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Spectral Inclusions Between α-times Integrated Semigroups and Their Generators

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ABSTRACT: We interest to α-times integrated semigroups and we characterize the different spectra of α-times integrated semigroups using the spectra of their generators.

Key Words: α-times integrated semigroup, Descent, Ascent, Fredholm operator, Browder operator, Drazin invertible.

Contents

1. Introduction

Let X be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. We denote by $D(T)$, $R(T)$, $R^\infty(T) := \cap_{n\geq 1}R(T^n)$, $N(T)$, $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$ respectively the domain, the range, the hyper range, the kernel, the resolvent and the spectrum of T, where $\sigma(T) = \{\lambda \in \mathbb{C} \setminus \lambda - T \text{ is not bijective}\}$ and $\sigma_p(T) = \{\lambda \in \mathbb{C} \setminus \lambda - T \text{ is not one to one}\}$. The function resolvent of $T \in \mathcal{B}(X)$ is defined for all $\lambda \in \rho(T)$ by $R(\lambda, T) = (\lambda - T)^{-1}$. The ascent and descent of an operator T are defined by $a(T) = \inf\{k \in \mathbb{N} \setminus N(T^k) = N(T^{k+1})\}$ and $d(T) = \inf\{k \in \mathbb{N} \setminus R(T^k) = R(T^{k+1})\}$, respectively with the convention $inf(\varnothing) = +\infty$. An $T \in \mathcal{B}(X)$ is called Drazin invertible if $a(T)$ and $d(T)$ are finite; in this case $a(T) = d(T) = p$ and by [11, Theorem 7.9], we have $X = N(T^p) \oplus R(T^p)$.
For a closed linear operator A, we say that A is Drazin invertible if there exists an operator $A^D \in \mathcal{B}(X)$ such that $R(A^D) \subset D(A)$, $R(I - AB) \subset D(A)$, $A^DAA^D = A^D$, $A^DA = AA^D$ and $A(I - AA^D)$ is nilpotent. Moreover, from [6], A is Drazin invertible if and only if $A = A_1 \oplus A_2$ such that A_1 is closed and invertible and A_2 is bounded and nilpotent. The ascent, descent and Drazin spectra are defined by

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : a(\lambda - T) = +\infty\},$$

$$\sigma_{dsc}(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = +\infty\},$$

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Drazin invertible}\}.$$

An operator $T \in \mathcal{B}(X)$ is called Fredholm operator, in symbol $T \in \Phi(X)$, if $\delta(T) = dimN(T)$ and $\beta(T) = codimT(X)$ are finite. We say that an operator

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$T \in \mathcal{B}(X)$ is Browder, in symbol $T \in \mathcal{B}r(X)$, if T is Fredholm operator and has finite both ascent and descent. The essential and Browder spectra are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(X)\}.$$

$$\sigma_B(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}r(X)\}.$$

Let $\beta \geq -1$ and f be a continuous function. The convolution $j_\beta * f$ is defined for all $t \geq 0$ by

$$j_\beta * f(t) = \begin{cases} \int_0^t \frac{(t-s)^\beta}{\Gamma(\beta+1)} f(s)ds & \text{if } \beta > -1, \\ \int_0^t f(t-s)d\delta_0(s) & \text{if } \beta = -1, \end{cases}$$

where Γ is the Euler integral giving by $\Gamma(\beta+1) = \int_0^{+\infty} x^\beta e^{-x}dx$, $j_{-1} = \delta_0$ the Dirac measure and for all $\beta > -1$

$$\begin{array}{cccc} j_\beta & : &]0,+\infty[& \rightarrow & \mathbb{R} \\ & & t & \mapsto & \frac{t^\beta}{\Gamma(\beta+1)}. \end{array}$$

Let $\alpha \geq 0$. A strongly continuous $S(t)_{t\geq 0} \subseteq \mathcal{B}(X)$ is called an α-times integrated semigroup, if $S(0) = 0$ and for all $t, s \geq 0$

$$S_n(t)S_n(s) = \int_t^{t+s} \frac{(s+t-r)^{n-1}}{\Gamma(n)} S_n(r)dr - \int_0^s \frac{(s+t-r)^{n-1}}{\Gamma(n)} S_n(r)dr, \qquad (*)$$

where $n-1 < \alpha \leq n$ and $S_n(t)(x) = (j_{n-\alpha-1} * S)(x)$ for all $x \in X$.
By (*) we deduce for all $t, s \geq 0$

