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Symmetric Generalized Biderivations on Prime Rings *

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ABSTRACT: The purpose of the present paper is to prove some results concerning symmetric generalized biderivations on prime and semiprime rings which partially extend some results of Vukman [7]. Infact we prove that: let R be a prime ring of characteristic not two and I be a nonzro ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $[\Delta(x,x),\Delta(y,y)] = 0$ for all $x, y \in I$, then one of the following conditions hold

1. R is commutative.
2. Δ acts as a left bimultiplier on R.

Key Words: Prime ring, Symmetric biderivation, Generalized Biderivation.

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1. Introduction

Throughout the paper all ring will be associative. We shall denote by $Z(R)$ the centre of ring R. A ring R is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall write for any pair of elements $x, y \in R$ the commutator $xy - yx$ and anticommutator $x \circ y = xy + yx$. We make extensive use of the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$ and (ii) $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d_a(x) = [a, x]$, for all $x \in R$.

A mapping $D : R \times R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$, where D is a symmetric and biadditive (i.e. additive in both arguments) mapping, is called the trace of D. The trace f of D satisfies the relation $f(x+y) = f(x)+f(y)+2D(x, y)$, for all $x, y \in R$. A biadditive mapping $D : R\times R \longrightarrow R$ is called a biderivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of R, i.e., $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and

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$D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$. Typical examples are mapping of the form $(x, y) \longmapsto \lambda[x, y]$ where $\lambda \in C$, the extended centroid of R (see [4] for details). Such maps are called inner biderivations.

G. Maksa [5] introduced the notion of symmetric biderivations. In the mentioned paper it was shown that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a biderivation on R. As linearizing $[x, f(x)] = 0$ for all $x, y \in R$, $(x, y) \longmapsto [f(x), y]$ is a biderivation.

The concept of generalized symmetric biderivations was introduced in [3]. More precisely, a generalized symmetric biderivation is defined as follows: Let $D : R \times R \longrightarrow R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \longmapsto \Delta(x, y)$ is a generalized derivation of R associated with function D as well as if for every $y \in R$, the map $x \longmapsto \Delta(x, y)$ is a generalized derivation of R associated with D for all $x, y \in R$. It also satisfies that $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$.

Example 1 Let R be a ring. If D is any biderivation of R and $\gamma : R \times R \longrightarrow R$ is a biadditive map such that $\gamma(x, yz) = \gamma(x, y)z$ and $\gamma(xy, z) = \gamma(x, z)y$ for all $x, y, z \in R$, then $D + \gamma$ is a generalized biderivation of R with associated biderivation D.

An additive mapping $h : R \longrightarrow R$ is called left (resp. right) multiplier of R if $h(xy) = h(x)y$ (resp. $h(xy) = xh(y)$) for all $x, y \in R$. A biadditive mapping $\zeta : R \times R \longrightarrow R$ is said to be a left (resp. right) bi-multiplier of R if $\zeta(x, yz) = \zeta(x, y)z$ (resp. $\zeta(xz, y) = x\zeta(z, y)$) for all $x, y, z \in R$. In this paper, we prove some theorems on symmetric generalized biderivations satisfying certain condition on an ideal of prime ring.

2. Preliminaries

We begin with the following lemmas:

Lemma 2.1. *[1] Let R be a prime ring of characteristic different from two and I be a nonzero left ideal of R. If Δ is a symmetric generalized biderivation with associated a biderivation D such that $[\Delta(x, x), x] = 0$ for all $x \in I$, then either R is commutative or Δ acts as a left bimultiplier on I.*

Lemma 2.2. *[7] Let R be a 2-torsion free semiprime ring. Suppose that there exists a symmetric biderivation $D : R \times R \longrightarrow R$ such that $D(f(x), x) = 0$ for all $x \in R$, where f denotes the trace of D. In this case we have $D = 0$.*

Lemma 2.3. *Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $D(\Delta(x,y),z)=0$ for all $x,y,z\in I$, then either R is commutative or $D=0$, moreover Δ acts as a left bimultiplier on R..*

Proof Let

$$D(\Delta(x,y),z)=0 \text{ for all } x,y,z\in I. \tag{2.1}$$

Replace y by yw in (2.1) and using (2.1), we have

$$\Delta(x,y)D(w,z)+yD(D(x,w),z)+D(y,z)D(x,w)=0 \text{ for all } w,x,y,z\in I. \tag{2.2}$$

