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On Exact Sequences and Strict Bounded Group Homomorphisms

Dinamérico P. Pombo Jr.

ABSTRACT: Necessary and sufficient conditions for the exactness (in the algebraic sense) of certain sequences of bounded group homomorphisms are established.

Key Words: Bornological groups, Bounded group homomorphisms, Exact sequences.

Contents

1. Introduction

Two fundamental theorems of Linear Algebra [1, p.57 and p.59] assert that the exactness of certain sequences of linear mappings between modules is equivalent to the exactness of sequences of group homomorphisms between the corresponding abelian groups of linear mappings. In this work we prove the validity of analogous results in the context of bornological groups [2]. More precisely, we show that the notion of a strict bounded group homomorphism is the key ingredient which allows us to establish the equivalence between the exactness (in the algebraic sense) of certain sequences of bounded group homomorphisms and the exactness of sequences of group homomorphisms between the corresponding groups of bounded group homomorphisms. It should also be mentioned that the exactness (in the algebraic sense) of sequences of bounded group homomorphisms has already been discussed in [2].

2. Preliminaries

In this work the identity element of any group and a group reduced to its identity element will be denoted by e. For arbitrary groups B and C, the group homomorphism $x \in B \mapsto e \in C$ will also be denoted by e; and, if $u \colon B \to C$ is an arbitrary group homomorphism, its kernel (resp. image) will be represented by $\mathrm{Ker}(u)$ (resp. $\mathrm{Im}(u)$).

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A pair $(B, \mathcal{B})$ consisting of a group B and a bornology $\mathcal{B}$ on B is a bornological group if the mapping

$$(x, y) \in (B \times B, \mathcal{B} \times \mathcal{B}) \mapsto xy^{-1} \in (B, \mathcal{B})$$

is bounded, where $\mathcal{B} \times \mathcal{B}$ is the product bornology on $B \times B$; examples of bornological groups may be found in [2]. In what follows the group of all bounded group homomorphisms from the bornological group $(B, \mathcal{B})$ into the bornological group $(C, \mathcal{C})$ will be represented by $\mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$; $u \in \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$ is said to be *strict* if the corresponding bounded group isomorphism

$$\overline{u}\colon \big(B/\operatorname{Ker}(u), \mathcal{E}\big) \to \big(\operatorname{Im}(u), \mathcal{C}_{\operatorname{Im}(u)}\big)$$

is a bornological group isomorphism, where $\mathcal{E}$ is the (group) quotient bornology on the quotient group $B/\operatorname{Ker}(u)$ and $\mathcal{C}_{\operatorname{Im}(u)}$ is the (group) bornology induced by $\mathcal{C}$ on $\operatorname{Im}(u)$. For $u \in \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$ and a bornological group $(H, \mathcal{H})$, the group homomorphism

$$\varphi \in \mathrm{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \mapsto u \circ \varphi \in \mathrm{Hom}_b((H, \mathcal{H}), (C, \mathcal{C}))$$

$$(\text{resp. } \psi \in \mathrm{Hom}_b((C, \mathcal{C}), (H, \mathcal{H})) \mapsto \psi \circ u \in \mathrm{Hom}_b((B, \mathcal{B}), (H, \mathcal{H}))$$

will be denoted by u^* (resp. u_*).

3. The results

We shall first establish the following

Theorem 3.1. *Let $(B, \mathcal{B})$, $(C, \mathcal{C})$, $(D, \mathcal{D})$ be bornological groups and consider $u \in \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$ and $v \in \mathrm{Hom}_b((C, \mathcal{C}), (D, \mathcal{D}))$. Then the following conditions are equivalent:*

(a) *u is strict and the sequence*

$$e \to B \xrightarrow{u} C \xrightarrow{v} D$$

of group homomorphisms is exact;

(b) *for each bornological group $(H, \mathcal{H})$, the sequence*

$$e \to \mathrm{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H, \mathcal{H}), (C, \mathcal{C})) \xrightarrow{v^*} \mathrm{Hom}_b((H, \mathcal{H}), (D, \mathcal{D}))$$

of group homomorphisms is exact.

