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Dynamics and Stability of ψ-fractional Pantograph Equations with Boundary Conditions *

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abstract: This manuscript is devoted to obtain some adequate conditions for existence of at least one solution to fractional pantograph equation (FPE) involving the ψ -fractional derivative. The proposed problem is studied under some boundary conditions. Since stability is an important aspect of the qualitative theory. Therefore, we also discuss the Ulam-Hyers and Ulam-Hyers-Rassias type stabilities for the considered problem. Our results are based on some standard fixed point theorems. For the demonstration of our results, we provide an example.
List of abbreviations: Boundary value problems (BVP), nonlinear fractional differential equations (NFDEs), fractional pantograph equation (FPE), Ulam-Hyers stability (UHS), Ulam-Hyers-Rassias stability (UHRS).

Key Words: ψ-fractional derivative, PE, UH stability, UHR stability.

Contents

1. Introduction

In this manuscript, we develop some conditions for the existence of at least one solution for the ψ-FPE with boundary condition of the form

$$^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \mathscr{H}(t, \mathfrak{u}(t), \mathfrak{u}(\eta t)), \quad \text{for each } t \in \mathscr{I} := [0, T], \tag{1.1}$$

$$a\mathfrak{u}(0) + b\mathfrak{u}(T) = c, \tag{1.2}$$

where $^{c}\mathscr{D}^{\omega;\psi}$ is the ψ-Caputo fractional derivative of order ω, $0 < \omega < 1$ and $\eta \in (0, 1)$. Also $\mathscr{H}, \psi : \mathscr{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and a, b, c are real constants with $a + b \neq 0$. Further, we also establish some conditions for UH

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stability and UHR stability. Equivalent integral equation to equations (1.1)-(1.2) is given by

$$\begin{aligned}\mathfrak{u}(t) = & \frac{1}{\Gamma(\omega)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\omega)}\int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds - c\right].\end{aligned} \tag{1.3}$$

In last few decades, NFDEs with boundary/initial conditions have been studied very well by many researchers. This is because, FDEs describe many real world process, phenomenons related to memory and hereditary properties of various materials more accurately as compared to classical order differential equations, (see [2,5]). Therefore, the fractional-order models become more practical and realistic as compared to the integer-order models. FDEs arise in lots of engineering and clinical disciplines which includes biology, physics, chemistry, economics, signal and image processing, control theory and so on; see the monographs of Hilfer [10], Podlubny [17] and Samko et al. [20]. For a few recent improvement on the subject, see [4,6,7,8,9,16] and the references therein. In recent times, Ricardo Almeida [19] introduced the so-referred to as ψ-fractional derivative with respect to another function.
An important class of differential equations containing proportional delays are called pantograph equations. This important class was named after the study work of Ockendon and Tayler. Numbers of applications have been studied by many researchers of these equations in applied sciences including biology, physics, economics, and electrodynamics. Many authors have been investigated numerical and analytical solutions to PEs. For more details about the aforesaid equations, we refer to [3,22].

Since different aspects of FDEs have been investigated very well like qualitative theory, numerical analysis, etc. One of the important aspects which for NFDEs very recently attracted the attentions of researchers is devoted to stability analysis. The stability of functional equations was firstly discussed by Ulam in 1940 in a speech delivered at the University of Wisconsin. In 1941, Hyers answered the Ulam's question in the case of Banach spaces. In 1978, a significant generalization of the Ulam-Hyers stability was stated by Rassias as "the abstraction of stability for a functional equation arises if we alter the functional equation by an inequality which acts as a perturbation of the equation". Thus, the stability question for functional equations is stated now as "How do the solutions of the inequality differ from those of the given functional equation?" Considerable attention has been paid to the abstraction of the UH and UHR stabilities of all kinds for functional equations, see [1,11,12,13,14,15,18,23] and references cited therein. We study the Ulam stability of BVP for ψ-FDE (1.1)-(1.2). A similar idea can be found in [8], but there is no work on Ulam stability results for ψ- FDEs with boundary conditions.

The paper is arranged as follows. In Section 2, we recall some useful preliminaries. In Section 3, we give some sufficient conditions of the existence of the solutions and UHS and UHRS for the considered problem in Section 4.