Substitute ry for y in (2.2) to get

$$\begin{aligned}\Delta(x,r)yD(w,z) &+ rD(x,y)D(w,z)+ryD(D(x,w),z)+rD(y,z)D(x,w)\\ &+ D(r,z)yD(x,w)=0 \text{ for all } w,x,y,z\in I,\ r\in R.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we find

$$\begin{aligned}\Delta(x,r)yD(w,z) &+ rD(x,y)D(w,z)+D(r,z)yD(x,w)\\ &- r\Delta(x,y)D(w,z)=0 \text{ for all } w,x,y,z\in I,\ r\in R.\end{aligned} \tag{2.4}$$

In Particular Take $x=z$ and $r=y$ to obtain

$$\begin{aligned}\Delta(x,y)yD(w,x) &+ yD(x,y)D(w,x)+D(y,x)yD(x,w)\\ &- y\Delta(x,y)D(w,x)=0 \text{ for all } w,x,y\in I.\end{aligned} \tag{2.5}$$

Which yields that $\{\Delta(x,y)y+yD(x,y)+D(y,x)y-y\Delta(x,y)\}D(w,x)=0$ for all $w,x,y\in I$. That is, $\{[\Delta(x,y),y]+D(x,y^2)\}RD(w,x)=0$ for all $w,x,y\in I$. Primeness of R implies that either $\{[\Delta(x,y),y]+D(x,y^2)\}=0$ or $D(w,x)=0$ for all $w,x,y\in I$. Consider for each $y\in I$ the sets $H=\{x\in I:[\Delta(x,y),y]+D(x,y^2)\}$ and $K=\{x\in I:D(w,x)=0\}$. It is clear that H and K are the additive subgroups of I and $(I,+)=(H,+)\cup(K,+)$. But a group cannot be the union of two proper subgroups. Hence we are forced to conclude that for each $y\in I$ either $I=H$ or $I=K$.

Consider the first case

$$[\Delta(x,y),y]+D(x,y^2)=0 \text{ for all } x,y\in I. \tag{2.6}$$

Again Replace x by xy in above equation to find

$$[\Delta(x,y),y]y+[xD(y,y),y]+D(x,y^2)y+xD(y,y^2)=0 \text{ for all } x,y\in I. \tag{2.7}$$

using (2.6) and (2.7), we arrive at

$$x[D(y,y),y]+[x,y]D(y,y)+xD(y,y^2)=0 \text{ for all } x,y\in I. \tag{2.8}$$

Substitute rx for r in (2.8) to get $[r,y]xD(y,y)=0$ for all $x,y\in I, r\in R$. By the primeness of R and $I\neq(0)$, we arrive at either $[r,y]=0$ or $D(y,u)=0$ for all

$y, u \in I$. By using The Group Theory Argument as above we obtain that either R is commutative, as desired or $D(y, u) = 0$ for all $y, u \in I$. We can easily obtain by suitable replacing that $D(r, s) = 0$ for all $r, s \in R$. The last relation gives us that Δ acts as a left bimultiplier on R, as desired.

Lemma 2.4. *Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $\Delta(\Delta(x, y), z) = 0$ for all $x, y, z \in I$, then either $D = 0$ or R is commutative.*

Proof Consider

$$\Delta(\Delta(x, y), z) = 0 \text{ for all } x, y, z \in I. \tag{2.9}$$

Replacing z by zw in (2.9), we find

$$\Delta(\Delta(x, y), z)w + zD(\Delta(x, y), w) = 0 \text{ for all } w, x, y, z \in I. \tag{2.10}$$

In view of (2.9) and (2.10), we get

$$zD(\Delta(x, y), w) = 0 \text{ for all } w, x, y, z \in I. \tag{2.11}$$

Since R is prime, we have $D(\Delta(x, y), w) = 0$ for all $x, y, w \in I$. Application of Lemma 2.3, we get the desired result.

3. Main Theorems

In [7], author prove that the existence of a nonzero symmetric biderivation $D : R \times R \longrightarrow R$, where R is a prime ring of characteristic not two, with the property $D(x, x)x = xD(x, x)$, for all $x \in R$, forces R to be commutative. Ali et.al. [2] and Shujat et. al. [6] extend this result for generalized biderivations of prime rings. Now we obtained partial generalization of previously mentioned results as follows:

Theorem 3.1. *Let R be a prime ring of characteristic not two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $[\Delta(x, x), \Delta(y, y)] = 0$ for all $x, y \in I$, then one of the conditions hold*

