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Ornstein-Uhlenbeck Semigroup on the Dual Space of Gelfand-Shilov Spaces of Beurling Type

Hamed M. Obiedat and Lloyd Edgar S. Moyo

ABSTRACT: We use a previously obtained topological characterization of Gelfand-Shilov spaces Σ_α^β of Beurling type to characterize its dual $(\Sigma_\alpha^\beta)'$ using Riesz Representation Theorem. Using the characterization of the dual space $(\Sigma_\alpha^\beta)'$ equipped with the weak topology, we study the action of Ornstein-Uhlenbeck semigroup on the dual space $(\Sigma_\alpha^\beta)'$.

Key Words: Short-time Fourier transform, Tempered Ultradistributions, Structure Theorem, Gelfand-Shilov spaces.

Contents

1. Introduction

In mathematical analysis, distributions (generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations. They play a crucial role in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. The theory of generalized functions devised by L. Schwartz was to provide a satisfactory framework for the Fourier transform (see [9]).

Some other types of distributions called ultradistributions have also been studied by Gelfand and Shilov (see [4]) which are well-known in the theory of tempered ultradistribution. S. Pilipovic obtained structural theorems and defined the convolution for Gelfand-Shilov spaces of Roumieu and Beurling type (see [6], [7], [8]).

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In this paper, we use the characterization of Gelfand-Shilov spaces of Beurling type of test functions of tempered ultradistribution in terms of their Fourier transform obtained in [1] to prove structure theorem for functionals in dual space $(\Sigma_\alpha^\beta)'$. Using the structure theorem of the dual space$(\Sigma_\alpha^\beta)'$ equipped with the weak topology, we study the action of Ornstein-Uhlenbeck semigroup on the dual space $(\Sigma_\alpha^\beta)'$.

The symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \le p \le \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. The Fourier transform of a function f will be denoted by $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm.

2. Preliminary definitions and results

In [1], J. Chung et al proved symmetric characterizations for Gelfand-Shilov spaces via the Fourier transform in terms of the growth of the function and its Fourier transform which imposes no conditions on the derivative.

Theorem 2.1. *([1]) The space Σ_α^β can be described as a set as well as topologically by*

$$\Sigma_\alpha^\beta = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \ p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty. \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\| e^{k|x|^{1/\alpha}} \varphi \right\|_\infty$, $\pi_{k,0}(\varphi) = \left\| e^{k|\xi|^{1/\beta}} \widehat{\varphi} \right\|_\infty$.
The space Σ_α^β, equipped with the family of semi-norms

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Remark 2.2. *For $\alpha > 1$, the function $|\bullet|^{1/\alpha} : [0, \infty) \to [0, \infty)$ has the following properties:*

1. *$|\bullet|^{1/\alpha}$ is increasing, continuous and concave,*
2. *$|t|^{1/\alpha} \ge a + b \ln(1 + t)$ for some $a \in \mathbb{R}$ and some $b > 0$.*

Remark 2.3. *Let us observe for future use that if we take $N > \frac{n}{b}$ is an integer, then*

$$C_N = \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} dx < \infty, \text{ for all } \alpha > 1,$$

where b is the constant in Property 2 of Remark 2.2. Moreover, Property 1 in Remark 2.2 implies that $|\bullet|^{1/\alpha}$ is subadditive.

Remark 2.4. *If $\tau \in \mathbb{R}^n$ and $\alpha > 1$, then there exist $N \in \mathbb{N}$ and a constant $C > 0$ such that $|\tau| \le Ce^{N|\tau|^{1/\alpha}}$. In fact, since*

$$|\tau| \le 1 + |\tau| = e^{\ln(1+|\tau|)}$$

and applying Property 2 in Remark 2.2, there exist $a \in \mathbb{R}$ and $b > 0$ such that

$$\ln(1+|\tau|) \le \frac{|\tau|^{1/\alpha} - a}{b}.$$

Hence,

$$\begin{aligned} |\tau| &\le 1 + |\tau| = e^{\ln(1+|\tau|)} \\ &\le e^{\frac{|\tau|^{1/\alpha}-a}{b}} = e^{-\frac{a}{b}} e^{\frac{|\tau|^{1/\alpha}}{b}} \\ &\le Ce^{N|\tau|^{1/\alpha}} \end{aligned}$$

where $C = e^{-\frac{a}{b}} > 0$ and $N > \frac{n}{b}$ is an integer.