$$S(t)S(s) = S(s)S(t).$$

Conversely, let $\alpha \geq 0$ and let A be a linear operator on a Banach space X.
We recall that A is the generator of an α-times integrated semigroup [4] if for some $\omega \in \mathbb{R}$ we have $]\omega, +\infty[\subseteq \rho(A)$ and there exists a strongly continuous mapping $S : [0, +\infty[\rightarrow \mathcal{B}(X)$ satisfying

$$\begin{aligned} \|S(t)\| &\leq Me^{\omega t} \text{ for all } t \geq 0 \text{ and some } M > 0 \\ R(\lambda, A) &= \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} S(t)ds \text{ for all } \lambda > \max\{\omega, 0\}, \end{aligned}$$

in this case, $(S(t))_{t\geq 0}$ is called the α-times integrated semigroup and the domain of its generator A is defined by

$$D(A) = \{x \in X : \int_0^t S(s)Axds = S(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)}\}$$

and $(S(t))_{t\geq 0}$, from the uniqueness Theorem of Laplace transforms, is uniquely determined. In particulary, a C_0-semigroup and an integrated semigroup are also a 0-times integrated semigroup and an 1-times integrated semigroup, respectively.

An important example of generators of an $\alpha-$times integrated semigroups is the adjoint A^* on X^* for all $\alpha > 0$ where A is the generator of a C_0-semigroup on a Banach space X. In [10], the authors have studied the different spectra of the 1-times integrated semigroups. In this paper, we study α-times integrated semigroups for all $\alpha > 0$. We investigate the relationships between the different spectra of α-times integrated semigroups and their generators, precisely the ordinary, point, Fredholm, ascent, descent, Browder and Drazin spectra.

2. Main results

Lemma 2.1. *[4, Proposition 2.4] Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ where $\alpha \geq 0$. Then for all $x \in D(A)$ and all $t \geq 0$ we have*

1. *$S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*
2. *$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t S(s)Axds$.*

Moreover, for all $x \in X$ we get $\int_0^t S(s)xds \in D(A)$ and

$$A\int_0^t S(s)xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Now, we give the following lemma.

Lemma 2.2. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ where $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$ and all $t \geq 0$, we have*

1. *For all $x \in D(A)$*

$$D_\lambda(t)(\lambda - A)x = \Big[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)\Big]x.$$

where

$$D_\lambda(t)x = \int_0^t e^{\lambda(t-s)}S(s)xds.$$

2. *For all $x \in X$, $D_\lambda(t)x \in D(A)$ and*

$$(\lambda - A)D_\lambda(t)x = \Big[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)\Big]x.$$

Proof. 1. By Lemma 2.1, we know that for all $x \in D(A)$

$$S(s)x = \frac{s^\alpha}{\Gamma(\alpha+1)}x + \int_0^s S(r)Axdr.$$

Then, since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we obtain

$$S'(s)x = \frac{s^{\alpha-1}}{\Gamma(\alpha)}x + S(s)Ax.$$

Therefore, we conclude that

$$\begin{aligned} D_\lambda(t)Ax &= \int_0^t e^{\lambda(t-s)} S(s)Axds \\ &= \int_0^t e^{\lambda(t-s)} [S'(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}x]ds \\ &= \int_0^t e^{\lambda(t-s)} S'(s)xds - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)}xds \\ &= S(t)x + \lambda D_\lambda(t)x - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)}xds \end{aligned}$$

Finally, we obtain for all $x \in D(A)$

$$D_\lambda(t)(\lambda - A)x = \big[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)\big]x.$$

2. Let $\mu \in \rho(A)$. From proof of Lemma 2.1, we have for all $x \in X$

$$R(\mu, A)S(s)x = S(s)R(\mu, A)x.$$

Hence, for all $x \in X$ we conclude

$$\begin{aligned} R(\mu, A)D_\lambda(t)x &= R(\mu, A) \int_0^t e^{\lambda(t-s)} S(s)xds \\ &= \int_0^t e^{\lambda(t-s)} R(\mu, A)S(s)xds \\ &= \int_0^t e^{\lambda(t-s)} S(s)R(\mu, A)xds \\ &= D_\lambda(t)R(\mu, A)x. \end{aligned}$$

