Bol. Soc. Paran. Mat. (3s.) **v. 38** 4 (2020): 111–126.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i4.40535

A Posteriori Error Estimates for the Generalized Overlapping Domain Decomposition Method for Parabolic Equation With Mixed Boundary Condition

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ABSTRACT: The paper deals with a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary condition the interfaces for parabolic variational equation with Laplace boundary value problems are proved using theta time scheme combined with Galerkin spatial approximation.

Key Words: A posteriori error estimates; GODDM; mixed boundary condition condition, Parabolic equation

Contents

1. Introduction

The overlapping domain decomposition method can be used to solve stationary boundary value problems on domains which decomposes of two or more overlapping subdomains (see [1], [3], [14], [18], [19]). It has been invented by Herman Amandus Schwarz in 1890. These qualitative problem solution can be approximated by an infinite sequence of functions which results from solving a sequence of evolutionary boundary value problems in each of the subdomain. Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems have been intensively studied the last three decades (see [4]- [7]). In addition, Schwarz methods

2010 *Mathematics Subject Classification:* 65M60, 34A37, 65K15, 49J40, 49M25.
Submitted November 21, 2017. Published December 25, 2017

effectiveness for these problems, especially those in fluid mechanics has been given in many papers. See proceedings of the annual domain decomposition conference beginning with [5]. Moreover, the a priory estimate for stationary case is given in several works, see for instance [3] which a weak formulation of the classical Schwarz method is given. In [7], results geometry convergence are given. Also, in [5], the convergence for a circular geometries has been archive. These results can be found in the recent books on domain decomposition methods [8], [9]. In recent work [10], [11] an improved version of the Schwarz method for highly heterogeneous media has been given. Quite a few works on uniform norm error analysis of overlapping nonmatching grids methods for stationary problems are studied in the many works for example in [9]- [12].

The main propose of this paper, we shall proceed as in [10]. More precisely, we develop an approach which combines a result of geometrical convergence due to [6], [17], [18] and a lemma which consists of estimating the error in the uniform norm between the continuous and discrete Schwarz iterates. The optimal order of the their convergence is then proved using the standard Galerkin method and an error estimate on uniform norm for linear elliptic equations [3].

In recent works, in [21] the authors presented the error analysis in the maximum norm for a class of nonlinear elliptic problems in the context of overlapping nonmatching grids and they studied the optimal error estimate on uniform norm between the discrete Schwarz sequence and the exact solution of the partial differential equations, and in [22] the authors derived a posteriori error estimates for GODDM with Direchlet boundary conditions on the interfaces for Laplace boundary value problems, they have proved that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interfaces using Galerkin method.

In this work, we have interested to prove a posteriori error estimates for the generalized overlapping domain decomposition method (GODDM) for the following parabolic equation: find $u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C^2\left(0,T,H^{-1}(\Omega)\right)$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, \quad \text{in } \Sigma, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0,\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.

The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative verifies

$$\alpha \leq \beta, \quad \beta > 0. \tag{1.2}$$

f is a regular function satisfies

$$f \in L^2\left(0,T,L^2(\Omega)\right) \cap C^1\left(0,T,H^{-1}(\Omega)\right).$$

The symbol $(.,.)_{\Omega}$ stands for the inner product in $L^2(\Omega)$.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, then we prove a weak formulation of the presented problem. In section 3, a posteriori error estimate is proposed for the convergence of the discretized solution using theta time scheme combined with Galerkin method on subdomains.

2. The continuous problem

The problem (1.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ solution of

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + a(u, v) = (f, v) + (\varphi, v)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u^i(x, 0) = u_0^i \text{ in } \Omega, \end{cases}$$

where $a(.,.)$ is the bilinear form defined as:

$$u, v \in H_0^1(\Omega) : a(u, u) = (\nabla u, \nabla u) - (a_0 u, u)$$

and

$$a_0 \in L^2(0, T, L^\infty(\Omega)) \cap C^{\,0}\left(0, T, H^{-1}(\Omega)\right)$$

is sufficiently smooth functions and satisfy the following condition: $a_0(t, x) \geq \beta > 0$, β is a constant.

