THE MAXIMUM NORM ANALYSIS OF A NONMATCHING GRIDS METHOD FOR A CLASS OF PARABOLIC EQUATION

SALAH BOULAARAS[1,2], MOHAMEDD CHERIF BAHI[3], AND MOHAMED HAIOUR[4]

.

Abstract. Motivated by the idea which has been introduced by M. Haiour and S.Boulaaras (Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121,No. 4, November 2011,pp.481–493), We provide a maximum norm analysis of a theta scheme combined with finite element Schwarz alternating method for a class of parabolic equation on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a stability analysis on the theta scheme which given by our work in (App. Math. Comp., 217, 6443–6450 (2011).), we establish, on each subdomain, an optimal asymptotic behavior between the discrete Schwarz sequence and the asymptotic solution of parabolic differential equations.

1. Introduction

This paper deals with the error analysis in the maximum norm, in the context of the nonmatching grids method, of the following evolutionary equation: find $u \in L^2\left(0, T; H_0^1(\Omega)\right) \cap C^2\left(0, T, H^{-1}(\Omega)\right)$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, \quad \text{in } \Sigma, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0,\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0, T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.

The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative satisfies

$$\alpha \leq \beta, \quad \beta > 0. \tag{1.2}$$

f is a regular function such that

$$f \in L^2\left(0, T, L^2(\Omega)\right) \cap C^1\left(0, T, H^{-1}(\Omega)\right).$$

Date: January 10^{th}, 2017.

1991 *Mathematics Subject Classification.* 65N06, 65N12, 65F05.

Key words and phrases. Maximum norm analysis, Nonmatching grids method, Schwarz sequence, Parabolic differential equations.

Let $(.,.)_{\Omega}$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall\xi \ > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping sub-domains (see [1], [7], [22], [23]). We refer to ([1], [7]-[9]) , and the references therein for the analysis of the Schwarz alternating method for elliptic obstacle problems and to the proceedings of the annual domain decomposition conference beginning with [15]. For results on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems we refer, for example, to [4].

In [7], we studied the overlapping domain decomposition method combined with a finite element approximation for elliptic equation related for Laplace operator Δ, where on uniform norm of an overlapping Schwarz method on nonmatching grids has been used, where we proved that the discretization on every subdomain converges on uniform norm norm. Furthermore, a result of asymptotic behavior in uniform norm has been given. In this paper, similar to that in [7], we extend the last work for evolutionary equation with mixed boundary conditions, where we provide a maximum norm analysis of a theta scheme combined with finite element Schwarz alternating method for a linear parabolic equations on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a stability analysis on the theta scheme which given by our work in [3], we establish, on each subdomain, an optimal asymptotic behavior between the discrete Schwarz sequence and the asymptotic solution of parabolic differential equations.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, then we prove a full-discrete weak formulation of the presented problem using the theta time scheme combined with a finite element method. In section 3 we state a continuous alternating Schwarz sequences and define their respective finite element counterparts in the context of nonmatching overlapping grids. Section 4 is devoted to the asymptotic behavior of the method.

2. The discrete parabolic equation

The problem (1.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ solution of

$$
(2.1) \qquad \begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + a\left(u, v\right) = \left(f, v\right) + \left(\varphi, v\right)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u^i\left(x, 0\right) = u_0^i \text{ in } \Omega, \end{cases}
$$

where $a\left(.,.\right)$ is the bilinear form defined as:

$$
(2.2) \qquad u, v \in H_0^1\left(\Omega\right) : a\left(u, u\right) = \left(\nabla u, \nabla u\right) - \left(a_0 u, u\right)
$$

2.1. **The space discretization.** Let Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m\left(h\right)\}$ defined by $\varphi_i\left(M_j\right) = \delta_{ij}$ where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$
(2.3) \qquad V_h^{(\varphi)} = \left\{ \begin{array}{l} v \in \left(L^2\left(0, T, H_0^1\left(\Omega\right)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \text{such that } v_h \mid_K = P_1,\ k \in \tau_h, \\ v_h\left(., 0\right) = v_{h0} \text{ in } \Omega,\ \dfrac{\partial v_h}{\partial \eta} = \pi_h \varphi \text{ in } \Gamma_0, \\ v_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{array} \right\}
$$

where P_1 Lagrangian polynomial of degree less than or equal to 1 and π_h is an interpolation operator on Γ_0.

