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Multi-valued Fixed Point Theorem via F- contraction of Nadler Type and Application to Functional and Integral Equations *

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ABSTRACT: In this work, using F-contraction of Nadler type, common multi-valued fixed point results in the setting of b-metric space are established. With the assistance of the determined results sufficient conditions for the existence of common solutions to the systems of functional and integral equations are studied.

Key Words: b- metric space, F-contraction, Functional equations.

Contents

1. Introduction and Preliminaries

In this paper, $CB(\Lambda)$ denotes the family of non-empty bounded and closed subsets of Λ. $\mathbb{R}^+$, $\mathbb{N}_0$ and $\mathbb{N}$ signify the set of all non-negative real numbers, the set of non-negative integers and the set of positive integers respectively. Metric fixed point theory which is a vital class of non-linear analysis, is normally not only restricted to mathematical proposition, but also comes into action in most of the applied sides of pure sciences and technical fields. Among the top-listed significance of fixed points of contractive mappings defined for variety of spaces is the confirmation of the existence and uniqueness of solutions of differential, integral as well as functional equations.

Nadler [13] elaborated and extended the Banach contraction principle [3] to set-valued mapping by using the Pompeiu-Hausdorff metric. The variability of these non-linear problems pare the way for finding out some more innovated and authentic tools which is currently more highlighted in the literature. Among these tools which is considered to be a novel tool is by Wardowski [18], in which the author has shown another kind of contractive mapping called F-contraction. Vetro [17] demonstrated some fixed point results for multi-valued operator using F-contraction and studied functional and integral equations. Czerwik [7] and Bakhtin [4] genralized

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the idea of metric space and presented metric spaces called b-metric spaces. Several researchers took after the idea of Czerwik and illustrated interesting results. For detail see([2], [8], [9], [10], [14], [15], [16]).

In the current work we derived common fixed point result for F-contraction in b-metric space. Also provided some applications for validity of the established result.

Now we recall some basic results and definitions.

Theorem 1.1. *[13] Assume $\lambda : \Lambda \to CB(\Lambda)$ be multi-valued mapping on complete metric space (Λ, d) such that*

$$H(\lambda\xi_1, \lambda\xi_2) \leq \mu(d(\xi_1, \xi_2)) \; forall \; \xi_1, \xi_2 \in \Lambda.$$

Where $\mu \in (0, 1)$. Then λ has a fixed point.

Definition 1.2. *[7] Consider Λ be a non-empty set and $s \geq 1$. Assume $d : \Lambda \times \Lambda \to \mathbb{R}^+$ be a function satisfying the conditions:*

1. *$d(\xi_{11}, \xi_{22}) = 0$ if and only if $\xi_{11} = \xi_{22}$ for all $\xi_{11}, \xi_{22} \in \Lambda$;*
2. *$d(\xi_{11}, \xi_{22}) > 0$ for all $\xi_{11} \neq \xi_{22}, \xi_{11}, \xi_{22} \in \Lambda$*
3. *$d(\xi_{11}, \xi_{22}) = d(\xi_{22}, \xi_{11})$, where $\xi_{11}, \xi_{22} \in \Lambda$;*
4. *$d(\xi_{11}, \xi_{22}) \leq s(d(\xi_{11}, \xi_{33}) + d(\xi_{33}, \xi_{22}))$ for all $\xi_{11}, \xi_{22}, \xi_{33} \in \Lambda$.*

Then it is a b-metric on Λ and the pair (Λ, d, s) is called b-metric space.

Definition 1.3. *[7] Assume (Λ, d, s) is a b-metric space, here $s \geq 1$. Let ξ_n be a sequence in Λ. $\xi \in \Lambda$ is said to be the limit of the sequence ξ_n if*

$$\lim_{n\to\infty} d(\xi_n, \xi) = 0$$

and the sequence ξ_n is called to be convergent in Λ.