Let $(.,.)_{\Omega}$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as:

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi \ > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

In [7], we have treated the overlapping domain decomposition method combined with a finite element approximation for elliptic equation related for Laplace operator Δ, where a Sobolev norm analysis of an overlapping Schwarz method on nonmatching grids has been used, where we proved that the discretization on every subdomain converges in Soblev norm. Furthermore, a result of asymptotic behavior in uniform norm has been given.

In this paper, we extend the last work for parabolic equation with mixed boundary conditions where we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary conditions on the boundaries for the discrete solutions on subdomains using theta time scheme combined with a finite element spatial approximation, similar to that in [21], which investigated full elliptic operator with Dirichlet boundary condition.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, definitions and fundamental published propositions in the proposed problem then we give the variational formulation of our model. In section 3 and 4, a posteriori error estimate for both continuous and discrete cases are proposed for the convergence of the discrete solution using theta time scheme combined with a finite element method on subdomains.

3. The discrete parabolic equation

3.1. The space discretization

Let Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$V^h = \left\{ \begin{array}{l} v \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\bar{\Omega})\right)\right) \\ \text{such that } v_h \mid_K = P_1, \ k \in \tau_h, \\ v_h(., 0) = v_{h0} \text{ (initial data) in } \Omega, \ \dfrac{\partial v_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ v_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{array} \right\} \tag{3.1}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1.

We consider r_h be the usual interpolation operator defined by $r_h v = \sum\limits_{i=1}^{m(h)} v(M_i) \varphi_i(x)$.

3.1.1. The discrete maximum principle assumption. We assume the matrices whose coefficients $a(\varphi_i, \varphi_j)$ are M-matrix. For convenience in all the sequels, C will be a generic constant independent on h. [7].

It can be approximated the problem (1.1) by a weakly coupled system of the following parabolic equation $v \in H^1(\Omega)$

$$\left(\frac{\partial u}{\partial t}, v\right)_\Omega + a(u, v) = (f, v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{3.2}$$

We discretize in space, i.e., we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\bar{\Omega})\right)\right)$, we get the following semi-discrete system of parabolic equation

$$\left(\frac{\partial u_h}{\partial t}, v_h\right)_\Omega + a(u_h, v_h) = (f, v_h)_\Omega + (\varphi, v_h)_{\Gamma_0}. \tag{3.3}$$

3.2. The time discretization

Now we apply the θ-scheme in the semi-discrete approximation (3.3). Thus we have, for any $\theta \in [0, 1]$ and $k = 1, ..., p$

$$
\begin{aligned}
&\left(u_h^k - u_h^{k-1}, v_h\right)_\Omega + (\Delta t)\, a\left(u_h^{\theta,k}, v_h\right) = \\
&(\Delta t)\left[\left(f^{i,\ \theta,k}, v_h\right)_\Omega + \left(\varphi^{i,\theta,k}, v_h\right)_{\Gamma_0}\right],
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
u_h^{\theta,k} &= \theta u_h^k + (1-\theta)\, u_h^{k-1} \\
f^{\ \theta,k} &= \theta f^k + (1-\theta)\, f^{k-1}
\end{aligned}
\tag{3.5}
$$

and

$$
\varphi^{\ \theta,k} = \theta \varphi^k + (1-\theta)\, \varphi^{k-1}. \tag{3.6}
$$

By multiplying and dividing by θ and by adding $\left(\dfrac{u_h^{k-1}}{\theta\Delta t}, v_h - u_h^{\theta,k}\right)$ to both parties of the inequalities (3.4), we get

$$
\begin{aligned}
&\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right) = \left(f^{\ \theta,k} + \frac{u_h^{\theta,k-1}}{\theta\Delta t}, v_h\right)_\Omega + \\
&+ \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_{ih} \in V_h.
\end{aligned}
\tag{3.7}
$$