Proof: (a) $\Rightarrow$ (b): Obviously, the exactness of the sequence

$$e \to B \xrightarrow{u} C$$

is equivalent to the injectivity of u.

Let $(H, \mathcal{H})$ be an arbitrary bornological group. If $\varphi \in \operatorname{Ker}(u^*)$, $u \circ \varphi = e$, and the injectivity of u furnishes $\varphi = e$. Thus the sequence

$$e \to \operatorname{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \xrightarrow{u^*} \operatorname{Hom}_b((H, \mathcal{H}), (C, \mathcal{C}))$$

is exact. Let us show the exactness of the sequence

$$\operatorname{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \xrightarrow{u^*} \operatorname{Hom}_b((H, \mathcal{H}), (C, \mathcal{C})) \xrightarrow{v^*} \operatorname{Hom}_b((H, \mathcal{H}), (D, \mathcal{D})).$$

Indeed, since $\operatorname{Im}(u) \subset \operatorname{Ker}(v)$, it follows that $\operatorname{Im}(u^*) \subset \operatorname{Ker}(v^*)$. On the other hand, if $\psi \in \operatorname{Ker}(v^*)$, $v \circ \psi = e$, and hence $\operatorname{Im}(\psi) \subset \operatorname{Ker}(v) = \operatorname{Im}(u)$. Consequently, there is a unique mapping $\varphi \colon H \to B$ such that $\psi = u \circ \varphi$. We claim that $\varphi \in \operatorname{Hom}_b((H, \mathcal{H}), (B, \mathcal{B}))$. In fact, it is clear that φ is a group homomorphism. By the strictness of u, the group isomorphism $u(x) \in \operatorname{Im}(u) \mapsto x \in B$ belongs to $\operatorname{Hom}_b((\operatorname{Im}(u), \mathcal{C}_{\operatorname{Im}(u)}), (B, \mathcal{B}))$; moreover, if we view ψ as a group homomorphism from H into $\operatorname{Im}(u)$, it follows that $\psi \in \operatorname{Hom}_b((H, \mathcal{H}), (\operatorname{Im}(u), \mathcal{C}_{\operatorname{Im}(u)}))$. Thus $\varphi \in \operatorname{Hom}_b((H, \mathcal{H}), (B, \mathcal{B}))$, and hence $\psi = u^*(\varphi) \in \operatorname{Im}(u^*)$. Therefore $\operatorname{Ker}(v^*) \subset \operatorname{Im}(u^*)$, and the equality $\operatorname{Im}(u^*) = \operatorname{Ker}(v^*)$ is established.

(b) $\Rightarrow$ (a): By hypothesis, the sequence

$$e \to \operatorname{Hom}_b((\operatorname{Ker}(u), \mathcal{B}_{\operatorname{Ker}(u)}), (B, \mathcal{B})) \xrightarrow{u^*} \operatorname{Hom}_b((\operatorname{Ker}(u), \mathcal{B}_{\operatorname{Ker}(u)}), (C, \mathcal{C}))$$

is exact, and it is clear that the inclusion mapping $\varphi \colon \operatorname{Ker}(u) \to B$ belongs to $\operatorname{Ker}(u^*)$. Thus $\varphi = e$, which implies the exactness of the sequence

$$e \to B \xrightarrow{u} C.$$

On the other hand, by hypothesis, the sequence

$$\operatorname{Hom}_b((B, \mathcal{B}), (B, \mathcal{B})) \xrightarrow{u^*} \operatorname{Hom}_b((B, \mathcal{B}), (C, \mathcal{C})) \xrightarrow{v^*} \operatorname{Hom}_b((B, \mathcal{B}), (D, \mathcal{D}))$$

is exact. Moreover, $(v^* \circ u^*)(1_B) = v \circ u$. Consequently, $\operatorname{Im}(u) \subset \operatorname{Ker}(v)$. Since, by hypothesis, the sequence