Definition 1.4. *[7] If for each $\epsilon > 0$, there is a positive integer $\mathbb{N}$ such that $d(\xi_n, \xi_m) < \epsilon$ for all $n, m > \mathbb{N}$. Then a sequence ξ_n is said to be a Cauchy sequence.*

Definition 1.5. *[7] A metric space (Λ, d, s) is said to be complete (or a b-complete metric space) if every Cauchy sequence in (Λ, d, s) is convergent in Λ.*

Definition 1.6. *[6] Assume a real number $s \geq 1$ and $\mathbb{F}$ represent the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$, with the following conditions*

(F_1) *For each positive term sequence $\{\alpha_n\} \subset \mathbb{R}^+$, F is strictly increasing;*

(F_2) *$\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*

(F_3) *For each positive number sequence i.e $\{\alpha_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$, there exist $k \in (0, 1)$ such that $\lim_{n\to\infty} (\alpha_n)^k F(\alpha_n) = 0$;*

(F_4) *For* $\{\alpha_n\} \subset \mathbb{R}^+$, *such that* $\lambda + F(s\alpha_n) \leq F(\alpha_{n-1})$ *for all* $n \in \mathbb{N}$ *and some* $\lambda \in \mathbb{R}^+$.
Then $\lambda + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$.

Now, we give some definitions for multi-valued mappings defined in a b-metric space (Λ, d, s). Defined the mapping $H : CB(\Lambda) \times CB(\Lambda) \to \mathbb{R}^+$ for $\Delta_1, \Delta_2 \in CB(\Lambda)$ by

$$H(\Delta_1, \Delta_2) = \max\{\delta(\Delta_1, \Delta_2), \delta(\Delta_2, \Delta_1)\},$$

where

$$\delta(\Delta_1, \Delta_2) = \sup\{d(a, \Delta_2), a \in \Delta_1\}, \delta(\Delta_2, \Delta_1) = \sup\{d(b, \Delta_1), b \in \Delta_1\}$$

and

$$d(a, \Delta_3) = \inf\{d(a, \xi_1), \xi_1 \in \Delta_3\}.$$

Lemma 1.7. *[11,12] Assume a b-metric space* (Λ, d, s). *Suppose* $\Delta_1, \Delta_2, \Delta_3 \in CB(\Lambda)$ *and* $\xi_1, \xi_2 \in \Lambda$, *then the following holds:*

1. $d(\xi_1, \Delta_2) \leq d(\xi_1, b)$ *for any* $b \in \Delta_2$;
2. $\sigma(\Delta_1, \Delta_2) \leq H(\Delta_1, \Delta_2)$;
3. $d(\xi_1, \Delta_2) \leq H(\Delta_1, \Delta_2)$ *for any* $\xi_1 \in \Delta_1$;
4. $H(\Delta_1, \Delta_1) = 0$
5. $H(\Delta_1, \Delta_2) = H(\Delta_2, \Delta_1)$;
6. $H(\Delta_1, \Delta_3) \leq s[H(\Delta_1, \Delta_2) + H(\Delta_2, \Delta_3)]$;
7. $H(\xi_1, \Delta_1) \leq s[d(\xi_1, \xi_2) + d(\xi_2, \Delta_1)]$.

Lemma 1.8. *[13] Consider* (Λ, d) *be a metric space. Further* $\Delta_1, \Delta_2 \in CB(\Lambda)$ *then for* $\gamma > 1$ *and for each* $\kappa_1 \in \Delta_1$ *there exists* $\kappa_2(\kappa_1) \in \Delta_2$ *such that* $d(\kappa_1, \kappa_2) \leq \gamma H(\Delta_1, \Delta_2)$.

In [5,13] it is shown that the above lemma is also true for $\gamma \geq 1$.

Lemma 1.9. *[13] Let* (Λ, d) *be a metric space. Further* $\Delta_1, \Delta_2 \in CB(\Lambda)$ *then for* $\gamma \geq 1$ *and for each* $\kappa_1 \in \Delta_1$ *there exists* $\kappa_2(\kappa_1) \in \Delta_2$ *such that* $d(\kappa_1, \kappa_2) \leq \gamma H(\Delta_1, \Delta_2)$.

The following is a consequences of Lemma 1.9.

Lemma 1.10. *[13] Let* Δ_1 *and* Δ_2 *be two non-empty compact subsets of a metric space* (Λ, d) *and* $\lambda : \Delta_1 \to CB(\Lambda)$ *be a multi-valued mapping. Let* $\gamma \geq 1$ *then for* $\kappa_1, \kappa_2 \in \Delta_1$ *and* $\xi_1 \in \lambda\kappa_1$ *there exists* $\xi_2 \in \lambda\kappa_2$ *such that* $d(\xi_1, \xi_2) \leq \gamma H(\lambda\kappa_1, \lambda\kappa_2)$.