Then, the problem (3.7) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$
b\left(u_h^{\theta,k}, v_h\right) = \left(f^{i,\ \theta,k} + \mu u_h^{k-1}, v_h\right)_\Omega + \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_h,\ u_h^{\theta,k} \in V_h, \tag{3.8}
$$

where

$$
\left\{
\begin{array}{l}
b^i\left(u_h^{i,\theta,k}, v_h\right) = \mu\left(u_h^{i,\theta,k}, v_h\right)_\Omega + a\left(u_h^{i,\theta,k}, v_h\right),\ v_h \in V_h^i, \\
\mu = \dfrac{1}{\theta\Delta t} = \dfrac{p}{\theta T}.
\end{array}
\right. . \tag{3.9}
$$

3.3. The space continuous for the generalized Schwarz method

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \Omega_{12}$, $\partial\Omega_s \cap \Omega_t = \Gamma_s$, $s \neq t$ and $s,t = 1,2$. We need the spaces $V_s = H^1(\Omega) \cap H^1(\Omega_s) = \left\{v \in H^1(\Omega_i) : v_{\partial\Omega_i \cap \partial\Omega} = 0\right\}$ and $W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \left\{v_{\Gamma_s},\ v \in V_s \text{ and } v = 0 \text{ on } \partial\Omega_s \backslash \Gamma_s\right\}$,

which is a subspace of $H^{\frac{1}{2}}(\Gamma_s) = \left\{\psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \text{ for } \varphi \in V_s,\ s = 1,2\right\}$, with its norm $\|\varphi\|_{W_s} = \inf\limits_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}$.

We define the continuous counterparts of the continuous Schwarz sequences defined in (3.9), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$, $m = 0,1,2,...,i = 1,...,M$ solution of

$$\begin{aligned}
&c\left(u_1^{\theta,k,m+1}, v\right) = \\
&\left(F^\theta\left(u_1^{\theta,k-1,m+1}\right), v\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0}, \\
&u_1^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\
&\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_1^{\theta,k,m+1} = \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_1^{\theta,k,m} \text{ on } \Gamma_1
\end{aligned} \tag{3.10}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1, 2$.

In the next section, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions of generalized Schwarz alternating method, we get

$$\begin{aligned}
\left(-\Delta\ u_1^{k,m+1}, v_1\right)_{\Omega_1} &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1\right)\right)_{\Omega_1} \\
&- \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1\right)\right)_{\Omega_1} - \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1}
\end{aligned}$$

thus we can deduce

$$\begin{aligned}
&\left(-\Delta u_1^{\theta,k,m+1}, v_1\right)_{\Omega_1} = \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} \\
&- \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_2} + \alpha_1\ u_2^{\theta,k,m} - \alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\
&- \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1},
\end{aligned}$$

thus the problem 3.10 equivalent to; find $u_1^{\theta,k,m+1} \in V_1$ such that

$$\begin{aligned} & c(u_1^{\theta,k,m+1}, v_1) + \left(\alpha_1 u_1^{\theta,k,m}, v_1\right)_{\Gamma_1} \\ & = \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} \\ & + \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{3.11}$$

and we have $u_2^{\theta,k,m+1} \in V_2$

$$\begin{aligned} & c(u_2^{\theta,k,m+1}, v_2) + \left(\alpha_2 u_2^{\theta,k,m+1}, v_2\right)_{\Gamma_2} \\ & = \left(F(u_2^{\theta,k-1,m+1}), v_2\right)_{\Omega_2} + (\varphi, v)_{\Gamma_0} \\ & \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_1^{\theta,k,m}, v_2\right)_{\Gamma_2}. \end{aligned} \tag{3.12}$$

4. A posteriori error estimate in continuous case

We define these auxiliary problems by of (3.10) with another problem in a nonoverlapping way over Ω. These auxiliary problems are needed for analysis and not for the computation section.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \qquad \Omega = \Omega_2 \cup \Omega_4, \text{and } \Omega_2 \cap \Omega_4 = \varnothing.$$