$$\begin{aligned}\operatorname{Hom}_b((\operatorname{Ker}(v), \mathcal{C}_{\operatorname{Ker}(v)}), (B, \mathcal{B})) &\xrightarrow{u^*} \operatorname{Hom}_b((\operatorname{Ker}(v), \mathcal{C}_{\operatorname{Ker}(v)}), (C, \mathcal{C}))\\ &\xrightarrow{v^*} \operatorname{Hom}_b((\operatorname{Ker}(v), \mathcal{C}_{\operatorname{Ker}(v)}), (D, \mathcal{D}))\end{aligned}$$

is exact, and since the inclusion mapping $\psi \colon \operatorname{Ker}(v) \to C$ belongs to

$$\operatorname{Ker}(v^*) = \operatorname{Im}(u^*),$$

there exists a $\varphi \in \operatorname{Hom}_b((\operatorname{Ker}(v), \mathcal{C}_{\operatorname{Ker}(v)}), (B, \mathcal{B}))$ so that $\psi = u \circ \varphi$. Therefore $\operatorname{Ker}(v) \subset \operatorname{Im}(u)$, and the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D$$

is exact.

Finally, let us show that u is strict. Indeed, by hypothesis, the sequence

$$\mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (B, \mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (C, \mathcal{C})) \xrightarrow{v^*} \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (D, \mathcal{D}))$$

is exact. Since the inclusion mapping $\psi\colon \mathrm{Im}(u) \to C$ belongs to $\mathrm{Ker}(v^*) = \mathrm{Im}(u^*)$, there is a $\varphi \in \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (B, \mathcal{B}))$ so that $\psi = u \circ \varphi$. Consequently

$$u(x) = \psi(u(x)) = u(\varphi(u(x)))$$

for all $x \in B$, and the injectivity of u implies $\varphi(u(x)) = x$ for all $x \in B$. Thus the mapping $x \in (B, \mathcal{B}) \mapsto u(x) \in (\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)})$ is a bornological group isomorphism. Hence u is strict, thereby concluding the proof. □

The strictness of u is essential for the validity of the implication (a) ⇒ (b) in Theorem 3.1, as we shall see in the following

Example 3.2. *Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be the discrete (resp. trivial) bornology on the additive group $\mathbb{R}$ of real numbers. Let $u \in \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_1), (\mathbb{R}, \mathcal{B}_2))$ be given by $u(x) = x$ for all $x \in \mathbb{R}$, and let $v \in \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2))$ be given by $v(x) = e$ for all $x \in \mathbb{R}$. It is obvious that u is not strict and that the sequence*

$$e \to \mathbb{R} \xrightarrow{u} \mathbb{R} \xrightarrow{v} \mathbb{R}$$

is exact. Nevertheless, the sequence

$$\mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_1)) \xrightarrow{u^*} \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2)) \xrightarrow{v^*} \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2))$$

is not exact. For, if it were, since $1_{\mathbb{R}} \in \mathrm{Ker}(v^*) = \mathrm{Im}(u^*)$, there would exist a $\varphi \in \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_1))$ so that $u^*(\varphi) = u \circ \varphi = 1_{\mathbb{R}}$; but this would imply $\varphi(x) = x$ for $x \in \mathbb{R}$, which is not bounded as a mapping from $(\mathbb{R}, \mathcal{B}_2)$ into $(\mathbb{R}, \mathcal{B}_1)$.

If B, C are arbitrary groups, $\mathrm{Hom}(B, C)$ will denote the group of all group homomorphisms from B into C.

Corollary 3.3. *Let B, C, D be groups, $u \in \mathrm{Hom}(B, C)$ and $v \in \mathrm{Hom}(C, D)$. Then the following conditions are equivalent:*

(*a*) *the sequence*

$$e \to B \xrightarrow{u} C \xrightarrow{v} D$$

of group homomorphisms is exact;

(*b*) *for each group H, the sequence*

$$e \to \mathrm{Hom}(H, B) \xrightarrow{u^+} \mathrm{Hom}(H, C) \xrightarrow{v^+} \mathrm{Hom}(H, D)$$

of group homomorphisms is exact, where $u^+(\varphi) = u \circ \varphi$ for $\varphi \in \mathrm{Hom}(H, B)$ and $v^+(\psi) = v \circ \psi$ for $\psi \in \mathrm{Hom}(H, C)$.