Lemma 1.11. *[11,12] Assume a b-metric space* (Λ, d, s) *and* $\Delta_1, \Delta_2 \in CB(\Lambda)$. *Then for* $\gamma > 1$, $\kappa_1 \in \Delta_1$ *there exists* $\kappa_2(\kappa_1) \in \Delta_2$ *such that* $d(\kappa_1, \kappa_2) \leq \gamma H(\Delta_1, \Delta_2)$.

Lemma 1.12. *[1] Let Δ_n be a sequence in $CB(\Lambda)$ and $\lim_{n\to\infty} H(\Delta_n, \Delta_1) = 0$ for $\Delta_1 \in CB(\Lambda)$ if $\xi_n \in \Delta_n$ and $\lim_{n\to\infty} d(\xi_n, \xi_1) = 0$ then $\xi_1 \in \Delta_1$.*

Theorem 1.13. *[6] Let (Λ, d, s) be a b-metric space, where $s \geq 1$. For $F \in \mathbb{F}$ a multi-valued mapping $\lambda : \Lambda \to CB(\Lambda)$ is called an Nadler type F-contraction such that for $\tau \in \mathbb{R}^+$*

$$2\tau + F(sH(\lambda\xi_1, \lambda\xi_2)) \leq F(d(\xi_1, \xi_2)), \tag{1.1}$$

for all $\xi_1, \xi_2 \in \Lambda$ $\lambda\xi_1 \neq \lambda\xi_2$, then λ has a fixed point in Λ.

2. Main Results

A multi-valued maps $S, T : \Lambda \to CB(\Lambda)$ is called an F-contraction of Nadler type if there exist $F \in \mathbb{F}$ such that for $\tau \in \mathbb{R}^+$

$$2\tau + F(sH(T\eta, S\zeta)) \leq F(\alpha\Upsilon(\eta, \zeta)), \tag{2.1}$$

where, $s \geq 1$, $0 < \alpha < 1$, and for all $\eta, \zeta \in \Lambda$

$$\Upsilon(\eta, \zeta) = \max\Big\{d(\eta, \zeta), d(\eta, T\eta), d(\zeta, S\zeta), \frac{d(\eta, S\zeta) + d(\zeta, T\eta)}{2s}\Big\}. \tag{2.2}$$

Based on the above definition, we proved the following common fixed point result.

Theorem 2.1. *Assume a b-metric space (Λ, d, s), where $s \geq 1$. Suppose there exist a continues from the right function $F \in \mathbb{F}$. If $S, T : \Lambda \to CB(\Lambda)$ is F-contraction of Nadler type. Then S, T has a common fixed point. Moreover common fixed point is unique if T is single valued.*

Proof: Fix any $\zeta \in \Lambda$. Define $\zeta_0 = \zeta$ and let $\zeta_1 \in T\zeta_0$ by Lemma 1.11 there exist $\zeta_2 \in S\zeta_1$ such that

$$d(\zeta_1, \zeta_2) \leq hH(T\zeta_0, S\zeta_1).$$

Multiplying both side by s we have

$$sd(\zeta_1, \zeta_2) \leq shH(T\zeta_0, S\zeta_1),$$

which implies that

$$F(sd(\zeta_1, \zeta_2)) \leq F(shH(T\zeta_0, S\zeta_1)). \tag{2.3}$$

Since $F \in \mathbb{F}$ is continuous from the right there exist $h > 1$ such that

$$F(hsH(T\zeta_0, S\zeta_1)) \leq F(sH(T\zeta_0, S\zeta_1)) + \tau. \tag{2.4}$$

From (2.4) and (2.3) we get

$$F(sd(\zeta_1, \zeta_2)) \leq F(shH(T\zeta_0, S\zeta_1)) \leq F(sH(T\zeta_0, S\zeta_1)) + \tau.$$

Adding τ both side and using equation (2.1) we obtained

$$\tau + F(sd(\zeta_1, \zeta_2)) \leq F(\alpha\Upsilon(\zeta_0, \zeta_1)).$$

We construct a sequence by using this iterating procedure, $\zeta_n \in \Lambda$ such that $\zeta_{2n+1} \in T\zeta_{2n}$ and $\zeta_{2n+2} \in S\zeta_{2n+1}$ and

$$\tau + F(sd(\zeta_{2n+1}, \zeta_{2n+2})) \leq F(\alpha\Upsilon(\zeta_{2n}, \zeta_{2n+1})). \tag{2.5}$$