1. *R is commutative.*
2. *Δ acts as a left bimultiplier on R.*

Proof Let us suppose

$$[\Delta(x,x),\Delta(y,y)] = 0 \text{ for all } x,y \in I. \tag{3.1}$$

Linearization of (3.1) in x and use Char $R \neq 2$, we have

$$[\Delta(x,w),\Delta(y,y)] = 0 \text{ for all } x,y,w \in I. \tag{3.2}$$

Again linearize (3.2) in y to get

$$[\Delta(x,w),\Delta(y,u)] = 0 \text{ for all } u,x,y,w \in I. \tag{3.3}$$

Substitute xz for x in (3.2) we get

$$\begin{aligned}\Delta(x,w)[z,\Delta(y,u)] &+ [\Delta(x,w),\Delta(y,u)]z + x[D(z,w),\Delta(y,u)]\\ &+ [x,\Delta(y,u)]D(z,w) = 0 \text{ for all } u,x,y,z,w \in I.\end{aligned} \tag{3.4}$$

In view of (3.2), (3.3) reduces to

$$\begin{aligned}\Delta(x,w)[z,\Delta(y,u)] &+ x[D(z,w),\Delta(y,u)]\\ &+ [x,\Delta(y,u)]D(z,w) = 0 \text{ for all } u,x,y,z,w \in I.\end{aligned} \tag{3.5}$$

Replacing x by vx in (3.5) and using (3.5) we have

$$v(D(x,w)-\Delta(x,w))[z,\Delta(y,u)] + \Delta(v,w)x[z,\Delta(y,u)] + [v,\Delta(y,u)]xD(z,w) = 0, \tag{3.6}$$

for all $u,\ v,\ x,\ y,\ z,\ w \in I$. Replacing z by $z\Delta(y,u)$ in (3.6) and using (3.6) we obtain

$$[v,\Delta(y,u)]xzD(\Delta(y,u),w) = 0 \text{ for all } u,v,x,y,z,w \in I. \tag{3.7}$$

Last equations give us $[v,\Delta(y,u)]xRzD(\Delta(y,u),w) = 0$ for all $u,v,x,y,z,w \in I$. Primeness of R provide us either $[x,\Delta(y,y)]x = 0$ or $zD(\Delta(y,y),w) = 0$ for all $x,y,z,w \in I$. Since the left and right annihilator of prime rings are zero. Therefore, we can find either $[x,\Delta(y,y)] = 0$ or $D(\Delta(y,y),w) = 0$ for all $x,y,w \in I$. Consider the case $[x,\Delta(y,y)] = 0$, i.e., in particular we can write $[x,\Delta(x,x)] = 0$ for all $x \in I$. Applying Lemma 2.1, we get the result. Now take the later case $D(\Delta(y,y),w) = 0$ for all $y,w \in I$. An application of Lemma 2.3 yields that either $D = 0$ or R is commutative. If $D(y,u) = 0$ for all $y,u \in I$. We can easily obtain by suitable replacing that $D(r,s) = 0$ for all $r,s \in R$. For all $r,s,t \in R$, we have $\Delta(rs,t) = \Delta(r,t)s$ and $\Delta(s,rt) = \Delta(s,r)t$. That is Δ acts as a left bimultiplier on R. This completes the proof.

Theorem 3.2. *Let R be a prime ring of characteristic not two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $[\Delta(x,x),D(y,y)] = 0$ for all $x,y \in I$, then one of the conditions hold*

1. *R is commutative.*
2. *Δ acts as a left bimultiplier on R.*

Proof Assume that

$$[\Delta(x,x), D(y,y)] = 0 \text{ for all } x,y \in I. \tag{3.8}$$

Linearize (3.8) in x to get

$$[\Delta(x,x), D(y,y)] + [\Delta(z,z), D(y,y)] + 2[\Delta(x,z), D(y,y)] = 0 \text{ for all } x,y,z \in I. \tag{3.9}$$

Since R is not of Char 2 and using (3.8), we obtain

$$[\Delta(x,z), D(y,y)] = 0 \text{ for all } x,y,z \in I. \tag{3.10}$$

Substitute xu for x in (3.10) to find

$$\Delta(x,z)[u, D(y,y)] + x[D(u,z), D(y,y)] + [x, D(y,y)]D(u,z) = 0 \text{ for all } x,y,z \in I. \tag{3.11}$$

Replacing u by $uD(y,y)$ in (3.11), we have

$$\begin{aligned} &\Delta(x,z)[u, D(y,y)]D(y,y) + x[D(u,z), D(y,y)]D(y,y) \\ &+[x, D(y,y)]D(u,z)D(y,y) + [x, D(y,y)]uD(D(y,y),z) \\ &+xu[D(D(y,y),z), D(y,y)] + x[u, D(y,y)]D(D(y,y),z) = 0, \end{aligned} \tag{3.12}$$

for all $u,\ x,\ y,\ z \in I$. Using (3.11) and (3.12), we get

$$\begin{aligned} &[x, D(y,y)]uD(D(y,y),z) + xu[D(D(y,y),z), D(y,y)] \\ &+x[u, D(y,y)]D(D(y,y),z) = 0, \end{aligned} \tag{3.13}$$

for all $u,\ x,\ y,\ z \in I$. Again replace x by rx in(3.13) and using (3.13) we arrive at

$$[r, D(y,y)]xuD(D(y,y),z) = 0 \text{ for all } u,x,y,z \in I, r \in R. \tag{3.14}$$