3. Characterization of the dual space $(\Sigma_\alpha^\beta)'$

Theorem 3.1. *([5])Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ so that*

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

Theorem 3.2. *Given $L \in \Sigma_\alpha^\beta \to \mathbb{C}$, then the following statements are equivalent :*
(i) $L \in (\Sigma_\alpha^\beta)'$
(ii) There exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \mathbb{N}_0$ such that

$$L = e^{k|x|^{1/\alpha}} d\mu_1 + e^{k|\xi|^{1/\beta}} d\mu_2,$$

in the sense of $(\Sigma_\alpha^\beta)'$.

Proof: Proving (i) implies (ii): Given $L \in (\Sigma_\alpha^\beta)'$, there exist constants k and C so that

$$L(\varphi) \le C(\left\| e^{k|x|^{1/\alpha}} \varphi \right\|_\infty + \left\| e^{k|\xi|^{1/\beta}} \widehat{\varphi} \right\|_\infty),$$

for all $\varphi \in \Sigma_\alpha^\beta$. Moreover, the map

$$\begin{aligned} \Sigma_\alpha^\beta &\to \mathcal{C}_0 \times \mathcal{C}_0 \\ \varphi &\to (e^{k|x|^{1/\alpha}} \varphi, e^{k|\xi|^{1/\beta}} \widehat{\varphi}) \end{aligned}$$

is well-defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1(e^{k|x|^{1/\alpha}}\varphi, e^{k|\xi|^{1/\beta}}\widehat{\varphi}) = L(\varphi),$$

for a unique $\varphi \in \Sigma_\alpha^\beta$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn-Banach Theorem, there exists a functional L_1 in the topological dual $(\mathcal{C}_0 \times \mathcal{C}_0)'$ of $\mathcal{C}_0 \times \mathcal{C}_0$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1. Using Theorem 3.1, there exist regular complex Borel measures μ_1 and μ_2 of finite total variation so that

$$L_1(f,g) = \int_{\mathbb{R}^n} f d\mu_1 + \int_{\mathbb{R}^n} g d\mu_2,$$

for all $(f,g) \in \mathcal{C}_0 \times \mathcal{C}_0$. If $(f,g) \in \mathcal{R}$, we conclude that

$$L(\varphi) = \int_{\mathbb{R}^n} e^{k|x|^{1/\alpha}}\varphi d\mu_1 + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}}\widehat{\varphi} d\mu_2,$$

for all $\varphi \in \Sigma_\alpha^\beta$. In the sense of $(\Sigma_\alpha^\beta)'$, we have

$$L = e^{k|x|^{1/\alpha}} d\mu_1 + e^{k|\xi|^{1/\beta}} d\mu_2.$$

This completes the proof that (i) implies (ii). Next, we prove that (ii) implies (i). Proving (ii) implies (i): If μ_1 and μ_2 are regular complex Borel measures satisfying (ii) and $\varphi \in \Sigma_\alpha^\beta$, then

$$L(\varphi) = \int_{\mathbb{R}^n} e^{k|x|^{1/\alpha}}\varphi d\mu_1 + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}}\widehat{\varphi} d\mu_2.$$