Using strictly increasing property of F we have

$$sd(\zeta_{2n+1}, \zeta_{2n+2}) \leq \alpha\Upsilon(\zeta_{2n}, \zeta_{2n+1}).$$

Which implies that

$$d(\zeta_{2n+1}, \zeta_{2n+2}) < \Upsilon(\zeta_{2n}, \zeta_{2n+1}). \tag{2.6}$$

Now by using Lemma 1.7 we have

$$\begin{aligned}
&\Upsilon(\zeta_{2n}, \zeta_{2n+1}) = \\
&\max\Big\{d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n}, T\zeta_{2n}), \\
&d(\zeta_{2n+1}, S\zeta_{2n+1}), \frac{d(\zeta_{2n}, S\zeta_{2n+1}) + d(\zeta_{2n+1}, T\zeta_{2n})}{2s}\Big\} \\
&\leq \max\Big\{d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n+1}, \zeta_{2n+2}), \frac{d(\zeta_{2n}, \zeta_{2n+2})}{2s}\Big\} \\
&= \max\Big\{d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n+1}, \zeta_{2n+2}), \frac{d(\zeta_{2n}, \zeta_{2n+2})}{2s}\Big\} \\
&\leq \max\Big\{d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n+1}, \zeta_{2n+2}), \frac{d(\zeta_{2n}, \zeta_{2n+1}) + d(\zeta_{2n+1}, \zeta_{2n+2})}{2s}\Big\} \\
&= \max\Big\{d(\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n+1}, \zeta_{2n+2})\Big\}.
\end{aligned}$$

Suppose $d(\zeta_{2n}, \zeta_{2n+1}) < d(\zeta_{2n+1}, \zeta_{2n+2})$ then

$$\Upsilon(\zeta_{2n}, \zeta_{2n+1}) \leq d(\zeta_{2n+1}, \zeta_{2n+2}).$$

Which is contradiction with (2.6).

Therefore equation (2.5) implies that

$$\tau + F(sd(\zeta_{2n+1}, \zeta_{2n+2})) \leq F(d(\zeta_{2n}, \zeta_{2n+1})). \tag{2.7}$$

Let $P_n = d(\zeta_{2n+1}, \zeta_{2n+2}) > 0 \ \forall \ n \in \mathbb{N}$. It follow from (2.7) and axiom F_4 that

$$\tau + F(s^n d(\zeta_{2n+1}, \zeta_{2n+2})) \leq F(s^{n-1} d(\zeta_{2n}, \zeta_{2n+1}) \forall n \in \mathbb{N}. \tag{2.8}$$

Thus, by equation (2.8)

$$F(s^n P_n) \leq F(s^{n-1} P_{n-1}) - \tau,$$

$$F(s^{n-1}P_{n-1}) \leq F(s^{n-2}P_{n-2}) - 2\tau$$
$$\vdots$$
$$F(s^{n}P_{n}) \leq F(s^{o}P_{0}) - n\tau. \tag{2.9}$$

Which implies that
$$\lim_{n\to\infty} F(s^{n}P_{n}) = -\infty.$$

By using F_2, we have
$$\lim_{n\to\infty} s^{n}P_{n} = 0.$$

By F_3 property there exists $0 < k < 1$ such that
$$\lim_{n\to\infty} (s^{n}P_{n})^{k} F(s^{n}P_{n}) = 0.$$

Equation (2.9) implies that
$$F(s^{n}p_{n}) \leq F(p_{0}) - n\tau. \tag{2.10}$$

Multiplying (2.10) by $(s^{n}p_{n})^{k}$ we have
$$(s^{n}p_{n})^{k}F(s^{n}p_{n}) \leq (s^{n}p_{n})^{k}F(p_{0}) - n\tau(s^{n}p_{n})^{k}.$$

Which implies that
$$(s^{n}p_{n})^{k}F(s^{n}p_{n}) - (s^{n}p_{n})^{k}F(p_{0}) \leq -n\tau(s^{n}p_{n})^{k} \leq 0.$$

Applying limit $n \to \infty$ we have
$$\lim_{n\to\infty} n(s^{n}p_{n})^{k} = 0. \tag{2.11}$$

From (2.11) there exist $n_1 \in \mathbb{N}$ such that $n(s^{n}P_{n})^{k} < 1$ such that
$$s^{n}P_{n} \leq \frac{1}{n^{\frac{1}{k}}} \; \forall n \geq n_1. \tag{2.12}$$