We consider r_h be the usual interpolation operator defined by

$$
r_h v = \sum_{i=1}^{m(h)} v\left(M_i\right) \varphi_i\left(x\right).
$$

2.1.1. ***The discrete maximum principle assumption (DMP)***. We assume the matrices whose coefficients $a\left(\varphi_i, \varphi_j\right)$ are M-matrix ([12] and [13]). For convenience in all the sequels, C will be a generic constant independent on h.It can be approximated the problem (1.1) by a weakly coupled system of the following parabolic equation $v \in H^1\left(\Omega\right)$

$$
(2.4) \qquad \left(\frac{\partial u}{\partial t}, v\right)_\Omega + a\left(u, v\right) = \left(f, v\right)_\Omega + \left(\varphi, v\right)_{\Gamma_0}.
$$

We discretize in space, i.e., we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset \left(L^2\left(0, T, H_0^1\left(\Omega\right)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right)$, we get the following semi-discrete system of parabolic equation

$$
(2.5) \qquad \left(\frac{\partial u_h}{\partial t}, v_h\right)_\Omega + a\left(u_h, v_h\right) = \left(f, v_h - u_h\right)_\Omega + \left(\varphi, v_h\right)_{\Gamma_0}.
$$

2.2. **The time discretization.** Now we apply the θ-scheme in the semi-discrete approximation (2.5). Thus we have, for any $\theta \in [0,1]$ and $k = 1, ..., p$

$$\begin{aligned} &\left(u_h^k - u_h^{k-1}, v_h\right)_\Omega + (\Delta t)\, a\left(u_h^{\theta,k}, v_h\right) = \\ &(\Delta t)\left[\left(f^{i,\ \theta,k}, v_h\right)_\Omega + \left(\varphi^{i,\theta,k}, v_h\right)_{\Gamma_0}\right], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} u_h^{\theta,k} &= \theta u_h^k + (1-\theta)\, u_h^{k-1}, \\ f^{\ \theta,k} &= \theta f^k + (1-\theta)\, f^{k-1} \end{aligned} \tag{2.7}$$

and

$$\varphi^{\ \theta,k} = \theta\varphi^k + (1-\theta)\,\varphi^{k-1}. \tag{2.8}$$

By multiplying and dividing by θ and by adding $\left(\dfrac{u_h^{k-1}}{\theta\Delta t}, v_h\right)$ to both parties of the inequalities (1.1), we get

$$\begin{aligned} &\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right) = \left(f^{\ \theta,k} + \frac{u_h^{\theta,k-1}}{\theta\Delta t}, v_h\right)_\Omega + \\ &+ \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_h \in V_h^{(\varphi)}. \end{aligned} \tag{2.9}$$

Then, the problem (2.9) can be reformulated into the following coercive discrete system of parabolic quasi-variational inequalities

$$b\left(u_h^{\theta,k}, v_h\right) = \left(f^{\theta,k} + \mu u_h^{k-1}, v_h\right)_\Omega + \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_h,\ u_h^{\theta,k} \in V_h^{(\varphi)}, \tag{2.10}$$

where

$$\left\{\begin{aligned} &b\left(u_h^{\theta,k}, v_h\right) = \mu\left(u_h^{\theta,k}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right),\ v_h \in V_h^{(\varphi)}, \\ &\mu = \frac{1}{\theta\Delta t} = \frac{p}{\theta T}.. \end{aligned}\right. . \tag{2.11}$$

Theorem 1. *see* [7] . *Under suitable regularity of the solution of problem (1.1), there exists a constant C independent of h such that*

$$\left\|\zeta_h^\infty - \zeta\right\| \le Ch^2 \left|\log h\right|. \tag{2.12}$$

Lemma 1. *(see* [18]*) Let $w \in H^1(\Omega) \cap C\left(\bar{\Omega}\right)$ satisfies $a(w,\phi) + \lambda(w,\phi) \ge 0$ or all nonnegative $\phi \in H^1(\Omega)$ and $w \ge 0$ on Γ, then $w \ge 0$ on $\bar{\Omega}$.*

Notation 1. $(F^{\theta,k}, \varphi^{\theta,k}); (\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *be a pair of data and $\zeta^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k}); \widetilde{\zeta}^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ the corresponding solutions to (2.10) .*

Proposition 1. *Under the previous notation, we have*

$$\left\|\zeta_h^{\theta,k}-\zeta^{\theta,k}\right\|_{L\infty(\Omega)} \leq \max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k}-\widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Omega)}\}. \tag{2.13}$$