Therefore, we obtain for all $x \in X$

$$\begin{aligned} D_\lambda(t)x =& \int_0^t e^{\lambda(t-s)} S(s)xds \\ =& \int_0^t e^{\lambda(t-s)} S(s)(\mu - A)R(\mu, A)xds \\ =& \mu \int_0^t e^{\lambda(t-s)} S(s)R(\mu, A)xds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)xds \\ =& \mu \int_0^t e^{\lambda(t-s)} R(\mu, A)S(s)xds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)xds \\ =& \mu R(\mu, A) \int_0^t e^{\lambda(t-s)} S(s)xds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)xds \\ =& \mu R(\mu, A)D_\lambda(t)x - D_\lambda(t)AR(\mu, A)x \end{aligned}$$

$$
\begin{aligned}
&= \mu R(\mu, A) D_\lambda(t)x - [S(t)R(\mu, A)x + \lambda D_\lambda(t) R(\mu, A)x \\
&\quad - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} R(\mu, A)x ds] \\
&= \mu R(\mu, A) D_\lambda(t)x - [R(\mu, A)S(t)x + \lambda R(\mu, A) D_\lambda(t)x \\
&\quad - R(\mu, A) \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds] \\
&= R(\mu, A)\big[(\mu - \lambda) D_\lambda(t)x - S(t)x + \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds\big]
\end{aligned}
$$

Therefore, for all $x \in X$ we have $D_\lambda(t)x \in D(A)$ and

$$
(\mu - A) D_\lambda(t)x = (\mu - \lambda) D_\lambda(t)x + \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds - S(t)x.
$$

Finally, for all $x \in X$ and all $\lambda \in \mathbb{C}$ we obtain

$$
(\lambda - A) D_\lambda(t)x = \big[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\big]x.
$$

□

Thanks to Lemma 2.2, we obtain automatically the next corollaries.

Corollary 2.3. *[10, Lemma 2.1] Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $\lambda \in \mathbb{C}$, $t \geq 0$ and $x \in D(A)$*

$$
D_\lambda(t)(\lambda - A)x = \big[\int_0^t e^{\lambda(t-s)} ds - S(t)\big]x.
$$

Moreover, for all $x \in X$ we have

$$
(\lambda - A) D_\lambda(t)x = \big[\int_0^t e^{\lambda(t-s)} ds - S(t)\big]x.
$$

Corollary 2.4. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ and $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $t \geq 0$ we have*

1. *For all $x \in X$*

$$
(\lambda - A)^n [D_\lambda(t)]^n x = [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1} x}{\Gamma(\alpha)} ds - S(t)]^n x.
$$

2. *For all $x \in D(A^n)$*

$$
[D_\lambda(t)]^n (\lambda - A)^n x = [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1} x}{\Gamma(\alpha)} ds - S(t)]^n x.
$$

3. $N[\lambda - A] \subseteq N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]$.
4. $R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \subseteq R[\lambda - A]$.
5. $N[\lambda - A]^n \subseteq N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n$.
6. $R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n \subseteq R[\lambda - A]^n$.
7. $R^\infty[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \subseteq R^\infty[\lambda - A]$.

In the upcoming theorem, we characterize the different spectra of the α-times integrated semigroups.

Theorem 2.5. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ and $\alpha > 0$. Then for all $t \geq 0$*

1. $\int_0^t e^{(t-s)\sigma(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma(S(t))$.
2. $\int_0^t e^{(t-s)\sigma_p(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_p(S(t))$.
3. $\int_0^t e^{(t-s)\sigma_e(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_e(S(t))$.

Proof.

1. Let $\lambda \in \mathbb{C}$ such that for all $t \geq 0$
$$\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma(S(t)),$$
then the operator $\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)$ is invertible with $F_\lambda(t)$ its inverse. Using Lemma 2.2, we obtain for every $x \in D(A)$
$$\begin{aligned} x &= F_\lambda(t)[\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]x \\ &= F_\lambda(t)[D_\lambda(t)(\lambda - A)]x \\ &= [F_\lambda(t)D_\lambda(t)](\lambda - A)x. \end{aligned}$$
On the other hand, also from Lemma 2.2, we obtain for every $x \in X$
$$\begin{aligned} x &= [\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]F_\lambda(t)x; \\ &= [(\lambda - A)D_\lambda(t)]F_\lambda(t)x; \\ &= (\lambda - A)[D_\lambda(t)F_\lambda(t)]x. \end{aligned}$$
Since we know that $S(t)F_\lambda(t) = F_\lambda(t)S(t)$, then
$$F_\lambda(t)D_\lambda(t) = D_\lambda(t)F_\lambda(t).$$
Finally, we conclude that $\lambda - A$ is invertible and hence $\lambda \notin \sigma(A)$.