To show that ζ_n is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n > n_1$, using triangular inequality, and using (2.12) we have

$$\begin{aligned}
d(\zeta_{2n}, \zeta_{2m}) &\leq sd(\zeta_{2n}, \zeta_{2n+1}) + s^{2}d(\zeta_{2n+1}, \zeta_{2n+2}) + ... + s^{m-n}d(\zeta_{2m-1}, \zeta_{2m}) \\
&= sP_{n-1} + s^{2}P_{n} + ... + s^{m-n}P_{m-2} \\
&= \sum_{j=n-1}^{m-2} s^{j-n+2}P_{j} \\
&\leq \sum_{j=n-1}^{\infty} s^{j-n+2}P_{j} \\
&\leq \sum_{j=n-1}^{\infty} s^{2-n}\frac{1}{j^{\frac{1}{k}}}.
\end{aligned}$$

By taking limit we get $d(\zeta_n, \zeta_m) \to 0$.

Hence ζ_n is a Cauchy sequence but a b-metric space (Λ, d, s) is a complete so there exist $\zeta \in \Lambda$ such that $\zeta_n \to \zeta$ as $n \to \infty$.

Next step is to show that ζ is a common fixed point of the mapping S and T. We have

$$d(\zeta_{2n+2}, S\zeta) \leq H(T\zeta_{2n+1}, S\zeta) \leq sH(T\zeta_{2n+1}, S\zeta).$$

Which implies that

$$d(\zeta_{2n+2}, S\zeta) \leq sH(T\zeta_{2n+1}, S\zeta).$$

Since F is strictly increasing, therefore

$$F(d(\zeta_{2n+2}, S\zeta)) \leq F(sH(T\zeta_{2n+1}S\zeta)).$$

Adding 2τ both side and using equation (2.1) we have

$$2\tau + F(d(\zeta_{2n+2}, S\zeta)) \leq 2\tau + F(sH(T\zeta_{2n+1}S\zeta)) \leq F(\alpha\Upsilon(\zeta_{2n+1}, \zeta)).$$

Since, $\tau \in \mathbb{R}^+$ we have

$$F(d(\zeta_{2n+2}, S\zeta)) \leq F(\alpha\Upsilon(\zeta_{2n+1}, \zeta)).$$

Since F is strictly increasing, therefore

$$d(\zeta_{2n+2}, S\zeta) \leq \alpha\Upsilon(\zeta_{2n+1}, \zeta).$$

Applying limit $n \to \infty$, we get

$$d(\zeta, S\zeta) \leq \alpha d(\zeta, S\zeta).$$

Which implies $d(\zeta, S\zeta) = 0$. Hence $\zeta \in S\zeta$. Similarly we can show that $\zeta \in T\zeta$.

Moreover, the common fixed point is unique if T is single valued. If μ_1 and ζ are two common fixed points of the T and S , then

$$\begin{aligned}
F(d(\mu_1, \zeta)) &\leq F(sH(\mu_1, S\zeta)) + 2\tau \\
&= F(sH(\{T\mu_1\}, S\zeta)) + 2\tau \\
&\leq F(\alpha\Upsilon(\mu_1, \zeta)) \\
&= F\Big(\alpha \max\Big\{d(\mu_1, \zeta), d(\mu_1, T\mu_1), d(\zeta, S\zeta), \frac{d(\mu_1, S\zeta) + d(\zeta, T\mu_1)}{2s}\Big\}\Big) \\
&\leq F\Big(\alpha \max\Big\{d(\mu_1, \zeta), 0, 0, \frac{d(\mu_1, \zeta) + d(\zeta, \mu_1)}{2s}\Big\}\Big) \\
&\leq F(\alpha d(\mu_1, \zeta)).
\end{aligned}$$

Which implies $d(\mu_1, \zeta) \leq \alpha d(\mu_1, \zeta)) < d(\mu_1, \zeta)$.

Hence $d(\mu_1, \zeta) = 0 \Rightarrow \mu_1 = \zeta$.

□

Let $\Lambda = [0, 1]$. *Define* $d : \Lambda \times \Lambda \to \mathbb{R}^+$ *by,*

$$d(\nu, \omega) = |\nu - \omega|^2.$$

Then (Λ, d, s) is b-metric space.