Proof. First, putting

$$\mu^{\theta,k} = \max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k}-\widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Gamma)}\}, \tag{2.14}$$

then

$$\begin{cases} \widetilde{F}^{\theta,k} \leq F^{\theta,k} + \left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \left(\frac{\lambda}{\beta}\right)\left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \lambda\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k}-\widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Gamma)}\} \\ \leq F^{\theta,k} + \lambda\mu^{\theta,k}. \end{cases}$$

So

$$b\left(\widetilde{\zeta}^{\theta,k},\phi\right) \leq b\left(\zeta^{\theta,k},\phi\right) + \lambda\left(\mu^{\theta,k},\phi\right), \text{ for all } \phi \geq 0,\ \phi \in H_0^1(\Omega) \tag{2.15}$$

and thus

$$b\left(\widetilde{\zeta}^{\theta,k},\phi\right) \leq b\left(\zeta^{\theta,k}+\mu^{\theta,k},\phi\right) = \left(F^{\theta,k}+\lambda\mu^{\theta,k},\phi\right).$$

On the other hand,we have

$$\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k} \geq 0 \text{ on } \Gamma_0. \tag{2.16}$$

So

$$b(\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k} \geq 0. \tag{2.17}$$

By using the result of lemma 1, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega} \tag{2.18}$$

Similarly, interchanging the roles of the couples $(F^{\theta,k}, \varphi^{\theta,k})$ and $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega}, \tag{2.19}$$

which completes the proof. □

Remark 1. *Proposition 1 stays true for the discrete case.*

Lemma 2. *([18]) Let $w \in V_h$ satisfy $b(w^{\theta,k}, \phi_s) > 0$ for $s = 1, 2...m(h)$ and $w^{\theta,k} \geq 0$ on Γ_0.then $w^{\theta,k} \geq 0$ on $(\overline{\Omega})$.*

Notation 2. $(F^{\theta,k}, \varphi^{\theta,k})$; $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *be a pair of data and* $\zeta_h^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k})$; $\widetilde{\zeta}_h^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *the corresponding solutions to (2.10) .*

Proposition 2. *Let DMP hold, we have*

(2.20)
$$\left\|\zeta_h^{\theta,k}-\widetilde{\zeta}_h^{\theta,k}\right\|_{L\infty(\Omega)} \leq \max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}-\widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k}-\widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Gamma_0)}\}$$

Proof. The proof is similar to that of the continuous case. □

3. Schwarz Alternating Methods for parabolic equation

We decompose (Ω) in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$;$i \neq j$ is empty. Let

$$V_i^{(w_j^{\theta,k})} = \begin{cases} v \in \left(L^2\left(0,T,H_0^1(\Omega)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \text{such that } v = w_j \text{ on } \Gamma_i. \end{cases}$$

We associate with problem (2.10) the following system: find $(u_1^{\theta,k}, u_2^{\theta,k}) \in V_1^{\theta,k} \times V_2^{\theta,k}$ solution to

$$\begin{cases} b_1(u_1^{\theta,k}, v) = (F^{\theta,k}, v)_{\Omega 1} + (\varphi^{\theta,k}, v)_{\Gamma_{01}}, \\ b_2(u_2^{\theta,k}, v) = (F^{\theta,k}, v)_{\Omega 2} + (\varphi^{\theta,k}, v)_{\Gamma_{02}}, \end{cases} \tag{3.1}$$

where

$$b_i(u_i^{\theta,k}, v) = \int_{\Omega_i} (\bigtriangledown u^{\theta,k}.\nabla v^{\theta,k} + \alpha u^{\theta,k}.v^{\theta,k})dx \tag{3.2}$$

and

$$u_i^{\theta,k} = u^{\theta,k}/\Omega_i; i = 1, 2$$

3.1. **The Continuous Schwartz Sequences.** Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$b(u_0, v) = (F^{\theta,k}, v). \tag{3.3}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences$\left(u_1^{n+1}\right)$on Ω_1 such that