2. Let $\lambda \in \sigma_p(A)$, then there exists $x \neq 0$ such that

$$x \in N(\lambda - A).$$

From Corollary 2.4, we get

$$x \in N[\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)].$$

Therefore we conclude that

$$\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \in \sigma_p(S(t)).$$

3. Let $\lambda \in \mathbb{C}$ such that

$$\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_e(S(t)).$$

Then we have

$$\delta[\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] < +\infty \text{ and } \beta[\int_0^t e^{(t-s)\lambda} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] < +\infty.$$

Therefore, by Corollary 2.4, we conclude that

$$\delta[\lambda - A] < +\infty \text{ and } \beta[\lambda - A] < +\infty,$$

and hence

$$\lambda \notin \sigma_e(A).$$

□

The important following lemma concerning the α-times integrated semigroups.

Lemma 2.6. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ where $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$, all $t \geq 0$ and all $x \in X$*

1. *We have the identity*

$$(\lambda - A)L_\lambda(t) + \varphi_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I,$$

where $L_\lambda(t) = \int_0^t e^{-\lambda s} D_\lambda(s)ds$, $\varphi_\lambda(t) = e^{\lambda t}$ and $\phi_\lambda(t) = \int_0^t \int_0^\tau e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} dr d\tau$. Moreover, the operator $L_\lambda(t)$ is commute with each one of $D_\lambda(t)$ and $(\lambda - A)$.

2. *For all $n \in \mathbb{N}^*$, there exists an $L_{\lambda,n}(t) \in \mathcal{B}(X)$ such that*

$$(\lambda - A)L_{\lambda,n}(t) + [\varphi_\lambda(t)]^n [D_\lambda(t)]^n = [\phi_\lambda(t)]^n I.$$

Moreover, the operator $L_{\lambda,n}(t)$ is commute with each one of $D_\lambda(t)$ and $\lambda - A$.

3. *For all $n \in \mathbb{N}^*$, there exists an operator $D_{\lambda,n}(t) \in \mathcal{B}(X)$ such that*

$$(\lambda - A)^n [L_\lambda(t)]^n + D_{\lambda,n}(t) D_\lambda(t) = [\phi_\lambda(t)]^n I.$$

Moreover, the operator $D_{\lambda,n}(t)$ is commute with each one of $D_\lambda(t)$, $L_\lambda(t)$ and $\lambda - A$.

4. *For all $n \in \mathbb{N}^*$, there exists an operator $K_{\lambda,n}(t) \in \mathcal{B}(X)$ such that*

$$(\lambda - A)^n K_{\lambda,n}(t) + [D_{\lambda,n}(t)]^n [D_\lambda(t)]^n = [\phi_\lambda(t)]^{n^2} I,$$

Moreover, the operator $K_{\lambda,n}(t)$ is commute with each one of $D_\lambda(t)$, $D_{\lambda,n}(t)$ and $\lambda - A$.

Proof. 1. Let $\mu \in \rho(A)$. By Lemma 2.2, for all $x \in X$ we have $D_\lambda(s)x \in D(A)$ and hence

$$\begin{aligned} L_\lambda(t)x &= \int_0^t e^{-\lambda s} D_\lambda(s) x ds \\ &= \int_0^t e^{-\lambda s} R(\mu, A)(\mu - A) D_\lambda(s) x ds \\ &= R(\mu, A)[\mu \int_0^t e^{-\lambda s} D_\lambda(s) x ds - \int_0^t e^{-\lambda s} A D_\lambda(s) x ds] \\ &= R(\mu, A)[\mu L_\lambda(t) x - \int_0^t e^{-\lambda s} A D_\lambda(s) x ds] \end{aligned}$$

Therefore for all $x \in X$, we have $L_\lambda(t)x \in D(A)$ and

$$(\mu - A) L_\lambda(t) x = \mu L_\lambda(t) x - \int_0^t e^{-\lambda s} A D_\lambda(s) x ds.$$