Let $S, T : \Lambda \to CB(\Lambda)$ be defined by $S\omega = \{\frac{\omega}{16}e^{-\tau}\}$ and $T\nu = [0, \frac{\nu}{16}e^{-\tau}]$. Obviously,

$$\begin{aligned} H(T\nu, S\omega) &= \max\{|\frac{\omega}{16}e^{-\tau} - \frac{\nu}{16}e^{-\tau}|^2, |\frac{\omega}{16}e^{-\tau}|^2\} \\ &\leq \frac{1}{4}e^{-2\tau}\max\{|\omega-\nu|^2, |\omega - \frac{\omega}{16}|^2\} \\ &\leq \frac{1}{4}e^{-2\tau}\max\{|\omega-\nu|^2, |\omega - \frac{\omega}{16}e^{-\tau}|^2\} \\ &= \frac{1}{4}e^{-2\tau}\max\{d(\nu,\omega), d(\omega, S\omega))\} \\ &\leq \frac{1}{4}e^{-2\tau}\Upsilon(\nu,\omega). \end{aligned}$$

Which implies that

$$2H(T\nu, S\omega) \leq \frac{1}{2}e^{-2\tau}\Upsilon(\nu,\omega).$$

By taking natural log both side and then consider $s = 2$, $\alpha = \frac{1}{2}$, $F(\nu) = \ln(\nu)$. All axiom of Theorem 2.1 are hold, therefore T and S have a common fixed point $\nu = 0$. Further since S is single valued, so the fixed point is unique.

3. Application

In this section, we study solvability of functional equations using the established fixed point theorem.

Let Δ_1, Δ_2 be Banach spaces, $\Theta_1 \subset \Delta_1$, $\Theta_2 \subset \Delta_2$ and $\mathbb{R}$ is the field of real numbers. Suppose $\Lambda = B(\Theta_1)$ represent the set of all functions defined on Θ_1 which is bounded and real valued. Define $d : \Lambda \times \Lambda \to \mathbb{R}^+$ by,

$$d(\nu,\zeta) = |\nu - \zeta|^2.$$

Then (Λ, d, s) is called b-metric space. Consider the system of functional equations given as

$$p(\nu) = \sup_{\zeta\in\Theta_2}\Big\{g(\nu,\zeta) + \Phi_1(\nu,\zeta,P(\tau(\nu,\zeta)))\Big\}, \quad \nu \in \Theta_1, \tag{3.1}$$

$$q(\nu) = \sup_{\zeta\in\Theta_2}\Big\{g(\nu,\zeta) + \Phi_2(\nu,\zeta,P(\tau(\nu,\zeta)))\Big\}, \quad \nu \in \Theta_1. \tag{3.2}$$

Here $g : \Theta_1 \times \Theta_2 \longrightarrow \mathbb{R}$, $\Phi_1, \Phi_2 : \Theta_1 \times \Theta_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ are functions which are bounded. Θ_1 and Θ_2 is the state and decision spaces, respectively $\tau : \Theta_1 \times \Theta_2 \longrightarrow \Theta_1$ show the transformation of the process, $p(\nu)$ and $q(\nu)$ signify Sup return functions with initial state Λ.

Let $S, T : B(\Theta_1) \to B(\Theta_1)$ defined by

$$S(h_1(\nu)) = \sup_{\zeta\in\Theta_2}\Big\{g(\nu,\zeta) + \Phi_1(\nu,\zeta,P(\tau(\nu,\zeta)))\Big\}, \quad \nu \in \Theta_1, \tag{3.3}$$

$$T(h_1(\nu)) = \sup_{\zeta\in\Theta_2}\Big\{g(\nu,\zeta) + \Phi_2(\nu,\zeta,P(\tau(\nu,\zeta)))\Big\}, \quad \nu \in \Theta_1. \tag{3.4}$$

Theorem 3.1. *$\Phi_1, \Phi_2 : \Theta_1 \times \Theta_2 \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Theta_1 \times \Theta_2 \longrightarrow \mathbb{R}$ are continues and bounded, and satisfying the following assumption.*

$$\Big|\Phi_i(\nu, \zeta, h(\nu)) - \Phi_i(\nu, \zeta, k(\nu))\Big| \leq \frac{e^{-\tau}}{\sqrt{s}} \beta(\sqrt{(\Upsilon(h,k))}), i = 1, 2 \tag{3.5}$$

where $i, j = 1, 2$ and $0 < \beta < 1$ for all $\nu \in \Theta_1$ and $\zeta \in \Theta_2$, where

$$\Upsilon(h,k) = \Big\{d(h,k), d(h,Th), d(k,Sk), \frac{d(h,Sk) + d(k,Th)}{2s}\Big\}. \tag{3.6}$$

Then functional equations (3.3),(3.4) has a common solution.

