N)^2)$). Also, M is said to be *fully idempotent* (resp. *fully coidempotent*) if every submodule of M is idempotent (resp. coidempotent) [6].

Corollary 3.8. *Let M be a faithful R-module. Then we have the following.*

(a) If M is a fully coidempotent R-module and N is a proper submodule of M, then N is a weakly strongly 2-absorbing second submodule of M if and only if N is a strongly 2-absorbing second submodule.

(b) If M is a fully idempotent R-module and N is a non-zero submodule of M, then N is a weakly 2-absorbing submodule if and only if N is a 2-absorbing submodule.

Proof. (a) The sufficiency is clear. Conversely, assume on the contrary that $N \neq M$ is a weakly strongly 2-absorbing second submodule of M which is not a strongly 2-absorbing second submodule. Then by Theorem 3.7, $Ann^3_R(N) \subseteq Ann_R(M)$. Hence as M is faithful, $Ann^3_R(N) = 0$. Since N is a coidempotent submodule of M, this implies that $N = (0 :_M Ann_R(N)^2) = (0 :_M Ann_R(N)^3) = M$, a contradiction.

(b)The proof is similar to the part (a) by using [15, 2.5]. □

Theorem 3.9. *Let $t \in R$ and M be an R-module. Then we have the following.*

(a) If $(0 :_M t) \subseteq tM$, then $(0 :_M t)$ is a strongly 2-absorbing second submodule if and only if it is a weakly strongly 2-absorbing second submodule.

(b) If $(tM :_R M) \subseteq Ann_R(tM)$, then the submodule tM is strongly 2-absorbing second if and only if it is weakly strongly 2-absorbing second.

Proof. (a) Suppose that $(0 :_M t)$ is a weakly strongly 2-absorbing second submodule of M, $a, b \in R$, and K is a submodule of M such that $ab(0 :_M t) \subseteq K$. If $abM \not\subseteq K$, then since $(0 :_M t)$ is weakly strongly 2-absorbing second, we have $a(0 :_M t) \subseteq K$ or $b(0 :_M t) \subseteq K$ or $ba \in Ann_R((0 :_M t))$ which implies $(0 :_M t)$ is strongly 2-absorbing second. Therefore we may assume that $abM \subseteq K$. Clearly, $a(b + t)(0 :_M t) \subseteq K$. If $a(b + t)M \not\subseteq K$, then we have $(b + t)(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $a(b + t) \in Ann_R((0 :_M t))$. Since $at \in Ann_R((0 :_M t))$ therefore $b(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $ab \in Ann_R((0 :_M t))$. Now suppose that $a(b + t)M \subseteq K$. Then since $abM \subseteq K$, we have $taM \subseteq K$ and so $tM \subseteq (K :_M a)$. Now $(0 :_M t) \subseteq tM$ implies that $(0 :_M t) \subseteq (K :_M a)$. Thus $a(0 :_M t) \subseteq K$ as needed. The converse is clear.

(b) Let tM be a weakly strongly 2-absorbing second submodule of M and assume that $a, b \in R$ and K be a submodule of M with $abtM \subseteq K$. Since tM is a weakly strongly 2-absorbing second submodule, we can suppose that $abM \subseteq K$, otherwise tM is strongly 2-absorbing second. Now $abtM \subseteq tM \cap K$. If $abM \not\subseteq tM \cap K$, then as tM is a weakly strongly 2-absorbing second submodule, we are done. Now let $abM \subseteq tM \cap K$. Then $abM \subseteq tM$. Thus $(tM :_R M) \subseteq Ann_R(tM)$ implies that $ab \in Ann_R(tM)$ as requested. The converse is clear. □

Theorem 3.10. *Consider the following statements for an R-module M.*

(a) Every non-zero submodule of M is a weakly strongly 2-absorbing second submodule of M.

(b) Every proper submodule of M is a weakly 2-absorbing submodule of M.

Then $(a) \Rightarrow (b)$. Moreover, $(b) \Rightarrow (a)$ if M is faithful.

Proof. $(a) \Rightarrow (b)$. Let N be a proper submodule of M, $a, b \in R$, and $m \in M$ with $0 \neq abm \in N$. If $abM \subseteq N$, then we are done. So suppose that $abM \not\subseteq N$. Since $0 \neq abm \in Rm$, we have $Rm \neq 0$. By assumption, Rm is weakly strongly 2-absorbing second. Thus $aRm \subseteq N$ or $bRm \subseteq N$ or $abRm = 0$. Since, $abm \neq 0$, $am \in N$ or $bm \in N$ as desired.