Thus

$$A L_\lambda(t) x = \int_0^t e^{-\lambda s} A D_\lambda(s) x ds.$$

Hence, we conclude that

$$
\begin{aligned}
(\lambda - A)L_\lambda(t)x =& \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s} A D_\lambda(s) x ds \\
=& \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}\big[\lambda D_\lambda(s)x \\
& - \int_0^s e^{\lambda(s-r)} \frac{r^{\alpha-1}}{\Gamma(\alpha)} x dr + S(s)x\big] ds \\
=& \lambda L_\lambda(t)x - \lambda \int_0^t e^{-\lambda s} D_\lambda(s) x ds \\
& + \int_0^t e^{-\lambda s} \int_0^s e^{\lambda(s-r)} \frac{r^{\alpha-1}}{\Gamma(\alpha)} x dr ds - \int_0^t e^{-\lambda s} S(s) x ds \\
=& \lambda L_\lambda(t)x - \lambda L_\lambda(t)x + \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} x dr ds \\
& - e^{-\lambda t} \int_0^t e^{\lambda(t-s)} S(s) x ds \\
=& \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} x dr ds - e^{-\lambda t} D_\lambda(t) x \\
=& \big[\phi_\lambda(t) I - \varphi_\lambda(t) D_\lambda(t)\big] x,
\end{aligned}
$$

where $\phi_\lambda(t) = \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} dr ds$ and $\varphi_\lambda(t) = e^{-\lambda t}$.
Therefore, we obtain

$$(\lambda - A)L_\lambda(t) + \varphi_\lambda(t) D_\lambda(t) = \phi_\lambda(t) I.$$

Since $S(s)S(t) = S(t)S(s)$ for all $s, t \geq 0$, then $D_\lambda(s)S(t) = S(t)D_\lambda(s)$.
Hence

$$
\begin{aligned}
D_\lambda(t) D_\lambda(s) &= \int_0^t e^{\lambda(t-r)} S(r) D_\lambda(s) dr \\
&= \int_0^t e^{\lambda(t-r)} S(r) D_\lambda(s) dr \\
&= \int_0^t e^{\lambda(t-r)} D_\lambda(s) S(r) dr \\
&= D_\lambda(s) \int_0^t e^{\lambda(t-r)} S(r) dr \\
&= D_\lambda(s) D_\lambda(t).
\end{aligned}
$$

Thus, we deduce that

$$\begin{aligned} D_\lambda(t)L_\lambda(t) &= D_\lambda(t)\int_0^t e^{-\lambda s}D_\lambda(s)ds \\ &= \int_0^t e^{-\lambda s}D_\lambda(t)D_\lambda(s)ds \\ &= \int_0^t e^{-\lambda s}D_\lambda(s)D_\lambda(t)ds \\ &= \int_0^t e^{-\lambda s}D_\lambda(s)dsD_\lambda(t) \\ &= L_\lambda(t)D_\lambda(t). \end{aligned}$$

Since for all $x \in X$ $AL_\lambda(t)x = \int_0^t e^{-\lambda s}AD_\lambda(s)xds$ and for all $x \in D(A)$ $AD_\lambda(s)x = D_\lambda(s)Ax$, then we obtain for all $x \in D(A)$

$$\begin{aligned} (\lambda - A)L_\lambda(t)x &= \lambda L_\lambda(t)x - AL_\lambda(t)x \\ &= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}AD_\lambda(s)xds \\ &= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}AD_\lambda(s)xds \\ &= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}D_\lambda(s)Axds \\ &= \lambda L_\lambda(t)x - L_\lambda(t)Ax \\ &= L_\lambda(t)(\lambda - A)x. \end{aligned}$$

2. Since $(\lambda - A)L_\lambda(t) + \varphi_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$, then for all $n \in \mathbb{N}^*$ we obtain

$$\begin{aligned} [\varphi_\lambda(t)D_\lambda(t)]^n &= [\phi_\lambda(t)I - (\lambda - A)L_\lambda(t)]^n \\ &= \sum_{i=0}^{n} C_n^i[\phi_\lambda(t)]^{n-i}[-(\lambda - A)L_\lambda(t)]^i \\ &= [\phi_\lambda(t)]^n I - (\lambda - A)\sum_{i=1}^{n} C_n^i[\phi_\lambda(t)]^{n-i}[-(\lambda - A)]^{i-1}[L_\lambda(t)]^i \\ &= [\phi_\lambda(t)]^n I - (\lambda - A)L_{\lambda,n}(t), \end{aligned}$$

where

$$L_{\lambda,n}(t) = \sum_{i=1}^{n} C_n^i[\phi_\lambda(t)]^{n-i}[-(\lambda - A)]^{i-1}[L_\lambda(t)]^i.$$

Therefore, we have

$$(\lambda - A)L_{\lambda,n}(t) + [\varphi_\lambda(t)]^n[D_\lambda(t)]^n = [\phi_\lambda(t)]^n I.$$

Finally, for commutativity, it is clear that $L_{\lambda,n}(t)$ commute with each one of $D_\lambda(t)$ and $\lambda - A$.

