The Image of Jordan Left Derivations on Algebras

Amin Hosseini* and Ajda Fošner

ABSTRACT: Let $\mathcal{A}$ be an algebra, and let $\mathcal{I}$ be a semiprime ideal of $\mathcal{A}$. Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a Jordan left derivation such that $d(\mathcal{I}) \subseteq \mathcal{I}$. We prove that if $\dim\{d(a) + \mathcal{I} \mid a \in \mathcal{A}\} \leq 1$, then $d(\mathcal{A}) \subseteq \mathcal{I}$. Additionally, we consider several consequences of this result.

Key Words: Left derivation, Jordan left derivation, Derivation, Jacobson radical.

Contents

1. Introduction and Preliminaries

Throughout the paper, $\mathcal{A}$ denotes an associative complex algebra and $\mathcal{I}$ an ideal of $\mathcal{A}$. Recall that $\mathcal{I}$ is prime if for $a, b \in \mathcal{A}$, $a \in \mathcal{I}$ or $b \in \mathcal{I}$ whenever $a\mathcal{A}b \subseteq \mathcal{I}$. Moreover, an ideal $\mathcal{I}$ is said to be semiprime if for $a \in \mathcal{A}$, $a\mathcal{A}a \subseteq \mathcal{I}$ implies that $a \in \mathcal{I}$. Obviously, every prime ideal is semiprime. Before describing the aim of the paper, let us recall some basic definitions and set the notations which we use in the sequel. As usual, the set of all primitive ideals is denoted by $\Pi(\mathcal{A})$ and $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. Recall that every primitive ideal is prime ([4], Proposition 1.4.34) and that the Jacobson radical is the intersection of all primitive ideals of an algebra.

Before introducing a background of our study we need three definitions. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called:

- a derivation on $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$;

- a Jordan derivation on $\mathcal{A}$ if $d(a^2) = d(a)a + ad(a)$ for all $a \in \mathcal{A}$;

- a left derivation on $\mathcal{A}$ if $d(ab) = ad(b) + bd(a)$ for all $a, b \in \mathcal{A}$;

- a Jordan left derivation if $d(a^2) = 2ad(a)$ for all $a \in \mathcal{A}$.

The concepts of a left derivation and a Jordan left derivation were introduced by Brešar and Vukman in [3]. In the past three decades, there has been considerable interest for Jordan left derivations and related mappings (see, e.g., [1,3,18]) which are in a close connection with so-called commuting mappings. The main motivation

comes from the Posner's fundamental result which states that if a prime ring admits a commuting nonzero derivation, then it must be commutative (see [14]).

Singer and Wermer [16] obtained a fundamental result which started investigations on the range of derivations on Banach algebras. The so-called Singer-Wermer theorem states that any continuous derivation on a commutative Banach algebra maps the algebra into the Jacobson radical. In the same paper, they made a very insightful conjecture that the assumption of continuity is superfluous. This conjecture, known as the Singer-Wermer conjecture, was proved in 1988 by Thomas [17]. According to this result, every derivation on a commutative semisimple Banach algebra is zero.

A number of authors have presented many non-commutative versions of the Singer-Wermer theorem (see, e.g., [2,12,13]). Moreover, the question under which conditions all derivations are zero on a given Banach algebra has attracted much attention of authors (see, e.g., [5,6,9,14,15]). In the following, some significant works on the range of left derivations are reviewed. In 1998, Jung [10] proved that every spectrally bounded left derivation maps algebra into its Jacobson radical. Vukman [19] showed that every Jordan left derivation on a semisimple Banach algebra is identically zero. In the same paper he conjectured that every Jordan left derivation on a Banach algebra maps the algebra into its radical. Furthermore, in [11] the authors obtained a result as follows:

If d is a Jordan left derivation on a unital Banach algebra $\mathcal{A}$ with the condition $\sup\{r(c^{-1}d(c)) \,:\, c \in \mathcal{A} \text{ invertible}\} < \infty$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $r(a)$ denotes the spectral radius of $a \in \mathcal{A}$.