$(b) \Rightarrow (a)$. Let $0 \neq N$ be a submodule of M, $a, b \in R$, and K be a submodule of M with $abN \subseteq K$, where $abM \not\subseteq K$. If $abN = 0$, then we are done. So suppose that $abN \neq 0$. Clearly, K is a proper submodule of M. By assumption, K is weakly 2-absorbing. Thus by [18, 3.4], $aN \subseteq K$ or $bN \subseteq K$ as needed. □

Corollary 3.11. *Let M be a non-zero R-module such that every non-zero submodule of M is weakly strongly 2-absorbing second. Then R has at most three maximal ideals containing $Ann(M)$.*

Proof. This follows from [21, 6.1] and Theorem 3.10 $(a) \Rightarrow (b)$. □

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References

1. D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, **39**, 1646-1672, (2011).

2. H. Ansari-Toroghy and F. Farshadifar, *Some generalizations of second submodules*, Palestine Journal of Mathematics, 8 (2) (2019), 159–168.

3. H. Ansari-Toroghy and F. Farshadifar, *Classical and strongly classical 2-absorbing second submodules*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 69 (1) (2020), 123-136.

4. H. Ansari-Toroghy and F. Farshadifar, *2-absorbing and strongly 2-absorbing secondary submodules of modules*, Le Matematiche **72** (11), 123-135, (2017).

5. H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math. **11** (4), 1189–1201, (2007).

6. H. Ansari-Toroghy and F. Farshadifar, *Fully idempotent and coidempotent modules*, Bull. Iranian Math. Soc. **38** (4), 987-1005, (2012).

7. H. Ansari-Toroghy, and F. Farshadifar, *On the dual notion of prime radicals of submodules*, Asian Eur. J. Math. **6** (2), 1350024 (11 pages), (2013).

8. H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules*, Algebra Colloq. **19** (Spec 1), 1109-1116, (2012).

9. H. Ansari-Toroghy and F. Farshadifar, *The dual notions of some generalizations of prime submodules*, Comm. Algebra **39** (7)(2011), 2396-2416.

10. H. Ansari-Toroghy, F. Farshadifar, and S. S. Pourmortazavi, *On the P-interiors of submodules of Artinian modules*, Hacet. J. Math. Stat. **45** (3), 675-682, (2016).

11. A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75**, 417-429, (2007).

12. A. Barnard, *Multiplication modules*, J. Algebra **71**, 174-178, (1981).

13. M.P. Brodmann and R.Y. Sharp, *Local cohomology an algebraic introduction with geometric applications*, Cambridge Univercity Press, 1998.

14. S. Ceken, M. Alkan, and P.F. Smith, *The dual notion of the prime radical of a module*, J. Algebra **392**, 265-275, (2013).

15. A. Y. Darani and F. Soheilnia, *2-absorbing and weakly 2-absorbing submoduels*, Thai J. Math. **9** (3), 577–584, (2011).

16. A. Y. Darani and F. Soheilnia, *On n-absorning submodules*, Math. Commun. **17**, 547-557, (2012).

17. J. Dauns, *Prime submodules*, J. Reine Angew. Math. **298**, 156–181, (1978).

18. M. K. Dubey and P. Aggarwal, *On n-absorbing submodules of modules over commutative rings*, Beitr. Algebra Geom. **57** (3), 679-690, (2016).

19. L. Fuchs, W. Heinzer, and B. Olberding, *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math. **249**, 121–145, (2006).

20. H. Matsumara, *Commutative Ring Theory*, Cambridge University Press, Cambridge 1986.

21. S. Moradi and A. Azizi, *Weakly 2-absorbing submodules of modules*, Turkish J. Math. **40** (2), 350-364, (2016).

22. H. Mostafanasab, U. Tekir, and K.H. Oral, *Classical 2-absorbing submodules of modules over commutative rings*, Eur. J. Pure Appl. Math. **8** (3), 417-430, 2015.

23. H. Mostafanasab, E. Yetkin, U. Tekir and A. Yousefian Darani, *On 2-absorbing primary submodules of modules over commutative rings*, An. Şt. Univ. Ovidius Constanta **24** (1), 335-351, 2016.

24. Sh. Payrovi and S. Babaei, *On 2-absorbing submodules*, Algebra Collq. **19**, 913-920, (2012).

25. Sh. Payrovi and S. Babaei, *On the 2-absorbing submodules*, Iran. J. Math. Sci. Inform. **10** (1), 131-137, (2015).

26. S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno) **37**, 273–278, (2001).

27. S. Yassemi, *The dual notion of the cyclic modules*, Kobe. J. Math. **15**, 41–46, (1998).

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