This implies that

$$\begin{aligned}
|L(\varphi)| &\leq \left| \int_{\mathbb{R}^n} e^{k|x|^{1/\alpha}}\varphi d\mu_1 + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}}\widehat{\varphi} d\mu_2 \right| \\
&\leq |\mu_1|(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} e^{k|x|^{1/\alpha}}\varphi d\mu_1 \right| + |\mu_2|(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}}\widehat{\varphi} d\mu_2 \right| \\
&\leq |\mu_1|(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} e^{(N+k)|x|^{1/\alpha}}\varphi d\mu_1 \right| \\
&\quad + |\mu_2|(\mathbb{R}^n) \left| \int_{\mathbb{R}^n} e^{-N|\xi|^{1/\beta}} e^{(N+k)|\xi|^{1/\beta}}\widehat{\varphi} d\mu_2 \right| \\
&\leq C(|\mu_1|(\mathbb{R}^n \left\| e^{(N+k)|x|^{1/\alpha}}\varphi \right\|_\infty \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} d\mu_1 \\
&\quad + |\mu_2|(\mathbb{R}^n) \left\| e^{(N+k)|\xi|^{1/\beta}}\widehat{\varphi} \right\|_\infty \int_{\mathbb{R}^n} e^{-N|\xi|^{1/\beta}} d\mu_2) \\
&\leq C(\left\| e^{(N+k)|x|^{1/\alpha}}\varphi \right\|_\infty + \left\| e^{(N+k)|\xi|^{1/\beta}}\widehat{\varphi} \right\|_\infty).
\end{aligned}$$

It may be noted that μ_1 and μ_2, employed to obtain the above inequality, are of finite total variation. This shows that $L \in (\Sigma_\alpha^\beta)'$. This completes the proof that (ii) implies (i). This completes the proof Theorem 3.2.

□

As an application of the structure theorem of $(\Sigma_\alpha^\beta)'$ stated in Theorem 3.2, we prove the following corollary.

Corollary 3.3. *If $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T * \varphi$ defined by*

$$\langle T * \varphi, \phi \rangle = \langle T_y, (\varphi_z, \phi(x+y)) \rangle$$

coincides with the functional given by integration against the function $\psi(x) = \langle T_y, \varphi(x-y) \rangle$.

Proof: Using Theorem 3.2, we can write, for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \psi(x) &= \langle T_y, \varphi(x-y) \rangle = \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} \varphi(x+y) d\mu_1(y) \\ &\quad + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} e^{-2\pi i y.\xi} \mathcal{F}^{-1}(\varphi)(\xi) d\mu_2(\xi). \end{aligned}$$

So,

$$\begin{aligned} \langle T * \varphi, \phi \rangle &= \langle T_y, (\varphi_z, \phi(x+y)) \rangle \\ &= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} \Big(\int_{\mathbb{R}^n} \varphi(x-y)\phi(y) d\mu_1(y) \Big) \\ &\quad + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} \mathcal{F}^{-1}(\varphi)(\xi) \widehat{\phi}(\xi) d\mu_2(\xi) \\ &= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} \Big(\int_{\mathbb{R}^n} \varphi(x-y)\phi(y) d\mu_1(y) \Big) \\ &\quad + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} \mathcal{F}(\check{\varphi} * \phi)(\xi) d\mu_2(\xi) \\ &= \langle e^{k|\cdot|^{1/\alpha}} \mu_1(y), \langle \varphi(x-y), \phi(x) \rangle \rangle \\ &\quad + \langle \mathcal{F}(e^{k|\cdot|^{1/\beta}} \mu_2)(y), \langle \varphi(x-y), \phi(x) \rangle \rangle \\ &= \langle e^{k|\cdot|^{1/\alpha}} \mu_1(y) + \mathcal{F}(e^{k|\cdot|^{1/\beta}} \mu_2)(y), \langle \varphi(x-y), \phi(x) \rangle \rangle \\ &= \langle T_y, \langle \varphi(x-y), \phi(x) \rangle \rangle, \end{aligned}$$

for all $\phi \in \Sigma_\alpha^\beta$. This completes the proof of Corollary 3.3. □