By getting idea and using the techniques of [7,8,15], the current study aims to prove the following result:

Let $\mathfrak{I}$ be a semiprime ideal of an algebra $\mathcal{A}$, and let $d : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation such that $d(\mathfrak{I}) \subseteq \mathfrak{I}$. If $\dim\{d(a) + \mathfrak{I} \mid a \in \mathcal{A}\} \leq 1$, then $d(\mathcal{A}) \subseteq \mathfrak{I}$. From this result, we obtain a consequence as follows:

Let every primitive ideal of an algebra $\mathcal{A}$ has codimension 1, and let $d : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation such that $d(\mathcal{P}) \subseteq \mathcal{P}$ for any $\mathcal{P} \in \Pi(\mathcal{A})$. Then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$. Moreover, in this paper, we present some examples of a Jordan left derivation which is not a Jordan derivation. We also establish some examples of a Jordan derivation which is not a Jordan left derivation.

2. Main Results

Our first result reads as follows. This theorem has been motivated by [7].

Theorem 2.1. *Let $\mathfrak{I}$ be a semiprime ideal of an algebra $\mathcal{A}$, and let $d : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation such that $d(\mathfrak{I}) \subseteq \mathfrak{I}$. If $\dim\{d(a) + \mathfrak{I} \mid a \in \mathcal{A}\} \leq 1$, then $d(\mathcal{A}) \subseteq \mathfrak{I}$.*

Proof: If $\dim\{d(a) + \mathfrak{I} \mid a \in \mathcal{A}\} = 0$, then, obviously, $d(\mathcal{A}) \subseteq \mathfrak{I}$. Now, suppose that $\dim\{d(a) + \mathfrak{I} \mid a \in \mathcal{A}\} = 1$. Hence, there exists an element $x \in \mathcal{A}$, $x \notin \mathfrak{I}$, such that $\{d(a) + \mathfrak{I} \mid a \in \mathcal{A}\} = \{\alpha(x + \mathfrak{I}) \mid \alpha \in \mathbb{C}\} = \{\alpha x + \mathfrak{I} \mid \alpha \in \mathbb{C}\}$. In what follows, we use the standard notation $\widehat{a} = a + \mathfrak{I}$ for $a \in \mathcal{A}$. We define a mapping

$D : \frac{\mathcal{A}}{\mathcal{I}} \to \frac{\mathcal{A}}{\mathcal{I}}$ by $D(\widehat{a}) = \widehat{d(a)}$, $a \in \mathcal{A}$. It is clear that D is linear. Next, we show that D is well-defined. Suppose that $\widehat{a} = \widehat{b}$ for $a, b \in \mathcal{A}$. Then $a - b \in \mathcal{I}$ and, thus $d(a - b) \in \mathcal{I}$ since $d(\mathcal{I}) \subseteq \mathcal{I}$ and, consequently, $D(\widehat{a}) = D(\widehat{b})$. Moreover, we have

$$\begin{aligned}
D(\widehat{a}^2) = D(\widehat{a^2}) = \widehat{d(a^2)} &= d(a^2) + \mathcal{I} \\
&= 2ad(a) + \mathcal{I} \\
&= 2(a + \mathcal{I})(d(a) + \mathcal{I}) \\
&= 2\widehat{a}D(\widehat{a}).
\end{aligned}$$

It means that D is a Jordan left derivation on $\frac{\mathcal{A}}{\mathcal{I}}$.

Suppose that $d(\mathcal{A}) \not\subseteq \mathcal{I}$. Then there exists an element $a_0 \in \mathcal{A}$ such that $d(a_0) \notin \mathcal{I}$. Thus, $D(\widehat{a_0}) = d(a_0) + \mathcal{I} \neq \mathcal{I}$. Since $\dim\{D(\widehat{a}) \mid a \in \mathcal{A}\} = 1$, we can consider the functional $f : \frac{\mathcal{A}}{\mathcal{I}} \to \mathbb{C}$ satisfying $D(\widehat{a}) = f(\widehat{a})\widehat{x}$ for all $a \in \mathcal{A}$. Clearly, $f(\widehat{a_0}) \neq 0$ since $f(\widehat{a_0})\widehat{x} = D(\widehat{a_0}) \neq \mathcal{I}$. Let us write $\widehat{b_0} = \frac{1}{f(\widehat{a_0})}\widehat{a_0}$. Then $D(\widehat{b_0}) = D(\frac{1}{f(\widehat{a_0})}\widehat{a_0}) = \frac{1}{f(\widehat{a_0})}f(\widehat{a_0})\widehat{x} = \widehat{x}$ and this implies that $f(\widehat{b_0}) = 1$.