Let $(u_1^{k,m}, u_2^{k,m})$ be the solution of problems (3.10), we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems

$$\begin{cases} \dfrac{u_1^{k,m+1} - u_1^{k-1,m+1}}{\Delta t} - \Delta\, u_1^{\theta,k,m+1} + a_0^k u_1^{\theta,k,m+1} = F^{\,\theta}\left(u_1^{\theta,k-1,m+1}\right) \text{ in } \Omega_1, \\ u_1^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\ k = 1, ..., n, \\ \dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \alpha u_1^{i,\theta,k,m} = \dfrac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, \text{ on } \Gamma_1 \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dfrac{u_3^{k,m+1} - u_3^{k-1,m+1}}{\Delta t} - \Delta\ u_3^{\theta,k,m+1} + a_0^k u_3^{\theta,k,m+1} = F^\theta\left(u_3^{\theta,k-1,m+1}\right) \text{ in } \Omega_3, \\ u_3^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_3^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{\theta,k,m} \text{on } \Gamma_2 = \dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}, \text{ on } \Gamma_1. \end{cases} \tag{4.2}$$

It can be taken $\epsilon_1^{\theta,k,m} = u_2^{\theta,k,m+1} - u_3^{\theta,k,m+1}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ of the problem (3.10) and (resp.,(4.1) and (4.2)) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [21] that is, $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ tend to $u_3^{\theta,k}$ (resp. $u_3^{\theta,k}$), then $\epsilon_1^{\theta,k,m}$ should tend to naught when m tends to infinity in V_2 .

By taking

$$\begin{aligned} \Lambda_3^{k,m} &= \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, \ \Lambda_1^{k,m} = \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}, \\ \Lambda_3^{k,m} &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{\theta,k,m}, \\ \Lambda_1^{k,m} &= \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}. \end{aligned} \tag{4.3}$$

Using Green formula, (4.1) and (4.2) can be reformulated to the following system of elliptic variational equations

$$\begin{aligned} & c(u_1^{\theta,k,m+1}, v_1) + \left(\alpha_1 u_1^{\theta,k,m}, v_1\right)_{\Gamma_1} \\ & = \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} \\ & + \left(\Lambda_3^{k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} & c(u_3^{\theta,k,m+1}, v_3) + \left(\alpha_3 u_3^{\theta,k,m+1}, v_3\right)_{\Gamma_1} \\ & = \left(F^\theta(u_3^{\theta,k-1,m+1}), v_3\right)_{\Omega 3} + (\varphi, v)_{\Gamma_0} \\ & + \left(\Lambda_1^{k,m}, v_3 - u_3^{;\theta,k,m+1}\right)_{\Gamma_1}, \forall v_3 \in V_3. \end{aligned} \tag{4.5}$$

On the other hand by taking

$$\theta_1^{k,m} = \frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{\theta,k,m}, \tag{4.6}$$

we get

$$\Lambda_3^{\theta,k,m} = \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial(u_2^{\theta,k,m} - u_3^{\theta,k,m})}{\partial \eta_1} + \alpha_1(u_2^{\theta,k,m} - u_3^{\theta,k,m})$$

$$= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial \epsilon_1^{k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m} \tag{4.7}$$

$$= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m}.$$

Using (4.6) we have

$$\begin{aligned}
\Lambda_3^{k,m+1} &= \frac{\partial u_3^{,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i\theta,k,m} + \theta_1^{k,m+1} \\
&= -\frac{\partial u_3^{\theta,k,m}}{\partial \eta_3} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\
&= \alpha_3 u_3^{\theta,k,m} - \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} - \alpha_3 u_1^{\theta,k,m} \\
&\quad + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\
&= (\alpha_1 + \alpha_3) u_3^{\theta,k,m} - \Lambda_1^{k,m} + \theta_1^{k,m+1}
\end{aligned} \tag{4.8}$$