3. For all $n \in \mathbb{N}^*$, we obtain

$$\begin{aligned}[(\lambda - A)L_\lambda(t)]^n &= [\phi_\lambda(t)I - \varphi_\lambda(t)D_\lambda(t)]^n \\ &= \sum_{i=0}^{n} C_n^i [\phi_\lambda(t)]^{n-i}[-\varphi_\lambda(t)D_\lambda(t)]^i \\ &= [\phi_\lambda(t)]^n I - D_\lambda(t)\sum_{i=1}^{n} C_n^i [\phi_\lambda(t)]^{n-i}[\varphi_\lambda(t)]^i[-D_\lambda(t)]^{i-1} \\ &= [\phi_\lambda(t)]^n I - D_\lambda(t)D_{\lambda,n}(t),\end{aligned}$$

where

$$D_{\lambda,n}(t) = \sum_{i=1}^{n} C_n^i [\phi_\lambda(t)]^{n-i}[\varphi_\lambda(t)]^i[-D_\lambda(t)]^{i-1}.$$

Therefore, we have

$$(\lambda - A)^n[L_\lambda(t)]^n + D_\lambda(t)D_{\lambda,n}(t) = [\phi_\lambda(t)]^n I.$$

Finally, for commutativity, it is clear that $D_{\lambda,n}(t)$ commute with each one of $D_\lambda(t)$, $L_\lambda(t)$ and $\lambda - A$.

4. Since we have $D_\lambda(t)D_{\lambda,n}(t) = [\phi_\lambda(t)]^n I - (\lambda - A)^n[L_\lambda(t)]^n$, then for all $n \in \mathbb{N}$

$$\begin{aligned}[D_\lambda(t)D_{\lambda,n}(t)]^n &= \big[[\phi_\lambda(t)]^n I - (\lambda - A)^n[L_\lambda(t)]^n\big]^n \\ &= [\phi_\lambda(t)]^{n^2} I - \sum_{i=1}^{n} C_n^i \big[[\phi_\lambda(t)]^n\big]^{n-i}\big[(\lambda - A)^n[L_\lambda(t)]^n\big]^i \\ &= [\phi_\lambda(t)]^{n^2} I \\ &\quad -(\lambda - A)^n \sum_{i=1}^{n} C_n^i \big[[\phi_\lambda(t)]^{n(n-i)}(\lambda - A)^{n(i-1)}[L_\lambda(t)]^{ni} \\ &= [\phi_\lambda(t)]^{n^2} I - (\lambda - A)^n K_{\lambda,n}(t),\end{aligned}$$

where $K_{\lambda,n}(t) = \sum_{i=1}^{n} C_n^i[\phi_\lambda(t)]^{n(n-i)}(\lambda - A)^{n(i-1)}[L_\lambda(t)]^{ni}$. Hence we obtain

$$[D_\lambda(t)]^n[D_{\lambda,n}(t)]^n + (\lambda - A)^n K_{\lambda,n}(t) = [\phi_\lambda(t)]^{n^2} I.$$

Finally, the commutativity is clear.

□

Now, we interest to the relation between the ascent and the descent of the α-times integrated semigroups and their generators.

Proposition 2.7. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ and $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$ and all $t \geq 0$, we have*

1. *$d[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha+1}}{\Gamma(\alpha+)} ds - S(t)] = n$, then $d[\lambda - A] \leq n$.*

2. *$a[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)] = n$, then $a[\lambda - A] \leq n$.*

Proof.

1. Let $y \in R[\lambda - A]^n$, then there exists $x \in D(A^n)$ satisfying

$$(\lambda - A)^n x = y.$$

Since $d[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)] = n$, therefore

$$R[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)]^n = R[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)]^{n+1}.$$

Hence there exists $z \in X$ such that

$$[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)]^n x = \int_0^t e^{\lambda(t-s)}\frac{s^{\alpha+1}}{\Gamma(\alpha)}ds - S(t)]^{n+1} z.$$