4. Ornstein-Uhlenbeck Semigroup action on $(\Sigma_\alpha^\beta)'$

The second-order differential operator defined by

$$A = -\frac{1}{2}\Delta - x \cdot \nabla,$$

where Δ denotes the Laplacian and ∇ denotes the gradient, is called Ornstein-Uhlenbeck operator. The semi-group generated by Ornstein-Uhlenbeck operator A is Ornstein-Uhlenbeck semi-group acting on the Hilbert space $L^2(\gamma)$ where γ is the normalized Gaussian measure. The Ornstein-Uhlenbeck semi-group $(P_t)_{t\geq 0} = (e^{At})_{t\geq 0}$ is given by

$$P_t\varphi(x) = \int_{\mathbb{R}^n} M_t(x,y)\varphi(y)d\gamma(y) = \langle M_t(x,y), \varphi(y)\rangle,$$

where $M_t(x,y)$ and $t > 0$ is the Mehler kernel and P_0 is the identity. The closed expression of the Mehler kernel $M_t(x,y)$ is given by

$$M_t(x,y) = \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} e^{-\frac{\left|y-e^{-t}x\right|^2}{1-e^{-2t}}}.$$

Observe that $M_t(x,\cdot)$ and $M_t(\cdot,y)$ are both in Σ_α^β for all $\alpha,\beta > 1$ because both have exponential decay which implies that the operator P_t is well defined. Then for $T \in (\Sigma_\alpha^\beta)'$ and $P_t = e^{At}$, where $A = -\frac{1}{2}\Delta - x\cdot\nabla$, we can write the action of P_t on $(\Sigma_\alpha^\beta)'$ as

$$\langle T * P_t, \varphi\rangle = \langle T_y, < M_t(x,y), \varphi(x)\rangle\rangle, \ \varphi \in \Sigma_\alpha^\beta.$$

To prove that $T * P_t \to T$ as $t \to 0^+$ in strong dual topology, it is enough to prove the following result.

Theorem 4.1. *Let B be a bounded subset of Σ_α^β and $\varphi \in \Sigma_\alpha^\beta$. Then $\varphi_t(x) = \langle M_t(x,\cdot), \varphi(x)\rangle \to \varphi$ in Σ_α^β as $t \to 0^+$ uniformly on B.*

Proof: Recall that $\int_{\mathbb{R}^n} M_t(x,y)dx = e^{nt}$. We can write

$$\begin{aligned}
I &= e^{k|y|^{1/\alpha}} \int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{k|y|^{1/\beta}}\varphi(y) \\
&= e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y)) \\
&= e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{-nt}\varphi(y)\int_{\mathbb{R}^n} M_t(x,y)dx) \\
&= e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} M_t(x,y)(\varphi(x) - e^{-nt}\varphi(y))dx) \\
&= e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} M_t(x,y)(\varphi(x) - \varphi(y) + \varphi(y) - e^{-nt}\varphi(y))dx.
\end{aligned}$$

Taking the absolute value for both sides and applying the triangle inequality, we get

$$\begin{aligned}|I| &= \left|e^{k|y|^{1/\alpha}}\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{k|y|^{1/\alpha}}\varphi(y)\right| \\ &\le \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} M_t(x,y)\left|\varphi(x)-\varphi(y)+\varphi(y)-e^{-nt}\varphi(y)\right| dx \\ &\le \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} M_t(x,y)\left|\varphi(x)-\varphi(y)\right| dx \\ &\quad +(1-e^{-nt})\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} M_t(x,y)\left|\varphi(y)\right| dx \\ &= I_1 + I_2.\end{aligned}$$

We estimate I_2 as follows:

$$\begin{aligned} I_2 &= (1-e^{-nt})\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} M_t(x,y)\left|\varphi(y)\right| dx \\ &\le e^{nt}(1-e^{-nt})\left|\left|e^{k|\cdot|^{1/\alpha}}\varphi\right|\right|_\infty .\end{aligned}$$

Using explicit formula for $M_t(x,y)$ and making the change of variable $u = \frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}}$, we estimate I_1 as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} M_t(x,y)\left|\varphi(x)-\varphi(y)\right| dx \\ &= \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\left|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}})-\varphi(y)\right| \frac{(1-e^{-2t})^{n/2}}{e^{-nt}}du \\ &= \frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\left|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}})-\varphi(\frac{y}{e^{-t}})+\varphi(\frac{y}{e^{-t}})-\varphi(y)\right| du \\ &\le \frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\left|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}})-\varphi(\frac{y}{e^{-t}})\right| du \\ &\quad +\frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\left|\varphi(\frac{y}{e^{-t}})-\varphi(y)\right| du.\end{aligned}$$