and the last equation in (4.8), we have

$$\begin{aligned}
&\Lambda_1^{k,m+1} = -\frac{\partial u_1^{\theta,k,m}}{\partial \eta_1} + \alpha_3 u_1^{\theta,k,m} = \alpha_1 u_1^{\theta,k,m} - \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} - \alpha_1 u_2^{\theta,k,m} + \\
&\alpha_3 u_1^{\theta,k,m} + \alpha_3 u_1^{\theta,k,m} = (\alpha_1 + \alpha_3) u_1^{\theta,k,m} - \Lambda_3^{k,m} + \theta_3^{k,m+1}.
\end{aligned} \tag{4.9}$$

Lemma 4.1. *Let $u_s^k = u_{\Omega s}^k$, $e_s^{\theta,k,m+1} = u_s^{\theta,k,m+1} - u_s^k$ and $\eta_s^{k,m+1} = \Lambda_s^{k,m+1} - \Lambda_s^k$. Then for $s,t = 1,3, s \neq t$, we have*

$$\begin{aligned}
&c_s(e_s^{i\theta,k,m+1}, v_s - e_s^{\theta,k,m+1}) + \left(\alpha_s e_s^{\theta,k,m+1}, v_s - e_s^{k,m+1}\right)_{\Gamma s} \\
&= \left(\eta_t^{k,m}, v_s - e_s^{i,k,m+1}\right)_{\Gamma_s}, \forall v_s \in V_s
\end{aligned} \tag{4.10}$$

and

$$\left(\eta_s^{k,m+1}, \psi^i\right)_{\Gamma_s} = \left((\alpha_s + \alpha_t) e_s^{k,m+1}, v_s\right)_{\Gamma_s} - \left(\eta_t^{k,m}, \psi\right)_{\Gamma_s} + \left(\theta_t^{k,m+1}, \psi\right)_{\Gamma s}, \forall \psi \in V_s. \tag{4.11}$$

Proof. The proof is very similar to that in [7]. □

Lemma 4.2. *By letting C be a generic constant which has different values at different places, we get for $s,t = 1,3, s \neq t$*

$$\left(\eta_s^{k,m-1} - \alpha_s e_s^{k,m}, w\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m}\right\|_{1,\Omega s} \|w\|_{W_1} \tag{4.12}$$

and

$$\left(\alpha_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1}\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m+1}\right\|_{1,\Omega_s} \|w\|_{W_1}. \tag{4.13}$$

where C is a constant independent ofh and k.

Proof. The proof is very similar to that in [7]. □

Proposition 4.3. *[7] For the sequences $(u_1^{\theta,k,m+1}, u_3^{\theta,k,m+1})_{m\in\mathbb{N}}$ solutions of (4.1) and (4.2) we have the following a posteriori error estimation*

$$\left\|u_1^{\theta,k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_3^{\theta,k,m+1} - u_3^k\right\|_{3,\Omega_3} \leqslant C \left\|u_1^{\theta k,m+1} - u_3^{k,m}\right\|_{W_1}.$$

where C is a constant independent ofh and k.

Proposition 4.4. *For the sequences $(u_2^{\theta,k,m+1}, u_4^{\theta,k,m+1})_{m\in\mathbb{N}}$. We get the the similar following a posteriori error estimation*

$$\left\|u_2^{\theta,k,m+1} - u_2^{i,\theta,k}\right\|_{2,\Omega_2} + \left\|u_4^{\theta,k,m+1} - u_4^{\theta,k}\right\|_{4,\Omega_4} \leqslant C \left\|u_2^{\theta,k,m+1} - u_4^{\theta,k,m+1}\right\|_{W_2}. \tag{4.14}$$

where C is a constant independent of h and k.