First, we show that $\widehat{a}\,\widehat{x}$ is a scalar multiple of $\widehat{x}$ for any a in $\mathcal{A}$. For an arbitrary element $a \in \mathcal{A}$, we have

$$D(\widehat{a}^2) = f(\widehat{a}^2)\widehat{x}. \tag{2.1}$$

On the other hand, since D is a Jordan left derivation and $\dim\{D(\widehat{a}) \mid a \in \mathcal{A}\} = 1$, we have

$$D(\widehat{a}^2) = 2\widehat{a}D(\widehat{a}) = 2\widehat{a}f(\widehat{a})\widehat{x}. \tag{2.2}$$

Comparing (2.1) and (2.2), we find that $f(\widehat{a}^2)\widehat{x} = 2f(\widehat{a})\widehat{a}\,\widehat{x}$. If $f(\widehat{a}) \neq 0$, then $\widehat{a}\,\widehat{x} = \frac{f(\widehat{a}^2)}{2f(\widehat{a})}\widehat{x}$. If $f(\widehat{a}) = 0$, then

$$\begin{aligned}
f(\widehat{a}\,\widehat{b_0} + \widehat{b_0}\,\widehat{a})\widehat{x} = D(\widehat{a}\,\widehat{b_0} + \widehat{b_0}\,\widehat{a}) &= 2\widehat{b_0}D(\widehat{a}) + 2\widehat{a}D(\widehat{b_0}) \\
&= 2\widehat{b_0}f(\widehat{a})\widehat{x} + 2\widehat{a}f(\widehat{b_0})\widehat{x} \\
&= 2\widehat{a}\,\widehat{x}.
\end{aligned}$$

Therefore, $\widehat{a}\,\widehat{x}$ is a scalar multiple of $\widehat{x}$ for any $a \in \mathcal{A}$.

Next, we show that $\widehat{x}^2 = 0$. We have $f(\widehat{b_0}^2)\widehat{x} = D(\widehat{b_0}^2) = 2\widehat{b_0}D(\widehat{b_0}) = 2\widehat{b_0}f(\widehat{b_0})\,\widehat{x} = 2\widehat{b_0}\,\widehat{x}$. It is a well-known fact in the abstract algebra that an ideal $\mathcal{J}$ of a ring $\mathcal{R}$ is semiprime if and only if the quotient ring $\frac{\mathcal{R}}{\mathcal{J}}$ is a semiprime ring. From this fact, we get that the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}}$ is a semiprime algebra. Now it follows from Theorem 2 of [19] that D is a left derivation on the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}}$. We can thus deduce that $D(\widehat{a}\,\widehat{b} + \widehat{b}\,\widehat{a}) = 2\widehat{a}D(\widehat{b}) + 2\widehat{b}D(\widehat{a}) = 2D(\widehat{ab})$ for all $a, b \in \mathcal{A}$. Hence, $D(\widehat{b_0}\,\widehat{x} + \widehat{x}\,\widehat{b_0}) = 2D(\widehat{b_0}\,\widehat{x}) = D(2\widehat{b_0}\,\widehat{x}) = D(f(\widehat{b_0}^2)\widehat{x}) = f(\widehat{b_0}^2)D(\widehat{x}) = f(\widehat{b_0}^2)f(\widehat{x})\widehat{x}$. We observe two cases.