2. Prerequisites

Denoting the space of all continuous functions from $\mathscr{I}$ to $\mathbf{R}$ by $\Phi = C(\mathscr{I}, \mathbf{R})$. Let us define the norm by

$$\|\mathfrak{u}\|_{\infty} := \sup \{|\mathfrak{u}(t)| : \ t \in \mathscr{I}\}.$$

Obviously Φ is a Banach space.
For complete study on ψ-fractional derivative, we refer to [19,21].

Definition 2.1. *Let $\omega > 0$, $\mathscr{I} = [0, T]$ be a finite or infinite interval, $\mathscr{H}$ an integrable function defined on $\mathscr{I}$ and $\psi \in C^1(\mathscr{I}, \mathbf{R})$ an increasing function such that $\psi^{'}(t) \neq 0$, for all $t \in \mathscr{I}$. Fractional integrals and derivatives of a function $\mathscr{H}$ with respect to another function ψ are defined as follows:*

$$I^{\omega;\psi}\mathscr{H}(t) := \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}\mathscr{H}(s)ds,$$

and

$$\begin{aligned} \mathscr{D}^{\omega;\psi}\mathscr{H}(t) &:= \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I^{n-\omega;\psi}\mathscr{H}(t) \\ &= \frac{1}{\Gamma(n-\omega)}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{n-\omega-1}\mathscr{H}(s)ds, \end{aligned}$$

respectively, where $n = [\omega] + 1$.

Lemma 2.2. *Let $\omega \in (0, 1)$ and let $\mathscr{H}, \psi : \mathscr{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be continuous. Then the solution $\mathfrak{u}$ of the ψ-FPE*

$$^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \mathscr{H}(t, \mathfrak{u}(t), \mathfrak{u}(\eta t)), \quad \text{for each } t \in \mathscr{I}, \tag{2.1}$$

$$\mathfrak{u}(0) = \mathfrak{u}_0, \tag{2.2}$$

where $0 < \eta < 1$, is provided as

$$\mathfrak{u}(t) = \mathfrak{u}_0 + \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}\mathscr{H}(s, \mathfrak{u}(s), \mathfrak{u}(\eta s))ds \tag{2.3}$$

As a consequence of Lemma 2.2, we require the following lemmas and theorems to establish our results.

Lemma 2.3. *Let $\omega \in (0, 1)$ and let $\mathscr{H}, \psi : \mathscr{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be continuous. A function $\mathfrak{u}$ is a solution of the ψ-fractional integral equation (2.3) if and only if $\mathfrak{u}$ is a solution of the ψ-FPE considered by us in (1.1)-(1.2).*

Theorem 2.4. *Let $\mathscr{P} : C(\mathscr{I}, \mathbf{R}) \to C(\mathscr{I}, \mathbf{R})$ completely continuous operator. If the set*

$$\kappa = \{u \in C(\mathscr{I}, \mathbf{R}) : \mathfrak{u} = \lambda\mathscr{P}(\mathfrak{u}), \quad \text{for some } \ \lambda \in (0, 1)\}$$

is bounded, then $\mathscr{P}$ has at least one fixed point.

We state some definitions of UH and UHR stability in the given sequel. Consider

$$ {}^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \mathscr{H}(t, \mathfrak{u}(t), \mathfrak{u}(\eta t)), \ \eta \in (0,1), \quad t \in \mathscr{I}, \tag{2.4} $$

and the following inequalities:

$$ \left| {}^{c}\mathscr{D}^{\omega;\psi}\mathcal{Z}(t) - \mathscr{H}(t, \mathcal{Z}(t), \mathcal{Z}(\eta t)) \right| \leq \epsilon, \quad t \in \mathscr{I}, \tag{2.5} $$

$$ \left| {}^{c}\mathscr{D}^{\omega;\psi}\mathcal{Z}(t) - \mathscr{H}(t, \mathcal{Z}(t), \mathcal{Z}(\eta t)) \right| \leq \epsilon\varphi(t), \quad t \in \mathscr{I}, \tag{2.6} $$

$$ \left| {}^{c}\mathscr{D}^{\omega;\psi}\mathcal{Z}(t) - \mathscr{H}(t, \mathcal{Z}(t), \mathcal{Z}(\eta t)) \right| \leq \varphi(t), \quad t \in \mathscr{I}. \tag{2.7} $$