Proof. The proof is very similar to proof of Proposition 2 which proved in our published paper on [7]. □

Theorem 4.5. *[7] Let $u_s^{\theta,k} = u_{\Omega_s}^{\theta,k}, s = 1, 2$. For the sequences $(u_1^{\theta,k,m+1}, u_2^{\theta,k,m+1})$ with $m \in \mathbb{N}$ solutions of problems (3.11) and (3.12), one have the following result*

$$\begin{aligned} &\left\|u_1^{\theta,k,m+1} - u_1^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_2^{\theta,k,m} - u_2^{\theta,k}\right\|_{2,\Omega_2} \leqslant \\ &C\ (\left\|u_1^{\theta,k,m+1} - u_2^{\theta,k,m}\right\|_{W_1} + \left\|u_1^{\theta,k,m} - ui_{,1}^{\theta,k,m+1}\right\|_{W_2} + \\ &+ \left\|e_1^{k,m}\right\|_{W_1} + \left\|e_2^{k,m+1}\right\|_{W_2}), \end{aligned}$$

where C is a constant independent of h and k.

5. A Posteriori Error Estimate: Discrete Case

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_s $s = \{1, ..., m(h)\}$ defined by $\varphi_l(M_j) = \delta_{lj}$, where M_j is a vertex of the considered triangulation.

In the first step, we approach the space H_0^1 by a suitable discretization space of finite dimensional $V^h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements $u_h^{\theta,n} \in V^h$ which approaches $u_h(t_n, .)$, $t_n = n\Delta t$, $k = 1, ..., n$, with initial data $u_h^0 = u_{0h}$.

Let $u_h^{\theta,k,m+1} \in V^h$ be the solution of the discrete problem associated with (3.10), $u_{s,h}^{\theta,k,m+1} = u_{h,\Omega_s}^{\theta,k,m+1}$.

We construct the sequences $(u_{s,h}^{\theta,k,m+1})_{m\in\mathbb{N}}, u_{s,h}^{\theta,k,m+1} \in V_s^h, (s = 1, 2)$ solutions of discrete problems associated with (4.4).

We define the discrete space K_h is a suitable set given by

$$K_h = \begin{cases} u_h \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\Omega)\right)\right), \\ \\ u_h = 0 \text{ in } \Gamma, \quad \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \; u_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{cases}$$

where r_h is the usual interpolation operator defined by $r_h v = \sum\limits_{i=1}^{m(h)} v(M_j)\,\varphi_i(x)$.

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for (Ω_1, Ω_3) the following full-discrete problems: find $u_{1,h}^{\theta,k,m+1} \in K_h$ solution of

$$\begin{cases} c(u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h}) + \left(\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h}\right)_{\Gamma_1}, \\ \\ \geq \left(F^\theta(u_{1,h}^{\theta,k-1,m+1}), \tilde{v}_{1,h}\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0}, \\ \\ u_{1,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \quad \tilde{v}_{1,h} \in K_h \\ \\ \dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0, \end{cases} \tag{5.1}$$

by taking the trial function $\tilde{v}_{1,h} = v_{1,h} - u_{1,h}^{i,\theta,k,m+1}$ in (5.1), we get

$$\begin{cases} c(u_{1,h}^{\theta,k,m+1}, v_{1,h}) + \left(\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} \\ = \left(F(u_{1,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_1} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{1,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \ \ v_{1,h} \in K_h, \\ \dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \ \text{ on } \Gamma_1 - \Gamma_0. \end{cases} \tag{5.2}$$

Similarly, we get

$$\begin{cases} c(u_{3,h}^{\theta,k,m+1}, v_{1,h}) + \left(\alpha_{3,h} u_{3,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} \\ = \left(F^{\theta}(u_{3,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_3} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{3,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_{3,h}^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{\theta,k,m+1} = \dfrac{\partial u_{1}^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_{1}^{\theta,k,m}, \ \text{ on } \Gamma_1 - \Gamma_0. \end{cases} \tag{5.3}$$

For (Ω_2, Ω_4), are similar in (5.2) and (5.3).