Using Mean Value Theorem, there is a point u' on the line segment L_1 from $\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}$ to $\frac{y}{e^{-t}}$ and a point u'' on the line segment L_2 from $\frac{y}{e^{-t}}$ to y such that

$$\left|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}})-\varphi(\frac{y}{e^{-t}})\right| = \frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}}\left|\nabla\varphi(u')\right|$$

and

$$\left|\varphi(\frac{y}{e^{-t}})-\varphi(y)\right| = \frac{|y|\,(1-e^{-t})}{e^{-t}}\left|\nabla\varphi(u'')\right|$$

respectively. Thus, the estimate for I_1 above now becomes

$$\begin{aligned} I_1 &\le \frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}}\left|\nabla\varphi(u')\right| du \\ &\quad +\frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} e^{-|u|^2}\frac{|y|\,(1-e^{-t})}{e^{-t}}\left|\nabla\varphi(u'')\right| du.\end{aligned}$$

Using $|y| \leq |u''|$ and applying Remark 2.4 for $|u''|$, then $|y| \leq Ce^{N|y|^{1/\alpha}} \leq Ce^{N|u''|^{1/\alpha}}$ for some integer N and constant $C>0$. Therefore,

$$\begin{aligned} I_1 &\leq \frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|u'|^{1/\alpha}} e^{-|u|^2}\frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}}|\nabla\varphi(u')|\,du \\ &\quad +\frac{e^{nt}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{k|u''|^{1/\alpha}} e^{-|u|^2}\frac{Ce^{N|u''|^{1/\alpha}}(1-e^{-t})}{e^{-t}}|\nabla\varphi(u'')|\,du \\ &\leq \pi^{-n/2}e^{(n+1)t}\sqrt{1-e^{-2t}}\left\|e^{k|\cdot|^{1/\alpha}}\nabla\varphi\right\|_\infty \left\|ue^{-|u|^2}\right\|_1 \\ &\quad +C\pi^{-n/2}e^{t}(1-e^{-t})\left\|e^{(N+k)|\cdot|^{1/\alpha}}\nabla\varphi\right\|_\infty \left\|e^{-|u|^2}\right\|_1. \end{aligned}$$

The estimates obtained for I_1 and I_2 imply that $I_1 \to 0$ and $I_2 \to 0$ as $t \to 0^+$ uniformly on B. Hence,

$$\left\|e^{k|\cdot|^{1/\alpha}}\left(\int_{\mathbb{R}^n} M_t(x,\cdot)\varphi(x)dx - \varphi(\cdot)\right)\right\|_\infty \to 0 \text{ as } t \to 0^+$$

uniformly on B as well. Now we prove that

$$\left\|e^{k|\cdot|^{1/\beta}}F\left(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y)\right)(\zeta)\right\|_\infty$$