Definition 2.5. *The equation (2.4) is UH stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathcal{Z} \in \Phi$ of the inequality (2.5) there exists a solution $\mathfrak{u} \in \Phi$ of equation (2.4) with*

$$ |\mathcal{Z}(t) - \mathfrak{u}(t)| \leq C_f \epsilon, \ t \in \mathscr{I}. $$

Definition 2.6. *The equation (2.4) is generalized UH stable if there exists*

$$ \psi_f \in C\left([0,\infty),[0,\infty)\right), \psi_f(0) = 0 $$

such that for each solution $\mathcal{Z} \in \Phi$ of the inequality (2.5) there exists a solution $\mathfrak{u} \in \Phi$ of equation (2.4) with

$$ |\mathcal{Z}(t) - \mathfrak{u}(t)| \leq \psi_f \epsilon, \quad t \in \mathscr{I}. $$

Definition 2.7. *The problem (2.4) is UHR stable with respect to $\varphi \in \Phi$, if there exists a real number $C_f > 0$ such that for every $\epsilon > 0$ and for each solution $\mathcal{Z} \in \Phi$ of the inequality (2.6) there exists a solution $\mathfrak{u} \in \Phi$ of equation (2.4) with*

$$ |\mathcal{Z}(t) - \mathfrak{u}(t)| \leq C_f \epsilon \varphi(t), \quad t \in \mathscr{I}. $$

Definition 2.8. *The equation (2.4) is generalized UHR stable with respect to $\varphi \in \Phi$ if there exists a real number $C_{f,\varphi} > 0$ such that for every solution $\mathcal{Z} \in \Phi$ of the inequality (2.7) there exists a solution $\mathfrak{u} \in \Phi$ of (2.4) with*

$$ |\mathcal{Z}(t) - \mathfrak{u}(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in \mathscr{I}. $$

Remark 2.9. *A function $\mathcal{Z} \in \Phi$ is a solution of (2.5) if and only if there exists a function $g \in \Phi$ (which depend on $\mathcal{Z}$) such that*

1. $|g(t)| \leq \epsilon$, $t \in \mathscr{I}$;
2. ${}^{c}\mathscr{D}^{\omega;\psi}\mathcal{Z}(t) = \mathscr{H}(t, \mathcal{Z}(t), \mathcal{Z}(\eta t)) + g(t)$, $t \in \mathscr{I}$.

Remark 2.10. *Let $\omega \in (0,]$ and if $\mathcal{Z} \in \Phi$ is a solution of the inequality (2.5), then $\mathcal{Z}$ is a solution of the given inequality*

$$ \left| \mathcal{Z}(t) - \mathfrak{A}_{\mathcal{Z}} - \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1} \mathscr{H}(s, \mathcal{Z}(s), \mathcal{Z}(\eta s)) ds \right| $$
$$ \leq \epsilon \frac{(\psi(T))^{\omega}}{\Gamma(\omega+1)} \left(1 + \frac{|b|}{|a+b|} \right). $$

In fact, by Remark 2.9, we have

$$^{c}\mathscr{D}^{\omega;\psi}\mathcal{Z}(t) = \mathscr{H}(t, \mathcal{Z}(t), \mathcal{Z}(\eta t)) + g(t), \quad t \in \mathscr{I}.$$

Then

$$\begin{aligned}\mathcal{Z}(t) &= \mathfrak{A}_{\mathcal{Z}} + \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}\mathscr{H}(s, \mathcal{Z}(s), \mathcal{Z}(\eta s))ds \\ &+ \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}g(s)ds \\ &- \left(\frac{b}{a+b}\right) \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}g(s)ds, \quad t \in \mathscr{I},\end{aligned}$$

with

$$\mathfrak{A}_{\mathcal{Z}} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}\mathscr{H}(s, \mathcal{Z}(s), \mathcal{Z}(\eta s))ds\right].$$