$u_1^{\theta,k,n+1} \in V_1^{\left(u_2^{\theta,k,n}\right)}$ solves of

$$b_1(u_1^{\theta,k,n+1}, v) = (F_1^{\theta,k}, v), \tag{3.4}$$

where

$$F_1^{\theta,k} = f^{\theta,k} + \lambda u_1^{\theta,k-1,n+1}$$

and $(u_2^{\theta,k,n+1})$on Ω_2 such that $u_2^{\theta,k,n+1} \in V_2^{\left(\theta,k,u_1^{\theta,k,n+1}\right)}$ solves

$$b_2(u_2^{\theta,k,n+1}, v) = (F_1^{\theta,k}, v), \tag{3.5}$$

where

$$F_1^{\theta,k} = f^{\theta,k} + \lambda u_2^{\theta,k-1,n+1}$$

Theorem 2. [7] *The sequences* $(u_h^{n+1});(u_h^{n+1})$, $n \geq 0$ *produced by the Schwarz alternating method converge geometrically to a solution* u *of the elliptic obstacle problem. More precisely, there exist* $k_1, k_2 \in (0,1)$ *which depend on* (Ω_1, γ_2) *and* $(\Omega_2, \gamma 1)$ *such that for all* $n \geq 0$,

$$\sup_{\overline{\Omega}_1} \left| u_h - u^{2n+1} \right| \leq \delta_1^n \delta_2^n \sup_{\gamma_1} \left| u_h - u_h^0 \right| \tag{3.6}$$

and

$$\sup_{\overline{\Omega}_2} \left| u_h - u^{2n} \right| \leq \delta_1^n \delta_2^{n-1} \sup_{\gamma_2} \left| u_h - u_h^0 \right| . \tag{3.7}$$

3.2. **The discrete Schwartz sequences.** As we have defined before, for $i = 1, 2$,

let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C(\Omega_i)$, we set

$$V_{hi}^{(w_j^{\theta,k})} = \begin{cases} v \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \text{such that } v = \phi \text{ on } \Gamma_{01} \cap \Gamma_{02};\ v = \pi_{h_i}(w) \text{ on } \Gamma_{0i}, \end{cases}$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.4) and (3.5) .

Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.3), we respectively, define by $u_{1h}^{\theta,k,n+1} \in V_{h1}^{\left(u_{2h}^{\theta,k,n}\right)}$ such that

$$b_1(u_{1h}^{\theta,k,n+1}, v) = (F^{\theta,k}(u_{1h}^{\theta,k,n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0 \tag{3.8}$$

and $u_{2h}^{\theta,k,n+1} \in V_{h2}^{\left(u_{1h}^{\theta,k,n+1}\right)}$ such that

$$b_2(u_{2h}^{\theta,k,n+1}, v) = (F^{\theta,k}(u_{2h}^{\theta,k,n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0. \tag{3.9}$$

4. Maximum norm analysis of asymptotic behavior

4.1. **Error Analysis for the stationary case.** We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

4.1.1. ***Two auxiliary Schwarz sequences.*** For $w_{2h}^0 = u_{2h}^0$, we define the sequences $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ such that $u_{1h}^{\theta,\infty,n+1} \in V_{h1}^{\left(u_2^{\theta,\infty,n}\right)}$ solves

$$b_1(w_{1h}^{\theta,\infty,n+1}, v) = (F^{\theta,k}(u_{1h}^{\theta,\infty,n+1}), v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{4.1}$$

and $w_{2h}^{\theta,\infty,n+1} \in V_{2h}^{\left(u_{1h}^{\theta,\infty,n+1}\right)}$ solves

$$b_2(w_{2h}^{\theta,\infty,n+1}, v) = (F^{\theta,k}(u_{2h}^{\theta,\infty,n+1}), v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{4.2}$$

respectively. It is then clear that $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ are the finite element approximation of $u_1^{\theta,\infty,n+1}$ and $u_2^{\theta,\infty,n+1}$ defined in (4.1), (4.2), respectively. Then, as $F^{\theta,k}(.)$ is continuous, $\left\|F^{\theta,k}\left(u_i^{\theta,k,n+1}\right)\right\|_\infty \leq \lambda \left\|u_i^{\theta,k,n+1}\right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear parabolic problems, we have

$$\left\|u_i^{\theta,k,n} - u_{ih}^{\theta,k,n}\right\|_{L\infty(\Omega_i)} \leq Ch^2 |\log h| \tag{4.3}$$

where C is a constant independent of both h and n.

Notation 3. *From now on, we shall adopt the following notations:* $|.|_1 = |.|_{L\infty(\Gamma_1)}$, $|.|_2 = |.|_{L\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L\infty(\Gamma_2)}$, *and we set* $\pi_{h_1} = \pi_{h_2} = \pi_h$.