Proof: For each group G let t_G be the trivial bornology on G. Then

$$\mathrm{Hom}(H,G) = \mathrm{Hom}_b((H,\mathcal{H}),(G,t_G))$$

for every bornological group $(H,\mathcal{H})$; in particular,

$$\mathrm{Hom}(H,G) = \mathrm{Hom}_b((H,t_H),(G,t_G))$$

for every group H.

Let us consider B (resp. C, D) endowed with t_B (resp. t_C, t_D).

(a) $\Rightarrow$ (b): It is obvious that the bounded group homomorphism

$$u\colon (B,t_B) \to (C,t_C)$$

is strict. Therefore, by (a) $\Rightarrow$ (b) of Theorem 3.1, the sequence

$$e \longrightarrow \mathrm{Hom}(H,B) = \mathrm{Hom}_b((H,t_H),(B,t_B)) \xrightarrow{u^+} \mathrm{Hom}(H,C) = \mathrm{Hom}_b((H,t_H),(C,t_C))$$
$$\xrightarrow{v^+} \mathrm{Hom}(H,D) = \mathrm{Hom}_b((H,t_H),(D,t_D))$$

is exact for every group H.

(b) $\Rightarrow$ (a): Let $(H,\mathcal{H})$ be an arbitrary bornological group. By hypothesis, the sequence

$$e \longrightarrow \mathrm{Hom}(H,B) = \mathrm{Hom}_b((H,\mathcal{H}),(B,t_B)) \xrightarrow{u^*} \mathrm{Hom}(H,C) = \mathrm{Hom}_b((H,\mathcal{H}),(C,t_C))$$
$$\xrightarrow{v^*} \mathrm{Hom}(H,D) = \mathrm{Hom}_b((H,\mathcal{H}),(D,t_D))$$

is exact. Therefore, by (b) $\Rightarrow$ (a) of Theorem 3.1, the sequence

$$e \longrightarrow B \xrightarrow{u} C \xrightarrow{v} D$$

is exact. □

Now let us prove the following

Theorem 3.4. *Let $(B,\mathcal{B})$, $(C,\mathcal{C})$, $(D,\mathcal{D})$ be bornological groups and consider $u \in \mathrm{Hom}_b((B,\mathcal{B}),(C,\mathcal{C}))$ and $v \in \mathrm{Hom}_b((C,\mathcal{C}),(D,\mathcal{D}))$. Then the following conditions are equivalent:*

(*a*) *v is strict and the sequence*

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact;

(*b*) *$\mathrm{Im}(u)$ is a normal subgroup of C, $\mathrm{Im}(v)$ is a normal subgroup of D and, for each bornological group $(H,\mathcal{H})$, the sequence*

$$e \to \mathrm{Hom}_b((D,\mathcal{D}),(H,\mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}_b((C,\mathcal{C}),(H,\mathcal{H})) \xrightarrow{u_*} \mathrm{Hom}_b((B,\mathcal{B}),(H,\mathcal{H}))$$

of group homomorphisms is exact.

Proof: (a) $\Rightarrow$ (b): First of all, to say that the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact is equivalent to saying that $\mathrm{Im}(u) = \mathrm{Ker}(v)$ and v is surjective; hence, in this case, $\mathrm{Im}(u)$ is a normal subgroup of C and $\mathrm{Im}(v)$ $(= D)$ is a normal subgroup of D.

Let $(H, \mathcal{H})$ be an arbitrary bornological group. The exactness of the sequence

$$e \to \mathrm{Hom}_b((D, \mathcal{D}), (H, \mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}_b((C, \mathcal{C}), (H, \mathcal{H}))$$

is equivalent to the injectivity of v_*, which follows from the surjectivity of v. Let us show the exactness of the sequence

$$\mathrm{Hom}_b((D, \mathcal{D}), (H, \mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}_b((C, \mathcal{C}), (H, \mathcal{H})) \xrightarrow{u_*} \mathrm{Hom}_b((B, \mathcal{B}), (H, \mathcal{H})).$$