From this it follows that

$$\begin{aligned}&\left|\mathcal{Z}(t) - \mathfrak{A}_{\mathcal{Z}} - \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}\mathscr{H}(s, \mathcal{Z}(s), \mathcal{Z}(\eta s))ds\right| \\ &= \left|\frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}g(s)ds \right. \\ &\left. - \left(\frac{b}{a+b}\right) \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}g(s)ds\right| \\ &\leq \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}|g(s)|\,ds \\ &- \left(\frac{b}{a+b}\right) \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\omega-1}|g(s)|\,ds \\ &\leq \epsilon\frac{(\psi(T))^{\omega}}{\Gamma(\omega+1)}\left(1 + \frac{|b|}{|a+b|}\right).\end{aligned}$$

Remark 2.11. *A solution of the ψ-FDE with boundary condition, the inequality (2.5) is called an ϵ-solution of the problem (2.4).*

The given generalized Grönwall's inequality plays a vital role in the proof of UH stability.

Lemma 2.12. *(see Theorem 3,([21])) Let $\mathcal{Z}, \beta : \mathscr{I} \to [0, \infty)$ be continuous functions where $T < \infty$. If β is nondecreasing function and there is a constant $k \geq 0$ and $0 < \omega \leq 1$ such that*

$$\mathcal{Z}(t) \leq \beta(t) + k \int_0^t \psi^{'}(t)(\psi(t) - \psi(s))^{\omega-1}\mathcal{Z}(s)ds, \quad t \in \mathscr{I},$$

then

$$\mathcal{Z}(t) \leq \beta(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\omega))^n}{\Gamma(n\omega)} (\psi(t) - \psi(s))^{n\omega-1} \beta(s) \right) ds, \quad t \in \mathscr{I}.$$

Remark 2.13. *Under the hypothesis of Lemma 2.12 and let $\beta(t)$ be a nondecreasing function on $\mathscr{I}$, then, we have*

$$\mathcal{Z}(t) \leq \beta(t) E_{\omega;\psi}(k\Gamma(\omega)(\psi(t))^{\omega}).$$

3. Existence results

This part of the manuscript is devoted to the main results about the problem (1.1)-(1.2).

Definition 3.1. *A function $\mathfrak{u} \in C^1(\mathscr{I}, \mathbf{R})$ is said to be a solution of (1.1)-(1.2) if $\mathfrak{u}$ satisfied the equation ${}^c\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \mathscr{H}(t, \mathfrak{u}(t), \mathfrak{u}(\eta t))$ on $\mathscr{I}$, and the condition $a\mathfrak{u}(0) + b\mathfrak{u}(T) = c$.*

We need the following hypothesis before to establish the main results.

(A1) The function $\mathscr{H} : \mathscr{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous.

(A2) There exists a constants $\mathfrak{K} > 0$ such that

$$|\mathscr{H}(t, \mathbf{x}, \mathbf{y}) - \mathscr{H}(t, \overline{\mathbf{x}}, \overline{\mathbf{y}})| \leq \mathfrak{K}\left(|\mathbf{x} - \overline{\mathbf{x}}| + |\mathbf{y} - \overline{\mathbf{y}}|\right),$$

for each $t \in \mathscr{I}$, $\forall\, \mathbf{x}, \overline{\mathbf{x}}, \mathbf{y}, \overline{\mathbf{y}} \in \mathbf{R}$.

(A3) There exists a constant $\mathfrak{M}$ such that

$$|\mathscr{H}(t, \mathbf{x}, \mathbf{y})| \leq \mathfrak{M} \quad \text{for each } t \in \mathscr{I} \text{ and } \forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}.$$

Let us use the notation $\Phi = C(\mathscr{I}, \mathbf{R})$ and consider the operator $\mathscr{P} : \Phi \to \Phi$ defined by

$$\begin{aligned} \mathscr{P}\mathfrak{u}(t) = {} & \frac{1}{\Gamma(\omega)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\omega-1} \mathcal{H}(s, \mathfrak{u}(s), \mathfrak{u}(\eta s)) ds \\ & \frac{1}{a+b}\left[\frac{b}{\Gamma(\omega)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\omega-1} \mathcal{H}(s, \mathfrak{u}(s), \mathfrak{u}(\eta s)) ds - c \right]. \end{aligned} \tag{3.1}$$

First, we prove that the operator $\mathscr{P}$ defined by (3.1) fulfill the conditions of Theorem 2.4.