On the other hand, by Lemma 2.6, we have

$$(\lambda - A)L_{\lambda,n}(t) + [\varphi_\lambda(t)]^n[D_\lambda(t)]^n = [\phi_\lambda(t)]^n I,$$

with $L_{\lambda,n}(t)$, $D_\lambda(t)$ and $(\lambda - A)$ are pairwise commute.
Thus, we have

$$\begin{aligned}
[\phi_\lambda(t)]^n y &= (\lambda - A)^n[\phi_\lambda(t)]^n x \\
&= (\lambda - A)^n[(\lambda - A)L_{\lambda,n}(t) + [\varphi_\lambda(t)]^n[D_\lambda(t)]^n]x \\
&= (\lambda - A)^n(\lambda - A)L_{\lambda,n}(t)x + [\varphi_\lambda(t)]^n(\lambda - A)^n[D_\lambda(t)^n]x \\
&= (\lambda - A)^{n+1}L_{\lambda,n}(t)x + [\varphi_\lambda(t)]^n[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n x \\
&= (\lambda - A)^{n+1}L_{\lambda,n}(t)x + [\varphi_\lambda(t)]^n[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^{n+1} z \\
&= (\lambda - A)^{n+1}L_{\lambda,n}(t)x + [\varphi_\lambda(t)]^n[(\lambda - A)^{n+1}[D_\lambda(t)]^{n+1} z] \\
&= (\lambda - A)^{n+1}[L_{\lambda,n}(t)x + [\varphi_\lambda(t)]^n[D_\lambda(t)]^{n+1} z].
\end{aligned}$$

Since $\phi_\lambda(t) \neq 0$ for $t > 0$, we conclude that $y \in R[\lambda - A]^{n+1}$ and hence

$$R[\lambda - A]^n = R[\lambda - A]^{n+1}.$$

Finally, we conclude that

$$d(\lambda - A) \leq n.$$

2. Let $x \in N(\lambda - A)^{n+1}$ and we suppose that $a[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)] = n$, then we obtain

$$N[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n = N[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^{n+1}.$$

From Corollary 2.4, we have

$$N(\lambda - A)^{n+1} \subseteq N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^{n+1},$$

hence

$$x \in N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n.$$

Thus, we have

$$\begin{aligned}
[\phi_\lambda(t)]^n(\lambda - A)^n x &= (\lambda - A)^n[(\lambda - A)L_{\lambda,n}(t) + [\varphi_\lambda(t)]^n[D_\lambda(t)]^n]x; \\
&= (\lambda - A)^n(\lambda - A)L_{\lambda,n}(t)x \\
&\quad +[\varphi_\lambda(t)]^n(\lambda - A)^n[D_\lambda(t)]^n x \\
&= (\lambda - A)^{n+1}L_{\lambda,n}(t)x \\
&\quad +[\varphi_\lambda(t)]^n[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n x \\
&= (\lambda - A)^{n+1}L_{\lambda,n}(t)x \\
&= L_{\lambda,n}(t)(\lambda - A)^{n+1}x \\
&= 0.
\end{aligned}$$

Therefore, since $\phi_\lambda(t) \neq 0$ for $t > 0$, we obtain $x \in N(\lambda - A)^n$ and hence

$$a(\lambda - A) \leq n.$$

□

Finally, we characterize the different spectra of the α-times integrated semigroups using the spectra of their generators.

Theorem 2.8. *Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ where $\alpha > 0$. Then for all $t \geq 0$*

1. $\int_0^t e^{(t-s)\sigma_{asc}(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_{asc}(S(t))$.
2. $\int_0^t e^{(t-s)\sigma_{dsc}(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_{dsc}(S(t))$.
3. $\int_0^t e^{(t-s)\sigma_B(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_B(S(t))$.
4. $\int_0^t e^{(t-s)\sigma_D(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_D(S(t))$.

Proof.

1. Let $\lambda \in \mathbb{C}$ such that
$$\int_0^t e^{(t-s)\lambda}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds \notin \sigma_{asc}(S(t)).$$
Then there is $n \in \mathbb{N}$ satisfying
$$a[\int_0^t e^{(t-s)\lambda}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)] = n.$$
Therefore, by Proposition 2.7, we obtain $a[\lambda - A] \leq n$ and hence
$$\lambda \notin \sigma_{asc}(A).$$

2. Let $\lambda \in \mathbb{C}$ such that
$$\int_0^t e^{(t-s)\lambda}\frac{s^{\alpha}}{\Gamma(\alpha+1)}ds \notin \sigma_{dsc}(S(t)),$$
then there is $n \in \mathbb{N}$ satisfying
$$d[\int_0^t e^{(t-s)\lambda}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)] = n.$$
Therefore, by Proposition 2.7, we obtain $d[\lambda - A] \leq n$ and hence
$$\lambda \notin \sigma_{dsc}(A).$$