<u>Case 1.</u> Suppose that $f(\widehat{x}) = 0$. Then $D(\widehat{b_0}\,\widehat{x} + \widehat{x}\,\widehat{b_0}) = 0$ and, hence,

$$\begin{aligned}
0 =& f(\widehat{b_0}^2)f(\widehat{x})\widehat{x} = f(\widehat{b_0}^2)D(\widehat{x}) = D(\widehat{b_0}\,\widehat{x} + \widehat{x}\,\widehat{b_0}) \\
=& 2\widehat{b_0}D(\widehat{x}) + 2\widehat{x}D(\widehat{b_0}) \\
=& 2\widehat{b_0}f(\widehat{x})\widehat{x} + 2f(\widehat{b_0})\widehat{x}^2 \\
=& 2\widehat{x}^2.
\end{aligned}$$

Consequently, $\widehat{x}^2 = 0$.

<u>Case 2.</u> Suppose that $f(\widehat{x}) \neq 0$. Then

$$f(\widehat{x}^2)\widehat{x} = D(\widehat{x}^2) = 2\widehat{x}D(\widehat{x}) = 2f(\widehat{x})\widehat{x}^2.$$

Thus,

$$f(\widehat{x}^2)\widehat{x} = 2f(\widehat{x})\widehat{x}^2. \tag{2.3}$$

If $f(\widehat{x}^2) = 0$, then, by (2.3), $\widehat{x}^2 = 0$. If $f(\widehat{x}^2) \neq 0$, then $\widehat{x}^2 = \frac{f(\widehat{x}^2)}{2f(\widehat{x})}\widehat{x}$. Note that $D(\alpha\widehat{a}) = \alpha D(\widehat{a})$. Hence, $f(\alpha\widehat{a})\widehat{x} = \alpha f(\widehat{a})\widehat{x}$, and, since we are assuming that $\widehat{x}$ is non-zero, it is concluded that $f(\alpha\widehat{a}) = \alpha f(\widehat{a})$ for all $a \in \mathcal{A}$, $\alpha \in \mathbb{C}$. Let $\lambda = \frac{f(\widehat{x}^2)}{2f(\widehat{x})}$. Replacing $\widehat{x}^2$ by $\lambda\widehat{x}$ in (2.3), we obtain $\lambda f(\widehat{x})\widehat{x} = f(\lambda\widehat{x})\widehat{x} = f(\widehat{x}^2)\widehat{x} = 2f(\widehat{x})\lambda\widehat{x}$. It follows that $\widehat{x} = 0$, which this is a contradiction. Therefore, $f(\widehat{x}^2)$ must be zero. Reusing the equation (2.3), we have $0 = f(\widehat{x}^2)\widehat{x} = 2f(\widehat{x})\widehat{x}^2$. So, $\widehat{x}^2 = 0$, as desired.

We already know that $\widehat{a}\,\widehat{x} = \alpha\widehat{x}$ for some scalar $\alpha \in \mathbb{C}$. Multiplying the equality by $\widehat{x}$ on the left side and using the fact that $\widehat{x}^2 = 0$, we see that $\widehat{x}\,\widehat{a}\,\widehat{x} = 0$ for any a in $\mathcal{A}$. It means that $xax \in \mathcal{I}$ for all $a \in \mathcal{A}$, and since $\mathcal{I}$ is a semiprime ideal, $x \in \mathcal{I}$. Thus, $\widehat{x} = x + \mathcal{I} = 0$, a contradiction. This contradiction shows that there is no element $a_0 \in \mathcal{A}$ such that $d(a_0) \notin \mathcal{I}$. Therefore, $d(\mathcal{A}) \subseteq \mathcal{I}$. The proof is completed.
□

The next corollary is a direct application of the preceding theorem.

Corollary 2.2. *Let every primitive ideal of an algebra $\mathcal{A}$ has codimension 1, and let $d : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation such that $d(\mathcal{P}) \subseteq \mathcal{P}$ for any $\mathcal{P} \in \Pi(\mathcal{A})$. Then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Proof: Let $\mathcal{P}$ be an arbitrary primitive ideal of $\mathcal{A}$. According to the aforementioned assumption, $\dim\{d(a) + \mathcal{P} \mid a \in \mathcal{A}\} \leq 1$, and it follows from Theorem 2.1 that $d(\mathcal{A}) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is an arbitrary primitive ideal of $\mathcal{A}$, we have $d(\mathcal{A}) \subseteq rad(\mathcal{A})$. □

Since every primitive ideal of a commutative algebra $\mathcal{A}$ has codimension 1, we have the next direct corollary.