Lemma 3.2. *The operator $\mathscr{P}$ is continuous.*

Proof: Let $\{\mathfrak{u}_n\}$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in Φ as $n \to \infty$. Then for every $t \in \mathscr{I}$, we take

$$\begin{aligned}
&|\mathscr{P}(\mathfrak{u}_n)(t) - \mathscr{P}(\mathfrak{u})(t)| \\
&\leq \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} |\mathscr{H}(s,\mathfrak{u}_n(s),\mathfrak{u}_n(\eta s)) - \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&+ \frac{C(a,b)}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1} |\mathscr{H}(s,\mathfrak{u}_n(s),\mathfrak{u}_n(\eta s)) - \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&\leq \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \sup_{s\in\mathscr{I}} |\mathscr{H}(s,\mathfrak{u}_n(s),\mathfrak{u}_n(\eta s)) - \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&+ \frac{C(a,b)}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1} \sup_{s\in\mathscr{I}} |\mathscr{H}(s,\mathfrak{u}_n(s),\mathfrak{u}_n(\eta s)) - \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&\leq \frac{\left\|\mathscr{H}(\cdot,\mathfrak{u}_n(\cdot),\mathfrak{u}_n(\cdot)) - \mathscr{H}(\cdot,\mathfrak{u}(\cdot),\mathfrak{u}(\cdot))\right\|_\infty}{\Gamma(\omega)} \\
&\times \left[\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} ds + \frac{|b|}{|a+b|}\int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1} ds\right] \\
&\leq \frac{(\psi(T))^\omega}{\Gamma(\omega+1)}\left(1+\frac{|b|}{|a+b|}\right) \left\|\mathscr{H}(\cdot,\mathfrak{u}_n(\cdot),\mathfrak{u}_n(\cdot)) - \mathscr{H}(\cdot,\mathfrak{u}(\cdot),\mathfrak{u}(\cdot))\right\|_\infty,
\end{aligned}$$

where $C(a,b) = \frac{|b|}{|a+b|}$. Since $\mathscr{H}$ is a continuous function, we have

$$\|\mathscr{P}(\mathfrak{u}_n) - \mathscr{P}(\mathfrak{u})\|_\infty \leq \frac{(\psi(T))^\omega}{\Gamma(\omega+1)}\left(1+\frac{|b|}{|a+b|}\right) \left\|\mathscr{H}(\cdot,\mathfrak{u}_n(\cdot),\mathfrak{u}_n(\cdot)) - \mathscr{H}(\cdot,\mathfrak{u}(\cdot),\mathfrak{u}(\cdot))\right\|_\infty$$

as $n \to \infty$. □

Lemma 3.3. *The operator $\mathscr{P}$ maps bounded sets into bounded sets in Φ.*

Proof: In fact, it is sufficient to prove that for any $q > 0$, there exists a positive constant ζ such that for each $\mathfrak{u} \in \mathscr{D}_q = \{\mathfrak{u} \in \Phi : \ \|\mathfrak{u}\|_\infty \leq q\}$, we have $\|\mathscr{P}(\mathfrak{u})\|_\infty \leq \zeta$.
Inview of (A3) and for every $t \in \mathscr{I}$, one has

$$\begin{aligned}
|\mathscr{P}(\mathfrak{u})(t)| &\leq \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} |\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&+ \frac{|b|}{|a+b|}\frac{1}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1} |\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))|\, ds \\
&+ \frac{|c|}{|a+b|} \\
&\leq \frac{\mathfrak{M}}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} ds \\
&+ \frac{|b|}{|a+b|}\frac{\mathfrak{M}}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1} ds + \frac{|c|}{|a+b|} \\
&\leq \frac{\mathfrak{M}}{\Gamma(\omega+1)}(\psi(T))^\omega + \frac{\mathfrak{M}\,|b|}{|a+b|\,\Gamma(\omega+1)}(\psi(T))^\omega + \frac{|c|}{|a+b|}.
\end{aligned}$$