Indeed, since $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$, we get $\mathrm{Im}(v_*) \subset \mathrm{Ker}(u_*)$. On the other hand, let $w \in \mathrm{Ker}(u_*)$. Hence $w \circ u = e$, which implies

$$\mathrm{Ker}(v) = \mathrm{Im}(u) \subset \mathrm{Ker}(w).$$

Put $w'(v(y)) = w(y)$ for $y \in C$; w' is well defined. We claim that w' is a bounded group homomorphism from $(D, \mathcal{D})$ into $(H, \mathcal{H})$. In fact, it is clear that w' is a group homomorphism. Let $R \in \mathcal{D}$ be arbitrary. Since v is strict, Theorem 3.16 of [2] ensures the existence of an $S \in \mathcal{C}$ so that $v^{-1}(R) \subset S\ \mathrm{Ker}(v)$. Consequently,

$$\begin{aligned} w'(R) &= w'(v(v^{-1}(R))) = w(v^{-1}(R)) \subset w(S\ \mathrm{Ker}(v)) \\ &= w(S)w(\mathrm{Ker}(v)) \subset w(S)w(\mathrm{Ker}(w)) = w(S). \end{aligned}$$

Since $w(S) \in \mathcal{H}$, it follows that $w'(R) \in \mathcal{H}$; thus $w' \in \mathrm{Hom}_b((D, \mathcal{D}), (H, \mathcal{H}))$. Finally, $w = v_*(w') \in \mathrm{Im}(v_*)$, and $\mathrm{Ker}(u_*) \subset \mathrm{Im}(v_*)$. Thus $\mathrm{Im}(v_*) = \mathrm{Ker}(u_*)$, and the proof of (b) is concluded.

(b) $\Rightarrow$ (a): Let $\mathcal{E}$ be the quotient bornology on the quotient group $D/\mathrm{Im}(v)$. By hypothesis, the sequence

$$e \to \mathrm{Hom}_b((D, \mathcal{D}), (D/\mathrm{Im}(v), \mathcal{E})) \xrightarrow{v_*} \mathrm{Hom}_b((C, \mathcal{C}), (D/\mathrm{Im}(v), \mathcal{E}))$$

is exact. But this implies the exactness of the sequence

$$C \xrightarrow{v} D \longrightarrow e\,,$$

that is, the surjectivity of v, because the canonical surjection

$$\pi\colon (D, \mathcal{D}) \to (D/\mathrm{Im}(v), \mathcal{E})$$

is bounded (because $\mathcal{E} = \{\pi(R); R \in \mathcal{D}\}$) and belongs to $\mathrm{Ker}(v_*) = e$.

Now, let us prove the exactness of the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D.$$

Indeed, by hypothesis, the sequence

$$\mathrm{Hom}_b((D,\mathcal{D}),(D,\mathcal{D})) \xrightarrow{v_*} \mathrm{Hom}_b((C,\mathcal{C}),(D,\mathcal{D})) \xrightarrow{u_*} \mathrm{Hom}_b((B,\mathcal{B}),(D,\mathcal{D}))$$

is exact, which furnishes

$$e = (u_* \circ v_*)(1_D) = v \circ u.$$

Consequently, $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$. On the other hand, by hypothesis, the sequence

$$\begin{aligned}\mathrm{Hom}_b((D,\mathcal{D}),(C/\mathrm{Im}(u),\mathcal{E})) &\xrightarrow{v_*} \mathrm{Hom}_b((C,\mathcal{C}),(C/\mathrm{Im}(u),\mathcal{E}))\\ &\xrightarrow{u_*} \mathrm{Hom}_b((B,\mathcal{B}),(C/\mathrm{Im}(u),\mathcal{E}))\end{aligned}$$

is exact, where $\mathcal{E}$ is the quotient bornology on the quotient group $C/\mathrm{Im}(u)$. Since the canonical surjection $\xi\colon C \to C/\mathrm{Im}(u)$ belongs to $\mathrm{Ker}(u_*) = \mathrm{Im}(v_*)$, there is a $\psi \in \mathrm{Hom}_b((D,\mathcal{D}),(C/\mathrm{Im}(u),\mathcal{E}))$ so that $\xi = v_*(\psi) = \psi \circ v$. Consequently,