4.2. **Iterative discrete algorithm.** We give our following discrete algorithm

$$u_{ih}^{\theta,k,n+1} = T_h u_{ih}^{k-1,n+1}, k = 1, ..., p, \ u_{ih}^{\theta,k,n+1} \in V_{hi}^{\left(u_2^{\theta,k,n}\right)} \tag{4.4}$$

where $u_h^{\theta,k}$ is the solution of the problem (2.10) and the first iteration u_h^0 is solution of (3.3).

Proposition 3. [3] *Under the previous hypotheses and notations, we have the following estimate of convergence if* $\theta \geq \frac{1}{2}$

$$\left\|u_h^{\theta,k,n+1} - u_h^\infty\right\|_\infty \leq \left(\frac{1}{1+\theta\Delta t}\right)^k \|u_h^\infty - u_{h_0}\|_\infty , \tag{4.5}$$

if $0 \leq \theta < \frac{1}{2}$, *we have*

$$\left\|u_h^{\theta,k,2n+1} - u_h^\infty\right\|_\infty \leq \left(\frac{2}{2+\theta(1-2\theta)\rho(A)}\right)^k \|u_h^\infty - u_{h_0}\|_\infty , \tag{4.6}$$

where $\rho(A)$ *is the spectral radius of the elliptic operator.*

Lemma 3. *Let* $\rho = \frac{\alpha}{\beta}$. *Then, under assumption (1.2), there exists a constant* C *independent of both* h *and* n *such that*

$$\left\|u_i^{\theta,\infty,n+1} - u_{ih}^{\theta,\infty,n+1}\right\|_i \leq \frac{Ch^2|\log h|}{1-\rho}, \quad i = 1, 2. \tag{4.7}$$

Proof. We know from standard error estimate on uniform norm for linear problem [17] that there exists a constant C independent of h such that

$$\left\|u^0 - u_h^0\right\|_{L=(\Omega)} \leq Ch^2 |\log h| . \tag{4.8}$$

Since $\frac{1}{2} < \rho < 1$, then $1 < \rho/(1-\rho)$ and

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq Ch^2 |\log h| \leq \frac{\rho Ch^2|\log h|}{1-\rho}. \tag{4.9}$$

Let us now prove (4.7) by induction. Indeed for $n=1$, using the result of Propsition1, we have in Ω_1

$$\begin{aligned}
\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1 &\le \left\|u_1^{\theta,k,1}-w_{1h}^{\theta,k,1}\right\|_1+\left\|w_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\\
&\le Ch^2\left|\log h\right|+\left\|w_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\\
&\le Ch^2\left|\log h\right|+\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}\left(u_1^{\theta,k,1}\right)-F^{\theta,k}\left(u_{1h}^{\theta,k,1}\right)\right\|_1,\left|u_2^0-u_{2h}^0\right|_1\\
&\le Ch^2\left|\log h\right|+\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}\left(u_1^{\theta,k,1}\right)-F^{\theta,k}\left(u_{1h}^{\theta,k,1}\right)\right\|_1,\left\|u_2^0-u_{2h}^0\right\|_2\\
&\le Ch^2\left|\log h\right|+\max\{\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1,\left\|u_2^0-u_{2h}^0\right\|_2.
\end{aligned}$$

We then have to distinguish between two cases

$$\max\{\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1,\left\|u_2^0-u_{2h}^0\right\|_2\}=\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1 \tag{4.10}$$

or

$$\max\{\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1,\left\|u_2^0-u_{2h}^0\right\|_2\}=\left\|u_2^0-u_{2h}^0\right\|_2. \tag{4.11}$$

(4.10) implies

$$\begin{cases}\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k1}\right\|_1\le Ch^2\left|\log h\right|+\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1,\\[2ex] \left\|u_2^0-u_{2h}^0\right\|_2\le\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1,\end{cases}$$

then

$$\begin{cases}\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\le\dfrac{Ch^2\left|\log h\right|}{1-\rho}.\\[2ex] \left\|u_2^0-u_{2h}^0\right\|_2\le\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\le\frac{\rho Ch^2\left|\log h\right|}{1-\rho}.\end{cases}$$

(4.11) implies

$$\begin{cases}\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\le Ch^2\left|\log h\right|+\left\|u_2^0-u_{2h}^0\right\|_2\\[2ex] \qquad\qquad\qquad\le\left\|u_2^0-u_{2h}^0\right\|_2,\end{cases}$$

so, by multiplying (4.11) by ρ we get

$$\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\le\rho Ch^2\left|\log h\right|+\rho\left\|u_2^0-u_{2h}^0\right\|_2. \tag{4.12}$$