Therefore,

$$\|\mathscr{P}(\mathfrak{u})\|_\infty \leq \frac{\mathfrak{M}}{\Gamma(\omega+1)}(\psi(T))^\omega + \frac{\mathfrak{M}\,|b|}{|a+b|\,\Gamma(\omega+1)}(\psi(T))^\omega + \frac{|c|}{|a+b|} := \zeta.$$

□

Lemma 3.4. *The operator $\mathscr{P}$ maps bounded sets into equi-continuous sets of Φ.*

Proof: Let $t_1, t_2 \in \mathscr{I}$, $t_1 < t_2$, $\mathscr{D}_q$ be a bounded set of Φ as in Lemma 3.3, and let $\mathfrak{u} \in \mathscr{D}_q$. Then

$$\begin{aligned}
&|\mathscr{P}(\mathfrak{u})(t_2) - \mathscr{P}(\mathfrak{u})(t_1)| \\
&\leq \left| \frac{1}{\Gamma(\omega)} \int_0^{t_1} \psi^{'}(s)\left[(\psi(t_2)-\psi(s))^{\omega-1} - (\psi(t_1)-\psi(s))^{\omega-1}\right] \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds \right. \\
&\qquad \left. + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} \psi^{'}(s)(\psi(t_2)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds \right| \\
&\leq \frac{\mathfrak{M}}{\Gamma(\omega)} \int_0^{t_1} \psi^{'}(s)\left[(\psi(t_2)-\psi(s)))^{\omega-1} - (\psi(t_1)-\psi(s))^{\omega-1}\right] ds \\
&\qquad + \frac{\mathfrak{M}}{\Gamma(\omega)} \int_{t_1}^{t_2} \psi^{'}(s)(\psi(t_2)-\psi(s))^{\omega-1} ds \\
&\leq \frac{\mathfrak{M}}{\Gamma(\omega+1)}(\psi(t_2)-\psi(t_1))^\omega + \frac{\mathfrak{M}}{\Gamma(\omega+1)}((\psi(t_1))^\omega - (\psi t_2))^\omega).
\end{aligned}$$

The right-hand side of above inequality tends to 0 by taking $t_1 \to t_2$. Hence by using Lemma 3.2-Lemma 3.4 together with Arzela-Ascoli theorem, we come across that $\mathscr{H} : \Phi \to \Phi$ is completely continuous. □

Theorem 3.5. *(Existence of at least one solution) Assume that(A1)-(A3) hold. Then the problem* (1.1)-(1.2) *has at least one solution on $\mathscr{I}$.*

Proof: Let $\mathscr{P} : \Phi \to \Phi$ be the operator defined in the beginning of this section. It is continuous and bounded (see Lemmas 3.2-3.4). Set

$$\kappa = \{\mathfrak{u} \in \Phi : \mathfrak{u} = \lambda\mathscr{P}(\mathfrak{u}) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $\mathfrak{u} \in \kappa$, then $\lambda\mathscr{P}(\mathfrak{u})$ for some $\lambda \in (0,1)$. Hence, for each $t \in \mathscr{I}$ we have

$$\begin{aligned}
\mathfrak{u}(t) = \lambda \Bigg[&\frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds \\
&- \frac{1}{a+b}\left[\frac{b}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds - c \right]\Bigg].
\end{aligned}$$

We complete this theorem by considering the estimation in Lemma 3.3. Thank to Schaefer's fixed point theorem, we finish the proof that $\mathscr{P}$ has at least one fixed point which is the solution of the problem (1.1)-(1.2). □

Theorem 3.6. *(Uniqueness of solution) Under the hypothesis (A1),(A2), if*

$$\frac{2\mathfrak{K}(\psi(T))^{\omega}}{\Gamma(\omega+1)}\left(1+\frac{|b|}{|a+b|}\right)<1, \tag{3.2}$$

then the problem (1.1)-(1.2) has only one solution on $\mathscr{I}$.