Corollary 2.3. *Let $\mathcal{A}$ be a commutative algebra, and let $d : \mathcal{A} \to \mathcal{A}$ be a derivation (or left derivation). If $d(\mathcal{P}) \subseteq \mathcal{P}$ for any $\mathcal{P} \in \Pi(\mathcal{A})$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

In the proof of Theorem 2.4, we omit the details since the main idea is the same as in the proof of Theorem 2.1. This theorem has been motivated by Theorem 2.1 of [8].

Theorem 2.4. *Let $\mathcal{P}$ be a semiprime ideal of an algebra $\mathcal{A}$, and let $d : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{P}}$ be a left derivation such that $d(\mathcal{P}) = \{0\}$. If $\dim\{d(a) \mid a \in \mathcal{A}\} \leq 1$, then d is identically zero.*

Proof: If $\dim\{d(a) \mid a \in \mathcal{A}\} = 0$, then, obviously, d is identically zero. Now, suppose that $\dim\{d(a) \mid a \in \mathcal{A}\} = 1$. Hence, there exists an element $x \in \mathcal{A}$, $x \notin \mathcal{P}$, such that $\{d(a) \mid a \in \mathcal{A}\} = \{\alpha(x + \mathcal{P}) \mid \alpha \in \mathbb{C}\} = \{\alpha x + \mathcal{P} \mid \alpha \in \mathbb{C}\}$. As before, we denote $\widehat{a} = a + \mathcal{P}$ for $a \in \mathcal{A}$. Suppose that $d(a_0) \neq 0$ for some $a_0 \in \mathcal{A}$. We consider the functional $f : \mathcal{A} \to \mathbb{C}$ satisfying $d(a) = f(a)\widehat{x}$, $a \in \mathcal{A}$. Clearly, $f(a_0) \neq 0$, since $f(a_0)\widehat{x} = d(a_0) \neq 0$. Let us write $b_0 = \frac{1}{f(a_0)}a_0$. Then $d(b_0) = d(\frac{1}{f(a_0)}a_0) = \frac{1}{f(a_0)}f(a_0)\widehat{x} = \widehat{x}$ and this implies that $f(b_0) = 1$.

Similar as above, we first show that $a\,\widehat{x}$ is a scalar multiple of $\widehat{x}$ for any $a \in \mathcal{A}$. For an arbitrary element $a \in \mathcal{A}$, we have

$$d(a^2) = f(a^2)\widehat{x}. \tag{2.4}$$

On the other hand, since d is a left derivation and $\dim\{d(a) \mid a \in \mathcal{A}\} = 1$, we have

$$d(a^2) = 2ad(a) = 2f(a)a\widehat{x}. \tag{2.5}$$

Comparing (2.4) and (2.5), we find that $f(a^2)\widehat{x} = 2f(a)a\,\widehat{x}$. If $f(a) \neq 0$, then $a\,\widehat{x} = \frac{f(a^2)}{2f(a)}\widehat{x}$. If $f(a) = 0$, then

$$
\begin{aligned}
f(ab_0 + b_0 a)\widehat{x} &= d(ab_0 + b_0 a) \\
&= 2(ad(b_0) + b_0 d(a)) \\
&= 2(af(b_0)\widehat{x} + b_0 f(a)\widehat{x}) \\
&= 2a\widehat{x}.
\end{aligned}
$$

Therefore, $a\,\widehat{x}$ is a scalar multiple of $\widehat{x}$ for any a in $\mathcal{A}$.