approaches 0 as $t \to 0^+$ uniformly on B. To do this, we write

$$\begin{aligned} I' &= \left|e^{k|\cdot|^{1/\beta}}F\left(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y)\right)(\zeta)\right| \\ &= \left|e^{k|\zeta|^{1/\beta}}F\left(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx\right)(\zeta) - e^{k|\zeta|^{1/\beta}}F(\varphi(y))(\zeta)\right| \\ &= \left|e^{k|\zeta|^{1/\beta}}\frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}}F\left(\int_{\mathbb{R}^n} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}\varphi(x)dx\right)(\zeta) - e^{k|\zeta|^{1/\beta}}\widehat{\varphi}(\zeta)\right| \\ &= \left|e^{k|\zeta|^{1/\beta}}\left(\frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}}e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}}\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right)\right| \\ &= \left|e^{k|\zeta|^{1/\beta}}\left(\frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}}e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}}\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(e^{-t}\zeta) + \widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right)\right| \\ &\leq \pi^{-n/2}e^{k|\zeta|^{1/\beta}}(1-e^{-2t})^{-n/2}\left|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}}\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(e^{-t}\zeta)\right| \\ &\quad + e^{k|\zeta|^{1/\beta}}\left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\ &= \pi^{-n/2}e^{k(e^t)^{1/\beta}|e^{-t}\zeta|^{1/\beta}}(1-e^{-2t})^{-n/2}\left|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\right|\left|\widehat{\varphi}(e^{-t}\zeta)\right| \\ &\quad + e^{k|\zeta|^{1/\beta}}\left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\ &\leq \pi^{-n/2}e^{k[(e^t)^{1/\beta}]|e^{-t}\zeta|^{1/\beta}}(1-e^{-2t})^{-n/2}\left|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\right|\left|\widehat{\varphi}(e^{-t}\zeta)\right| \\ &\quad + e^{k|\zeta|^{1/\beta}}\left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \end{aligned}$$

$$
\begin{aligned}
\leq & \pi^{-n/2} e^{k([(e^t)^{1/\beta}]+1)\left|e^{-t}\zeta\right|^{1/\beta}} (1-e^{-2t})^{-n/2} \left| e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1 \right| \left|\widehat{\varphi}(e^{-t}\zeta)\right| \\
& + e^{k|\zeta|^{1/\beta}} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
\leq & \pi^{-n/2} e^{2k\left|e^{-t}\zeta\right|^{1/\beta}} (1-e^{-2t})^{-n/2} \left| e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1 \right| \left|\widehat{\varphi}(e^{-t}\zeta)\right| \\
& + e^{k|\zeta|^{1/\beta}} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
\leq & \pi^{-n/2} (1-e^{-2t})^{-n/2} \left| e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1 \right| \left\| e^{2k|\cdot|^{1/\beta}} \widehat{\varphi} \right\|_\infty \\
& + e^{k|\zeta|^{1/\beta}} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
= & A_1 + A_2.
\end{aligned}
$$

Since $e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} \to 1$ as $t \to 0^+$ uniformly on compact subsets of $\mathbb{R}^n$, the first term A_1 converges to 0 uniformly on B. Applying the Mean Value Theorem for the second term A_2, there is a point τ on the line segment from $e^{-t}\zeta$ to ζ such that $|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)| = (1-e^{-t})\,|\zeta|\,|\nabla\widehat{\varphi}(\tau)|$. Using Remark 2.4, we estimate $|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)|$ as follows:

$$
\begin{aligned}
\left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| &= (1-e^{-t})\,|\zeta|\,|\nabla\widehat{\varphi}(\tau)| \\
&\leq (1-e^{-t})\,|\tau|\,|\nabla\widehat{\varphi}(\tau)| \\
&\leq C(1-e^{-t})e^{-t}e^{N|\tau|^{1/\beta}}\,|\nabla\widehat{\varphi}(\tau)| \\
&\leq C\left\| e^{N|\cdot|^{1/\beta}}\nabla\widehat{\varphi}\right\|_\infty (1-e^{-t})e^{-t},
\end{aligned}
$$

which implies that A_2 converges to 0 as $t \to 0^+$. Hence,

$$
\left\| e^{k|\cdot|^{1/\beta}} F\Big(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y)\Big)(\zeta) \right\|_\infty
$$

converges to 0 uniformly on B as $t \to 0^+$. This completes the proof of Theorem 4.1.

Remark 4.2. *We observe that the functionals in the dual space* $(\Sigma_\alpha^\beta)'$ *can be realized as boundary values to the differential equation* $\frac{\partial}{\partial t}u - Au = 0$, $t > 0$.

□

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Hamed M. Obiedat,
Department of Mathematics,
Faculty of Science,
The Hashemite University,
P.O.Box 330127,
Zarqa 13133,
Jordan.
E-mail address: hobiedat@hu.edu.jo

and

Lloyd E. Moyo,
Department of Mathematics,
Henderson State University,
Arkadelphia,
AR 71999, USA.
E-mail address: moyol@hsu.edu