So,$\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1$ is bounded by both $\rho Ch^2|\log h|+\rho\left\|u_2^0-u_{2h}^0\right\|_2$and $\left\|u_2^0-u_{2h}^0\right\|_2$,

this implies that

$$\rho\left\|u_2^0-u_{2h}^0\right\|_2\le\rho Ch^2\left|\log h\right|+\rho\left\|u_2^0-u_{2h}^0\right\|_2, \tag{4.13}$$

or

$$\rho Ch^2\left|\log h\right|+\rho\left\|u_2^0-u_{2h}^0\right\|_2\le\left\|u_2^0-u_{2h}^0\right\|_2, \tag{4.14}$$

that is (4.13) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \tag{4.15}$$

and (4.14) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}. \tag{4.16}$$

It follows that only the case (4.13) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}, \tag{4.17}$$

then

$$\begin{aligned}
\rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 &\leq Ch^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\
&\leq Ch^2 \left|\log h\right| + \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \\
&\leq \frac{Ch^2 \left|\log h\right|}{1-\rho}.
\end{aligned}$$

So, in both cases (4.10) and (4.11), we have

$$\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \leq \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{4.18}$$

Similarly, we have in Ω_2

$$\begin{aligned}
\left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 &\leq Ch^2 \left|\log h\right| + \left\|w_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|F^{\theta,k}\left(u_2^{\theta,k,1}\right) - F^{\theta,k}\left(u_{2h}^{\theta,k,1}\right)\right\|_2, \left|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right|_2\} \\
&\leq Ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\|F^{\theta,k}\left(u_2^{\theta,k,1}\right) - F^{\theta,k}\left(u_{2h}^{\theta,k,1}\right)\right\|_2, \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1\} \\
&\leq Ch^2 \left|\log h\right| + \max\{\rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2, \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1\}.
\end{aligned}$$

So

$$\max\{\rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2, \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1\} = \rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \tag{4.19}$$

or

$$\max\{\rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2, \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1\} = \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1. \tag{4.20}$$

cases (4.19) implies

$$\begin{aligned}
\left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 &\leq Ch^2 \left|\log h\right| + \rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2, \\
\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 &\leq \rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2
\end{aligned}$$

so

$$\begin{cases} \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \\ \qquad\qquad \leq \rho \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \\ \qquad\qquad \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \end{cases}$$

while case (4.20) implies

$$\begin{cases} \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \leq Ch^2 \left|\log h\right| + \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \\ \rho \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \leq \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 . \end{cases} , \tag{4.21}$$

So, by multiplying (4.21) by ρ we get

$$\rho \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 . \tag{4.22}$$

Hence $\rho \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2$ is bounded by both $\rho Ch^2 |logh| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1$ and $\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k1} \right\|_1$, then

$$\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \tag{4.23}$$

or

$$Ch^2 \left|\log h\right| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 , \tag{4.24}$$

which (4.23) implies

$$\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} < \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{4.25}$$

or (4.24) implies

$$\frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 < \frac{Ch^2 \left|\log h\right|}{1-\rho}. \tag{4.26}$$

Hence, (4.23) and (4.24) are true because they both coincide with (4.18). So, there is either a contradiction and thus case (4.19) is impossible or case (4.20) is possible only if

$$\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 = \rho Ch^2 \left|\log h\right| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 , \tag{4.27}$$

that is

$$\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 = \frac{\rho Ch^2 \left|\log h\right|}{1-\rho}, \tag{4.28}$$

thus

$$\begin{aligned}\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2 &\leq Ch^2\left|\log h\right|+\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\\ &\leq Ch^2\left|\log h\right|+\frac{\rho Ch^2\left|\log h\right|}{1-\rho}\\ &\leq \frac{Ch^2\left|\log h\right|}{1-\rho},\end{aligned}$$

that is, both cases (4.19) and (4.20) imply

$$\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2 \leq \frac{Ch^2\left|\log h\right|}{1-\rho}. \tag{4.29}$$

Now, let us assume that

$$\left\|u_2^{\theta,k,n}-u_{2h}^{\theta,k,n}\right\|_2 \leq \frac{Ch^2\left|\log h\right|}{1-\rho} \tag{4.30}$$

and prove that

$$\begin{cases}\left\|u_1^{\theta,k,n+1}-u_{1h}^{\theta,k,n+1}\right\|_1 \leq \dfrac{Ch^2\left|\log h\right|}{1-\rho}\\ \left\|u_2^{\theta,k,n+1}-u_{2h}^{\theta,k,n+1}\right\|_2 \leq \dfrac{Ch^2\left|\log h\right|}{1-\rho}\end{cases}$$