Next we show that $\widehat{x}^2 = 0$. We have $f(b_0{}^2)\widehat{x} = d(b_0{}^2) = 2b_0 d(b_0) = 2f(b_0)b_0\widehat{x} = 2b_0\,\widehat{x}$, i.e., $2b_0\widehat{x} - f(b_0{}^2)\widehat{x} = 0$. Therefore, $2b_0 x - f(b_0{}^2)x \in \mathcal{P}$. Based on the hypothesis, we have $d(2\,b_0 x - f(b_0{}^2)x) = 0$, which means that $2d(b_0 x) = f(b_0{}^2)d(x)$. This equation together with the fact that d is a left derivation imply that $f(b_0^2)d(x) = 2d(b_0 x) = 2(xd(b_0) + b_0 d(x)) = 2(x\widehat{x} + b_0 f(x)\widehat{x}) = 2(\widehat{x}^2 + f(x)b_0\widehat{x})$. Hence,

$$2f(x)b_0\widehat{x} + 2\widehat{x}^2 = f(b_0^2)f(x)\widehat{x}.$$

If $f(x) = 0$, then $\widehat{x}^2 = 0$. Now, suppose that $f(x) \neq 0$. We therefore have

$$f(x^2)\widehat{x} = d(x^2) = 2xd(x) = 2xf(x)\widehat{x} = 2f(x)\widehat{x}^2. \tag{2.6}$$

If $f(x^2) = 0$, then by the above, $\widehat{x}^2 = 0$. So, assume that $f(x^2) \neq 0$. In this case, we have $\widehat{x}^2 = \frac{f(x^2)}{2f(x)}\widehat{x}$. Let us denote $\lambda = \frac{f(\widehat{x}^2)}{2f(\widehat{x})}$. The equality $d(\alpha a) = \alpha d(a)$

implies that $f(\alpha a)\widehat{x} = \alpha f(a)\widehat{x}$ and since we are assuming that $\widehat{x}$ is non-zero, it is concluded that $f(\alpha a) = \alpha f(a)$ for all $a \in \mathcal{A}$, $\alpha \in \mathbb{C}$. We already know that $\widehat{x}^2 = \lambda\widehat{x}$, and hence $x^2 - \lambda x \in \mathcal{P}$. According to the assumption that $d(\mathcal{P}) = \{0\}$, we have $d(x^2 - \lambda x) = 0$, i.e., $d(x^2) = \lambda d(x)$. Therefore,

$$f(x^2)\widehat{x} = \lambda f(x)\widehat{x} = f(x)\widehat{x}^2. \tag{2.7}$$

Comparing (2.6) and (2.7), we obtain that $f(x)\widehat{x}^2 = 0$ and since $f(x) \neq 0$ we have $\widehat{x}^2 = 0$, as desired.

We already know that $a\widehat{x} = \alpha\widehat{x}$ for some scalar $\alpha \in \mathbb{C}$. Multiplying the equality by $\widehat{x}$ on the left side and using the fact that $\widehat{x}^2 = 0$, we see that $\widehat{x}\, a\, \widehat{x} = 0$ for any $a \in \mathcal{A}$. Thus, $xax \in \mathcal{P}$ for all $a \in \mathcal{A}$ and, since $\mathcal{P}$ is a semiprime ideal of $\mathcal{A}$, $x \in \mathcal{P}$. It implies that $\widehat{x} = x + \mathcal{P} = 0$. This contradiction shows that there is no element a_0 of $\mathcal{A}$ such that $d(a_0) \neq 0$. Therefore, d must be zero. This completes the proof of the theorem. $\qquad\square$

As a consequence of Theorem 2.4, we obtain the following corollary:

Corollary 2.5. *Let $d : \mathcal{A} \to \mathcal{A}$ be a left derivation, and let every semiprime ideal of $\mathcal{A}$ has codimension 1. If $d(\mathcal{P}) \subseteq \mathcal{P}$ for any semiprime ideal $\mathcal{P}$ of $\mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$. Moreover, if the intersection of all semiprime ideals of $\mathcal{A}$ is zero, then d is identically zero.*