$$\xi(y) = \psi(v(y)) = \psi(e) = e$$

for all $y \in \mathrm{Ker}(v)$, that is, $\mathrm{Ker}(v) \subset \mathrm{Ker}(\xi) = \mathrm{Im}(u)$. Therefore $\mathrm{Im}(u) = \mathrm{Ker}(v)$. Finally,

$$\psi(v(y)) = \xi(y) = y\,\mathrm{Ker}(v)$$

for all $y \in C$, that is, ψ is the inverse of the bounded group isomorphism

$$y\,\mathrm{Ker}(v) \in (C/\mathrm{Ker}(v),\mathcal{E}) \mapsto v(y) \in (D,\mathcal{D}).$$

Thus v is strict, thereby concluding the proof of the theorem. □

As before (recall Example 3.2) it is easily seen that the strictness of v is essential for the validity of the implication (a) ⇒ (b) in Theorem 3.4.

Corollary 3.5. *Let B, C, D be groups, $u \in \mathrm{Hom}(B,C)$ and $v \in \mathrm{Hom}(C,D)$. Then the following conditions are equivalent:*

(*a*) *the sequence*

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact;

(*b*) *$\mathrm{Im}(u)$ is a normal subgroup of C, $\mathrm{Im}(v)$ is a normal subgroup of D and, for each group H, the sequence*

$$e \to \mathrm{Hom}(D,H) \xrightarrow{v_+} \mathrm{Hom}(C,H) \xrightarrow{u_+} \mathrm{Hom}(B,H)$$

of group homomorphisms is exact, where $v_+(\varphi) = \varphi \circ v$ for $\varphi \in \mathrm{Hom}(D,H)$ and $u_+(\psi) = \psi \circ u$ for $\psi \in \mathrm{Hom}(C,H)$.

Proof: For each group G let d_G be the discrete bornology on G. Then

$$\mathrm{Hom}(G,H) = \mathrm{Hom}_b((G,d_G),(H,\mathcal{H}))$$

for every bornological group $(H,\mathcal{H})$; in particular,

$$\mathrm{Hom}(G,H) = \mathrm{Hom}_b((G,d_G),(H,d_H))$$

for every group H.

Let us consider B (resp. C, D) endowed with d_B (resp. d_C, d_D).

(a) $\Rightarrow$ (b): It is obvious that the bounded group homomorphism

$$v\colon (C,d_C) \to (D,d_D)$$

is strict. Therefore, by (a) $\Rightarrow$ (b) of Theorem 3.4, $\mathrm{Im}(u)$ (resp. $\mathrm{Im}(v)$) is a normal subgroup of C (resp. D) and the sequence

$$e \to \mathrm{Hom}(D,H) = \mathrm{Hom}_b((D,d_D),(H,d_H)) \xrightarrow{v_+} \mathrm{Hom}(C,H) = \mathrm{Hom}_b((C,d_C),(H,\mathcal{H}))$$
$$\xrightarrow{u_+} \mathrm{Hom}(B,H) = \mathrm{Hom}_b((B,d_B),(H,\mathcal{H}))$$

is exact for every group H.

(b) $\Rightarrow$ (a): Let $(H,\mathcal{H})$ be an arbitrary bornological group. By hypothesis, the sequence

$$e \to \mathrm{Hom}(D,H) = \mathrm{Hom}_b((D,d_D),(H,\mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}(C,H) = \mathrm{Hom}_b((C,d_C),(H,\mathcal{H}))$$
$$\xrightarrow{u_*} \mathrm{Hom}(B,H) = \mathrm{Hom}_b((B,d_B),(H,\mathcal{H}))$$

is exact. Therefore, by (b) $\Rightarrow$ (a) of Theorem 3.4, the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

is exact. □

Finally we would like to observe that the important part of the first (resp. second) result mentioned in the introduction of our work follows immediately from the implication (a)$\Rightarrow$(b) in Corollary 3.5 (resp. Corollary 3.3).

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Dinamérico P. Pombo Jr.,
Instituto de Matemática e Estatística,
Universidade Federal Fluminense,
Rua Mário Santos Braga, s/n^o,
24020-140 Niterói, RJ Brasil.
E-mail address: dpombo@terra.com.br