Proof: Let δ be an arbitrary positive number, $\nu \in \Theta_1$, $h_1, h_2 \in B(\Lambda)$ then there exist $\zeta_1, \zeta_2 \in \Theta_2$ such that

$$S(h_1(\nu)) < g(\nu, \zeta_1) + \Phi_1(\nu, \zeta_1, h_1(\tau_1)) + \delta \tag{3.7}$$

$$T(h_2(\nu)) < g(\nu, \zeta_2) + \Phi_2(\nu, \zeta_2, h_2(\tau_2)) + \delta \tag{3.8}$$

$$S(h_1(\nu)) \geq g(\nu, \zeta_2) + \Phi_1(\nu, \zeta_2, h_1(\tau_2)) \tag{3.9}$$

$$T(h_2(\nu)) \geq g(\nu, \zeta_1) + \Phi_2(\nu, \zeta_1, h_2(\tau_1)). \tag{3.10}$$

Using equations (3.7) and (3.10)

$$\begin{aligned} S(h_1(\nu)) - T(h_2(\nu)) &< \Phi_1(\nu, \zeta_1, h_1(\tau(\nu, \zeta_1))) - \Phi_2(\nu, \zeta_1, h_1(\tau(\nu, \zeta_1))) + \delta \\ &\leq \Big|\Phi_1(\nu, \zeta_1, h_2(\tau(\nu, \zeta_1))) - \Phi_2(\nu, \zeta_1, h_1(\tau(\nu, \zeta_1)))\Big| + \delta \\ &< \frac{e^{-\tau}}{\sqrt{s}} \beta(\sqrt{\Upsilon(h_1(\nu), h_2(\nu))}) + \delta. \end{aligned} \tag{3.11}$$

It follows that

$$S(h_1(\nu)) - T(h_2(\nu)) \leq \frac{e^{-\tau}}{\sqrt{s}} \beta(\sqrt{\Upsilon(h_1(\nu), h_2(\nu))}) + \delta. \tag{3.12}$$

Similarly using equations (3.8) and (3.9) we have

$$T(h_1(\nu)) - S(h_2(\nu)) \leq \frac{e^{-\tau}}{\sqrt{s}} \beta(\Upsilon(h_1(\nu), h_2(\nu))) + \delta. \tag{3.13}$$

Using equations (3.12) and (3.13) we get

$$\Big|T(h_1(\nu)) - S(h_2(\nu))\Big| \leq \frac{e^{-\tau}}{\sqrt{s}} \beta(\sqrt{\Upsilon(h_1(\nu), h_2(\nu))}) + \delta.$$

$\nu \in \Theta_1$ and $\delta > 0$ is arbitrary, therefore

$$s\Big(\Big|T(h_1(\nu)) - S(h_2(\nu))\Big|^2\Big) \leq e^{-2\tau} \beta^2(\Upsilon(h_1(\nu), h_2(\nu))).$$

Taking Logarithms we have

$$\ln\left(s\Big|T(h_1(\nu)) - S(h_2(\nu))\Big|^2\right) \leq \ln\left(e^{-2\tau}\beta^2(\Upsilon(h_1(\nu), h_2(\nu))\right).$$

After simple calculation we get

$$2\tau + \ln\left(sd(T(h_1(\nu)) - S(h_2(\nu)))\right) \leq \ln\left(\beta^2(\Upsilon(h_1(\nu), h_2(\nu)))\right).$$

By taking $\beta^2 = \alpha$, $F(\nu) = \ln(\nu)$ from Theorem 2.1 the functional equations (3.3) and (3.4) has common solution. □

On ward, we derive sufficient conditions for the solutions to the following general non-linear system of Fredholm integral equations of 2nd kind given by

$$\begin{cases} \nu(t_1) = \phi(t_1) + \displaystyle\int_{\varsigma}^{\sigma} K_1(t_1, s_1, \nu(s_1))ds_1, t_1 \in [\varsigma, \sigma], \\ \zeta(t_1) = \phi(t_1) + \displaystyle\int_{\varsigma}^{\sigma} K_2(t_1, s_1, \zeta(s_1))ds_1, t_1 \in [\varsigma, \sigma]. \end{cases} \tag{3.14}$$