Proof: Let $\mathcal{P}$ be an arbitrary semiprime ideal of $\mathcal{A}$, and let $D : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{P}}$ be a mapping defined by $D(a) = d(a) + \mathcal{P}$, $a \in \mathcal{A}$. It is easy to see that D is a left derivation with $\dim\{D(a) \mid a \in \mathcal{A}\} \leq 1$, since $\dim(\frac{\mathcal{A}}{\mathcal{P}}) = 1$. Note that $D(\mathcal{P}) = \{d(p) + \mathcal{P} \mid p \in \mathcal{P}\} = \{0\}$ and, by Theorem 2.4, D is identically zero, i.e., $d(\mathcal{A}) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is an arbitrary semiprime ideal of $\mathcal{A}$, this yields that $d(\mathcal{A})$ lies in the intersection of all semiprime ideals of $\mathcal{A}$. Recall that every primitive ideal is semiprime ([4], Proposition 1.4.34) and thus $d(\mathcal{A}) \subseteq \bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P} = rad(\mathcal{A})$. It is clear that if the intersection of all semiprime ideals of $\mathcal{A}$ is zero, then $\bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P} = rad(\mathcal{A}) = \{0\}$ and consequently, $d \equiv 0$. This proves the corollary. $\qquad\square$

In the rest of this paper, we present some examples of a Jordan left derivation which is not a Jordan derivation. We also establish some examples of a Jordan derivation which is not a Jordan left derivation. Such examples are as follows:

Example 2.6. *Let R be a ring such that the square of each element in R is zero, but the product of some nonzero elements in R is nonzero. Next, let*

$$\mathfrak{R} = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} : a, b \in R \right\}$$

Clearly, $\mathfrak{R}$ is a ring under matrix addition and matrix multiplication. Define the mapping $\Delta : \mathfrak{R} \to \mathfrak{R}$ by

$$\Delta\left(\begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A straightforward verification shows that Δ is a Jordan left derivation which is neither a left derivation nor a Jordan derivation.

Example 2.7. *Let S be a ring, and let*

$$\mathfrak{S} = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix} \; : \; a, b, c, d, e \in S \right\}$$

Clearly, $\mathfrak{S}$ is a ring. Define the mapping $\delta : \mathfrak{S} \to \mathfrak{S}$ by

$$\delta\left(\begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix}\right) = \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is evident that δ is a derivation, but it is not a Jordan left derivation.

Example 2.8. *Let T be a ring such that the square of each element in T is zero, but the product of some nonzero elements in T is nonzero. Let*

$$\mathfrak{T} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \; : \; a, b \in T \right\}$$

Clearly, $\mathfrak{T}$ is a ring. Define the mapping $\Psi : \mathfrak{T} \to \mathfrak{T}$ by

$$\Psi\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that Ψ is both a Jordan left derivation and a Jordan derivation. But, it is neither a left derivation nor a derivation.

Example 2.9. *Let U be a ring, and let*

$$\mathfrak{U} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \; : \; a, b \in U \right\}$$

Clearly, $\mathfrak{U}$ is a ring. Define the mapping $\Omega : \mathfrak{U} \to \mathfrak{U}$ by

$$\Omega\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

A simple calculation shows that Ω is a derivation, but it is not a Jordan left derivation.

Example 2.10. *Let V be a ring, and let*

$$\mathfrak{V} = \left\{ \left[\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right] \ : \ a, b, c \in V \right\}$$

Clearly, $\mathfrak{V}$ is a ring. Define the mapping $\Phi : \mathfrak{V} \to \mathfrak{V}$ by

$$\Phi \left(\left[\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right] \right) = \left[\begin{array}{ccc} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

It is straightforward to see that Φ is a left derivation, but it is not a Jordan derivation.

Acknowledgments

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Amin Hosseini,
Department of Mathematics,
Kashmar Higher Education Institute,
Kashmar, Iran.
E-mail address: `a.hosseini@kashmar.ac.ir, hosseini.amin82@gmail.com`

and

Ajda Fošner,
Faculty of Management,
University of Primorska,
Cankarjeva 5, SI-6000 Koper,
Slovenia.
E-mail address: `ajda.fosner@fm-kp.si`