3. It is automatically, by the previous assertions and Theorem 2.5.

4. Suppose that $\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)$ is Drazin invertible, then
$$a[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)] = d[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)] = n$$
and we have
$$X = N[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n \oplus R[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n.$$
Let $x \in N[\lambda - A]^n \cap R[\lambda - A]^n$, then $x \in N[\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n$ and there exists $y \in X$ such that $x = (\lambda - A)^n(y)$. From Lemma 2.6, we obtain
$$\begin{aligned}
[\phi_\lambda(t)]^{n^2}x &= (\lambda - A)^n[\phi_\lambda(t)]^{n^2}y \\
&= (\lambda - A)^n\left[[D_\lambda(t)]^n[D_{\lambda,n}(t)]^n y + (\lambda - A)^n K_{\lambda,n}(t)y\right] \\
&= [D_{\lambda,n}(t)]^n(\lambda - A)^n[D_\lambda(t)]^n y + K_{\lambda,n}(t)(\lambda - A)^n(\lambda - A)^n y] \\
&= [D_{\lambda,n}(t)]^n(\lambda - A)^n[D_\lambda(t)]^n y + K_{\lambda,n}(t)(\lambda - A)^n x] \\
&= (\lambda - A)^n[D_\lambda(t)]^n[D_{\lambda,n}(t)]^n y \\
&= [\int_0^t e^{\lambda(t-s)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds - S(t)]^n[D_{\lambda,n}(t)]^n y,
\end{aligned}$$

which implies that

$$x \in R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n.$$

Therefore, we have

$$x \in N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n \cap R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n = \{0\},$$

and hence

$$N(\lambda - A)^n \cap R(\lambda - A)^n = \{0\}.$$

Now let $x \in X$, then by supposition there exist $x, y \in X$ such that $x = y + z$, $y \in N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n$ and $z \in R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n$.
So $z \in R(\lambda - A)^n$ and by Lemma 2.6, we obtain

$$\begin{aligned} (\lambda - A)^n \left[[\phi_\lambda(t)]^{n^2} y\right] &= (\lambda - A)^n \left[[D_\lambda(t)]^n [D_{\lambda,n}(t)]^n y \right. \\ &\quad \left. + (\lambda - A)^n K_{\lambda,n}(t) y\right] \\ &= [D_{\lambda,n}(t)]^n (\lambda - A)^n [D_\lambda(t)]^n y \\ &\quad + (\lambda - A)^n (\lambda - A)^n K_{\lambda,n}(t) y \\ &= [D_{\lambda,n}(t)]^n [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^n y \\ &\quad + (\lambda - A)^n (\lambda - A)^n K_{\lambda,n}(t) y \\ &= (\lambda - A)^n (\lambda - A)^n K_{\lambda,n}(t) y. \end{aligned}$$

Therefore, we deduce

$$(\lambda - A)^n \left[[\phi_\lambda(t)]^{n^2} y - (\lambda - A)^n K_{\lambda,n}(t) y\right] = 0.$$

Then, we obtain

$$u = [\phi_\lambda(t)]^{n^2} y - (\lambda - A)^n K_{\lambda,n}(t) y \in N(\lambda - A)^n,$$

it follows that

$$[\phi_\lambda(t)]^{n^2} y = v + u,$$

where $v = (\lambda - A)^n K_{\lambda,n}(t) y \in R(\lambda - A)^n$ and $u \in N(\lambda - A)^n$.
Finally we get for $t \neq 0$,

$$x = y + z = \frac{u + v}{[\phi_\lambda(t)]^{n^2}} + z = u' + v',$$

where $u' = \frac{u}{[\phi_\lambda(t)]^{n^2}} \in N(\lambda - A)^n$ and $v' = \frac{v}{[\phi_\lambda(t)]^{n^2}} + z \in R(\lambda - A)^n$.
Therefore, we deduce that

$$X = N(\lambda - A)^n \oplus R(\lambda - A)^n.$$

Finally, it is clear that $(\lambda - A)_{|N(\lambda - A)^n}$ is nilpotent and since $R(\lambda - A)^n \cap N(\lambda - A)^n = \{0\}$, then $(\lambda - A)_{|R(\lambda - A)^n}$ is invertible and hence $\lambda - A$ is Drazin invertible.

□

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