Let $\Lambda = C[\varsigma, \sigma]$ be the set of all continuous function defined on $[\varsigma, \sigma]$. Define $d : \Lambda \times \Lambda \to \mathbb{R}^+$, by

$$d(\nu, \zeta) = \left(\sup_{t_1 \in [\varsigma, \sigma]} |\nu(t_1) - \zeta(t_1)|\right)^2.$$

Then (Λ, d, s) is a complete b metric space on Λ. For the derivation of aforesaid condition, we give the following theorem.

Theorem 3.2. *Assume the assumptions given below are holds.*

(A_1) $K_i : [\varsigma, \sigma] \times [\varsigma, \sigma] \times \mathbb{R}^+ \to \mathbb{R}^+$, *for* $i = 1, 2$ *and* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous;*

(A_2) *There exist a function* $G : [\varsigma, \sigma] \times [\varsigma, \sigma] \to [0, \infty)$ *which is a continuous such that,*

$$\Big|K_i(t_1, s_1, u) - K_j(t_1, s_1, v)\Big| \leq G(t_1, s_1)|u - v|$$

for each $t_1, s_1 \in [\varsigma, \sigma]$,

(A_3) $\sup\limits_{t_1, s_1 \in [\varsigma, \sigma]} \int_{\varsigma}^{\sigma} |G(t_1, s_1)|ds_1 \leq \sqrt{\frac{\alpha}{s}}.e^{-\tau}$, $0 < \alpha < 1$.

Then the system of integral equations (3.14) has a common solution in $C([\varsigma, \sigma])$.

Proof: Define $S, T : C([\varsigma, \sigma]) \to C([\varsigma, \sigma])$ by,

$$S\nu(t_1) = \phi(t_1) + \int_{\varsigma}^{\sigma} K_1(t_1, s_1, \nu(s_1))ds_1, t_1 \in [\varsigma, \sigma].$$

$$T\zeta(t_1) = \phi(t_1) + \int_{\varsigma}^{\sigma} K_2(t_1, s_1, \zeta(s_1))ds_1, t_1 \in [\varsigma, \sigma].$$

we have

$$\begin{aligned}
d(S\nu, T\zeta) &= \sup_{t_1\in[\varsigma,\sigma]} |S\nu - T\zeta|^2 \\
&\leq \sup_{t_1\in[\varsigma,\sigma]} \Big(\int_{\varsigma}^{\sigma} |k_1(t_1, s_1, \nu(s_1)) - k_2(t_1, s_1, \zeta(s_1))| ds_1 \Big)^2 \\
&\leq \sup_{t_1\in[\varsigma,\sigma]} \Big(\int_{\varsigma}^{\sigma} G(t_1, s_1)|\nu(s_1) - \zeta(s_1)| ds_1 \Big)^2 \\
&\leq \Big(\sup_{t_1\in[\varsigma,\sigma]} |\nu(t_1) - \zeta(t_1)| \Big)^2 \Big(\sup_{t_1\in[\varsigma,\sigma]} \int_{\varsigma}^{\sigma} G(t_1, s_1) ds_1 \Big)^2 \\
&\leq \Big(\sup_{t_1\in[\varsigma,\sigma]} |\nu(t_1) - \zeta(t_1)| \Big)^2 \frac{\alpha e^{-2\tau}}{s} \\
&= \frac{\alpha e^{-2\tau}}{s} d(\nu, \zeta).
\end{aligned}$$

Which implies that

$$d(S\nu, T\zeta) \leq \frac{\alpha e^{-2\tau}}{s} d(\nu, \zeta) \leq \frac{\alpha e^{-2\tau}}{s} \Upsilon(\nu, \zeta).$$

$$sd(S\nu, T\zeta) \leq e^{-2\tau} \alpha \Upsilon(\nu, \zeta).$$

By taking $F(\nu) = \ln(\nu)$ from Theorem 2.1 the system of integral equations (3.14) has common solution. □

4. Conclusion

In this paper, we proved common multi-valued fixed point result by using the notion of F type contraction of Nadler type. Also provide an example for the validity of the established result and further discuss the existence of solution to the system of functional and integral equations

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