Theorem 5.1. *[10]The solution of the system of parabolic equations (5.2) and (5.3) is the maximum element the set of discrete subsolutions.*

We can obtain the discrete counterparts of propositions 1 and 2 by doing almost the same analysis as in section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{3,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k} \right\|_{1,\Omega_3} \leqslant C \left\| u_{1,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k,m} \right\|_{W_1} \tag{5.4}$$

and

$$\left\| u_{2,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} + \left\| u_{4,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k} \right\|_{1,\Omega_4} \leqslant C \left\| u_{2,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k,m} \right\|_{W_2}. \tag{5.5}$$

Similar to that in the proof of Theorem 2 we get the following discrete estimates

$$\begin{aligned} &\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{\theta,k,m} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} \leqslant \\ &C\ (\left\| u_{1,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k,m} \right\|_{W_1} + \left\| u_{2,h}^{\theta,k,m} - u_{1,h}^{\theta,k,m} \right\|_{W_2} \\ &+ \left\| e_{1,h}^{k+1,m} \right\|_{W_1} + \left\| e_{2,h}^{k+1,m} \right\|_{W_2}). \end{aligned}$$

Next we will obtain an error estimate between the approximated solution $u_{s,h}^{\theta,k,m+1}$ and the semi discrete solution in time $u^{i,\theta,k}$. We introduce some necessary notations. We denote by $\varepsilon_h = \{E \in T : T \in \tau_h \text{ and } E \notin \partial\Omega\}$
and for every $T \in \tau_h$ and $E \in \varepsilon_h$, we define as $\omega_T = \{T' \in \tau_h : T' \cap T \neq \varnothing\}$, and $\omega_E = \{T' \in \tau_h : T' \cap E \neq \varnothing\}$.

The right hand side f is not necessarily continuous function across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E.

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

Theorem 5.2. *Let $u_s^{\theta,k} = u^{\theta,k} \mid_{\Omega_s}$ where u is the solution of problem (1.1), the sequences $\left(u_{1,h}^{\theta,k,m+1}, u_{2,h}^{\theta,k,m}\right)_{m\in\mathbb{N}}$ are solutions of the discrete problems (4.4) and (4.5). Then there exists a constant C independent of h such that*

$$\left\|u_{1,h}^{\theta,k,m+1} - u_1^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{\theta,k,m} - u_2^{\theta,k}\right\|_{1,\Omega_2} \leqslant C\left\{\sum_{i=1}^{2}\sum_{T\in\tau_h}\left(\eta_i^T\right) + \eta_{\Gamma_s}\right\},$$

where

$$\eta_{\Gamma_s} = \left\|u_{h,s}^{\theta,k,*} - u_{h,t}^{i,\theta,k,*-1}\right\|_{W_{h,s}} + \left\|\epsilon_{i,h}^{\theta,k,*}\right\|_{W_{h,s}}$$

and

$$\eta_s^T = h_T \left\| \begin{array}{c} F\left(u_{h,s}^{\theta,k-1,*}\right) + u_{h,s}^{\theta,k-1} + \\ \Delta\, u_{h,s}^{\theta,k,*} - \left(1+\lambda a_{h0}^k\right) u_{h,s}^{;\theta,k} \end{array} \right\|_{0,T} + \sum_{E\in\varepsilon_h} h_E^{\frac{1}{2}} \left\| \left[\frac{\partial u_{h,s}^{\theta,k,*}}{\partial \eta_E}\right] \right\|_{0,E},$$

where C is a constant independent of h and k and the symbol $*$ is corresponds to $m+1$ when $s=1$ and to m when $s=2$.