□

Theorem 3. *Let $h = max\left(h_1,h_2\right)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{\theta,k,n+1}-u_{ih}^{\theta,k,n+1}\right\|_1 \leq \frac{ch^2\left|\log h\right|}{1-\rho},\quad \forall i=1,2. \tag{4.31}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\delta = \delta_1\delta_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned}\left\|u_1^{\theta,k}-u_{1h}^{\theta,k,n+1}\right\|_1 &\leq \left\|u_1^{\theta,k}-u_1^{\theta,k,n+1}\right\|_1+\left\|u_1^{\theta,k,n+1}-u_{1h}^{\theta,k,n+1}\right\|_1\\ &\leq \delta_1^n\delta_2^n\left|u^0-u\right|_1+\frac{ch^2\left|\log h\right|}{1-\rho}\\ &\leq \delta^{2n}\left|u^0-u\right|_1+\frac{ch^2\left|\log h\right|}{1-\rho}.\end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{4.32}$$

and thus

$$\begin{aligned}\left\|u_1^{\theta,k}-u_{1h}^{\theta,k,n+1}\right\|_1 &\leq ch^2+ch^2\left|\log h\right|\\ &\leq ch^2\left|\log h\right|,\end{aligned}$$

which is the desired result. □

4.3. **Asymptotic behavior.** This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h(T)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution of (2.11)

Theorem 4. *According to the results of the proposition 3 and the theorem 3, we have*

for the first case $\theta \geq \frac{1}{2}$

$$\left\| u_{1h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{1}{1+\theta\Delta t} \right)^p \right], \tag{4.33}$$

and

$$\left\| u_{2h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{1}{1+\theta\Delta t} \right)^p \right], \tag{4.34}$$

and for the second case $0 \leq \theta < \frac{1}{2}$

$$\left\| u_{1h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right] \tag{4.35}$$

and

$$\left\| u_{2h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right], \tag{4.36}$$

where C is a constant independent of h and k.

Proof. We have

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq \left\| u_h^{\theta,p,2n+1} - u_h^\infty \right\|_\infty + \left\| u_h^\infty - u^\infty \right\|_\infty .$$

Using the proposition 3 and the theorem 3, we have for $\theta \geq \frac{1}{2}$

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right|^3 + \left(\frac{1}{1+\theta\Delta t} \right)^p \right],$$

and for $0 \leq \theta < \frac{1}{2}$ we have

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right|^3 + \left(\frac{2}{2+\theta(1-2\theta)\rho(\Delta)} \right)^p \right]$$

The proof for (4.35) and (4.36) case is similar. □

Remark 2. *It can be seen in the previous estimates (4.33) up to (4.36),* $\left(\frac{1}{1+\beta\theta\Delta t} \right)^p$, $\left(\frac{2}{2+\theta(1-2\theta)\rho(\Delta)} \right)^p$ *to* 0 *when p tend to infinity. Therefore, the estimation order for both the coercive and noncoercive problems is*

$$\left\| u^\infty - u_{1h}^{\infty,n+1} \right\|_{L^\infty(\bar{\Omega}_1)} \leq C h^2 \left|\log h\right|^3$$

and

$$\left\| u^{\infty} - u_{2h}^{\infty,n+1} \right\|_{L^{\infty}\left(\bar{\Omega}_2\right)} \leq Ch^2 \left|\log h\right|^3 .$$

References

[1] L. Badea. On the schwarz alternating method with more than two subd-omains for monotone problems. SIAM Journal on Numerical Analysis 28 (1991), no. 1, 179-204.

[2] A. Bensoussan & J. L. Lions. Contrôle impulsionnel et Inéquations Quasi-variationnelles. Dunod, 1982.

[3] S. Boulaaras, M. Haiour, "L^{∞}-asymptotic behavior for a finite element approximation in parabolic quasi-variational inequalities related to impulse control problem", App. Math. Comp., 217, 6443–6450 (2011).

[4] Cai, X-C, Mathew, TP, Sarkis, MV: Maximum norm analysis of overlapping nonmatching grid discretization of elliptic equations. SIAM J. Numer. Anal. 5, 1709-1728 (2000)

[5] J. Douglas, Jr. and C-S. Huang, An accelerated domain decomposition procedure based on Robin transmissionconditions, BIT Numerical Mathematics, vol. 37, no. 3, pp. 678–686, 1997.