Primeness of R yields that $[r, D(y,y)] = 0$ or $D(D(y,y),z) = 0$ for all $y,z \in I, r \in R$. If $D(D(y,y),z) = 0$ for all $y,z \in I$, then conclusion follows from Lemma 2.2. Now consider the case when $[r, f(y)] = 0$ for all $y \in I, r \in R$. Linearization implies that $[r, D(x,y)] = 0$ for all $x,y \in I, r \in R$. Replacing x by xz, we have$[r,x]D(z,y) + D(x,y)[r,z] = 0$ for all $x,y,z \in I, r \in R$. In particular, we get $[z,x]D(z,y) = 0$ for all $x,y,z \in I$.

By the primeness of R and $I \neq (0)$ we arrive at either $[x,z] = 0$ or $D(z,y) = 0$ for all $y,z,x \in I$. Setting $H = \{y \in I \mid y \in Z(R)\}$ and $K = \{y \in I \mid D(y,z) = 0\ for\ all\ z \in I\}$. It is clear that H and K are the additive subgroups of I and $(I,+) = (H,+) \cup (K,+)$. But a group cannot be the union of proper subgroups, hence we are forced to conclude that for each $y \in I$ either $I = H$ or $I = K$. The first case implies that R is commutative, as desired. Let R be not commutative. So we have $D(y,u) = 0$ for all $y,u \in I$. We can easily obtain by suitable replacing that $D(r,s) = 0$ for all $r,s \in R$. For all $r,s,t \in R$, we can find $\Delta(rs,t) = \Delta(r,t)s$ and $\Delta(s,rt) = \Delta(s,r)t$. That is, Δ acts as a left bimultiplier on R.

Theorem 3.3. *Let R be a prime ring of characteristic not two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $\Delta(x,x) \circ \Delta(y,y) = 0$ for all $x, y \in I$, then one of the conditions hold*

1. *R is commutative.*
2. *Δ acts as a left bimultiplier on R.*

Proof Consider the hypothesis

$$\Delta(x,x) \circ \Delta(y,y) = 0 \text{ for all } x, y \in I. \tag{3.15}$$

Linearization in x and using (3.15) yields that

$$2\Delta(z,x)\Delta(y,y) + 2\Delta(y,y)\Delta(z,x) = 0 \text{ for all } x, y, z \in I. \tag{3.16}$$

As Char $R \neq 2$, we get

$$\Delta(z,x)\Delta(y,y) + \Delta(y,y)\Delta(z,x) = 0 \text{ for all } x, y, z \in I. \tag{3.17}$$

Substitute zu for z in (3.17) and use (3.17), we obtain

$$\Delta(z,x)[u, \Delta(y,y)] + zD(u,x)\Delta(y,y) + \Delta(y,y)zD(u,x) = 0 \text{ for all } u, x, y, z \in I. \tag{3.18}$$

Replace u by $u\Delta(y,y)$ in above equation and use it to find

$$zuD(\Delta(y,y),x)\Delta(y,y) + \Delta(y,y)zuD(\Delta(y,y),x) = 0 \text{ for all } u, x, y, z \in I. \tag{3.19}$$

Again replacing z by rz in (3.19) and using (3.19), we arrive at

$$[\Delta(y,y), r]zuD(\Delta(y,y),x) = 0 \text{ for all } u, x, y, z \in I. \tag{3.20}$$

Arguing in the same manner as in the proof of Theorem 3.1 and using Lemma 2.3, we complete the proof.

Example 2. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$, Then R is a noncommutative ring. Consider $\Delta\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} a_1a_2 & 0 \\ 0 & 0 \end{pmatrix}$ and $D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ b_1b_2 & 0 \end{pmatrix}$. Therefore, $\Delta : R \times R \longrightarrow R$ is a generalized biderivation associated with a biderivation $D : R \times R \longrightarrow R$, Where Δ and D both are additive obviously. It can be easily seen that Δ satisfies $[\Delta(x,x), \Delta(y,y)] = 0$ for all $x, y \in R$. Hence primeness can not be omitted from Theorem 3.1.

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