Proof. The proof is based on the technique of the residual a posteriori estimation

see [21] and Theorem 3. We give the main steps by the triangle inequality we have

$$\sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k,*}\right\|_{1,\Omega_s} \leqslant \sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{1,\Omega_s} + \sum_{s=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s}. \tag{5.6}$$

The second term on the right hand side of (5.6) is bounded by

$$\sum_{s=1}^{2}\sum_{i=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s} \leqslant C\sum_{s=1}^{2}\eta_{\Gamma_s}.$$

To bound the first term on the right hand side of (5.6) we use the residual equation and apply the technique of the residual a posteriori error estimation [21], to get for $v_h \in V^h$

$$
\begin{aligned}
&c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) = c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s - v_{h,s})\\
&\leq \sum_{T\subset\Omega_s}\int_T \left(\begin{array}{c} F^\theta\left(u_{h,s}^{\theta,k-1}\right) + u_{h,s}^{\theta,k-1} + \mu\Delta\ u_{h,s}^{\theta,k} - \\ \left(1+\mu a_{h0}^k\right) u_{h,s}^{\theta k}\end{array}\right)(v_s - v_{h,s})\,ds\\
&-\sum_{E\subset\Omega_s}\int_E \left[\frac{\partial u_{h,s}^{\theta k}}{\partial \eta_{E}}\right](v_s - v_{h,s})\,ds\\
&-\sum_{E\subset\Gamma_s}\int_E \frac{\partial u_{h,s}^{\theta k}}{\partial \eta_{E}}(v_s - v_{h,s})\,ds'\\
&+\sum_{E\subset\Omega_s}\int_T \left(F^\theta\left(u_s^{\theta,k}\right) - F^\theta\left(u_{h,s}^{\theta k}\right)\right)(v_s - v_{h,s})d\sigma\\
&+\left(\frac{\partial u_{h,s}^{\theta k}}{\partial \eta_s}, v_s - v_{h,s}\right)_{\Gamma_s},
\end{aligned}
$$

where $F^\theta\left(u_{h,s}^{\theta,k}\right)$ is any approximation of $F^\theta\left(u_s^{\theta,k}\right)$. Therefore

$$
\begin{aligned}
&\sum_{s=1}^{2} c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s)\\
&\leq \sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\| \begin{array}{c} F^\theta\left(u_{h,s}^{\theta,k}\right) + u_{h,s}^{\theta,k-1} + \mu\Delta\ u_{h,s}^{\theta,k} \\ -\left(1+\mu a_{h0}^k\right) u_{h,s}^{\theta,k}\end{array}\right\|_{0,T} \|v_s - v_{h,s}\|_{0,T}\\
&+\sum_{s=1}^{2}\sum_{E\subset\Omega_s}\left\|\left[\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_{E}}\right]\right\|_{0,E}\|v_s - v_{h,s}\|_{0,E} + \sum_{s=1}^{2}\sum_{E\subset\Gamma_s}\left\|\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_{E}}\right\|_{0,E}\|v_s - v_{h,s}\|_{0,E}\\
&+\sum_{s=1}^{2}\sum_{T\subset\Omega_s} c\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{0,T}\|v_s - v_{h,s}\|_{0,T} + \sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\|\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_s}\right\|_{0,T}\|v_s - v_{h,s}\|_{0,T}.
\end{aligned}
\tag{5.7}
$$

Using the following fact

$$
\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{1,\Omega_s} \leqslant \sup_{v_s^i\in K}\frac{c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s + ch_s^T)}{\|v_s + ch_s^T\|_{1,\Omega_i}},
$$

we get

$$\sum_{s=1}^{2} c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s + ch_s^{i,T}) \leq \sum_{s=1}^{2} \left(\sum_{T \subset \Omega_s} \eta_s^{i,T} \right) \sum_{s=1}^{2} \|v_s\|_{1,\Omega_s} . \tag{5.8}$$

Finally, by combining (5.5), (5.6) and (5.7) the required result follows. □

Conclusion

In this paper, a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary boundary conditions on the interfaces for parabolic equation with second order boundary value problems are studied using theta time scheme combined with a Galerkin approximation. In future. The geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate for linear elliptic PDEs will be established and the results of some numerical experiments will be presented to support the theory.

Acknowledgement 5.1. *The authors wish to thank deeply the anonymous referees and the handling editor for their useful remarks and their careful reading of the proofs presented in this paper.*

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