[6] B. Engquist and H.-K. Zhao, Absorbing boundary conditions for domain decomposition, Applied Numerical Mathematics, vol. 27, no. 4, pp. 341–365, 1998.

[7] M. Haiour, S. Boulaaras, Overlapping domain decomposition methods for elliptic quasi-variational inequalities related to impulse control problem with mixed boundary conditions, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121,No. 4, November 2011,pp.481–493

[8] P.-L. Lions, On the Schwarz alternating method. I, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J.P ´eriaux,Eds.,pp. 1–42, SIAM, Philadelphia, USA,1988.

[9] T. F. Chan, T. Y. Hou, and P.-L. Lions, Geometry related convergence results for domain decomposition algorithms, SIAM Journal on Numerical Analysis, vol. 28, no. 2, pp. 378–391, 1991.

[10] A. Quarteroni and A. Valli, Domain Decomposition Methods for Partial Differential Equations, The Clarend on Press,Oxford, UK,1999.

[11] A. Toselli and O. Widlund, Domain Decomposition Methods Algorithms and Theory, vol. 34 of Springer Seriesin Computational Mathematics, Springer, Berlin, Germany, 2005.

[12] Y. Maday and F. Magoul'es, Improved ad hoc interface conditions for Schwarz solution procedure tuned to highly heterogeneous media, Applied Mathematical Modelling, vol. 30, no. 8, pp. 731–743, 2006.

[13] Y. Maday and F. Magoul'es, A survey of various absorbing interface conditions for the Schwarz algorithm tuned to highly heterogeneous media, in Domain Decomposition Methods: Theory and Applications, vol. 25 of Gakuto International Series. Mathematical Sciences Applications, pp. 65–93, Gakkotosho,Tokyo,Japan,2006.

[14] C. Farhat and P. Le Tallec, Vista in Domain Decomposition Methods," Computer Methods in Applied Mechanics and Engineering, vol. 184, no. 2-4, pp. 143–520, 2000.

[15] Glowinski, R, Golub, GH, Meurant, GA, Periaux, J: Domain Decomposition Methods for Partial Differential Equations. SIAM, Philadelphia (1988). ISBN:0-89871-220-3.

[16] F.Magoul'es and D. Rixen, Domain decomposition methods :recent advances and new challengesin engineering, Computer Methods in Applied Mechanics and Engineering, vol. 196, no. 8, pp. 1345–1346, 2007.

[17] J. Nitsche, L^{∞}-convergence of finite element approximations, in Proceedings of the Symposium on Mathematical Aspects of Finite Element Methods, vol. 606 of Lecture Notes in Mathematics, 261–274, 1977.

[18] S. H. Lui, "On linear monotone iteration and Schwarz methods for nonlinear elliptic PDEs," Numerische Mathematik, vol. 93, no. 1, pp. 109–129, 2002.

[19] A. Verf¨urth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley Teubner,Stuttgart,Germany, 1996.

[20] P. L. Lions. On the Schwarz alternating method. I. First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987). SIAM. Philadelphia, 1988. pp. 1-42.

[21] P. L. Lions. On the Schwarz alternating method. II.Stochastic interpretation and order properties, Domain Decomposition Methods (Los angeles, Calif, 1988). SIAM. Philadelphia 1989. pp 47-70.

[22] F.-C. Otto and G. Lube, A posteriori estimates for a non-overlapping domain decomposition method, Computing,vol. 62, no. 1, pp. 27–43, 1999.

[23] C. Bernardi, T. Chac´on Rebollo, E. Chac´on Vera, and D. Franco Coronil, A posteriori error analysis for two-overlapping domain decomposition techniques, Applied Numerical Mathematics, vol. 59, no.6, pp. 1214–1236, 2009.

[1]Department of Mathematics, College Of Science and Arts, Al-Ras, Qassim University, Kingdom Of Saudi Arabia., [2]Laboratory of Fundamental and Applied Mathematics, Oran University 1, Ahmed Benbella, Oran, Algeria.

E-mail address: saleh_boulaares@yahoo.fr

[3]Department of Mathematics and Computer Science, Larbi Tebessi University, 12002 Tebessa, Algeria

E-mail address: habitakhaled@gmail.com

[4]Department of Mathematics, Faculty of Sciences, Annaba University, Algeria

E-mail address: haiourm@yahoo.fr.