Proof: Here it is remarked that the fixed points of the operator $\mathscr{P}$ implies the solutions of the problem (1.1)-(1.2). With the help of Banach fixed point theorem, we shall demonstrate that $\mathscr{P}$ is a contraction. Let for every $t\in\mathscr{I}$ and $\mathfrak{u},\mathfrak{v}\in\Phi$, then we proceed as

$$\begin{aligned}
&|\mathscr{P}(\mathfrak{u})(t)-\mathscr{P}(\mathfrak{v})(t)|\\
&\leq\frac{1}{\Gamma(\omega)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1}\left|\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))-\mathscr{H}(s,\mathfrak{v}(s),\mathfrak{v}(\eta s))\right|ds\\
&-\frac{|b|}{|a+b|\,\Gamma(\omega)}\int_0^T\psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1}\left|\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))-\mathscr{H}(s,\mathfrak{v}(s),\mathfrak{v}(\eta s))\right|ds\\
&\leq\frac{2\mathfrak{K}\left\|\mathfrak{u}-\mathfrak{v}\right\|_{\infty}}{\Gamma(\omega)}\int_0^T\psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1}ds\\
&-\frac{|b|\,2\mathfrak{K}\left\|\mathfrak{u}-\mathfrak{v}\right\|_{\infty}}{|a+b|\,\Gamma(\omega)}\int_0^T\psi^{'}(s)(\psi(T)-\psi(s))^{\omega-1}ds\\
&\leq\left[\frac{2\mathfrak{K}(\psi(T))^{\omega}}{\Gamma(\omega+1)}\left(1+\frac{|b|}{|a+b|}\right)\right]\left\|\mathfrak{u}-\mathfrak{v}\right\|_{\infty}.
\end{aligned}$$

Therefore,

$$\left\|\mathscr{P}(\mathfrak{u})-\mathscr{P}(\mathfrak{v})\right\|_{\infty}\leq\left[\frac{2\mathfrak{K}(\psi(T))^{\omega}}{\Gamma(\omega+1)}\left(1+\frac{|b|}{|a+b|}\right)\right]\left\|\mathfrak{u}-\mathfrak{v}\right\|_{\infty}.$$

As a result by (3.2), $\mathscr{P}$ is a contraction. Therefore by Banach contraction theorem, the proposed problem (1.1)-(1.2) has unique solution. □

4. Stability analysis

Theorem 4.1. *Assume that $(A1)-(A2)$ and (3.2) hold. Then, the problem (1.1)-(1.2) is UH stable.*

Proof: Let $\epsilon>0$ and let $\mathfrak{Z}\in\Phi$ be a function which satisfies the inequality (2.5) and let $\mathfrak{u}\in\Phi$ be the unique solution of the following problem

$$\begin{aligned}
&{}^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t)=\mathscr{H}(t,\mathfrak{u}(t),\mathfrak{u}(\eta t)),\ \eta\in(0,1),\quad t\in\mathscr{I},\ \omega\in(0,1),\\
&\mathfrak{u}(0)=\mathfrak{Z}(0),\ \mathfrak{u}(T)=\mathfrak{Z}(T).
\end{aligned}$$

Applying Lemma 3.4, we obtain

$$\mathfrak{u}(t)=\mathfrak{A}_{\mathfrak{u}}+\frac{1}{\Gamma(\omega)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds.$$

Alternatively , if $\mathfrak{u}(0) = \mathcal{Z}(0),\ \mathfrak{u}(T) = \mathcal{Z}(T)$, then $\mathfrak{A}_{\mathfrak{u}} = \mathfrak{A}_{\mathcal{Z}}$.
In fact,

$$\begin{aligned}
&|\mathfrak{A}_{\mathfrak{u}} - \mathfrak{A}_{\mathcal{Z}}| \\
&\leq \frac{C(a,b)}{\Gamma(\omega)} \int_0^T \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \left|\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s)) - \mathscr{H}(s,\mathcal{Z}(s),\mathcal{Z}(\eta s))\right| ds \\
&\leq \frac{2\mathfrak{K}\,|b|}{|a+b|} I^{\omega;\psi} \left|\mathfrak{u}(T) - \mathcal{Z}(T)\right| \\
&= 0.
\end{aligned}$$

Therefore, $\mathfrak{A}_u = \mathfrak{A}_{\mathcal{Z}}$. We have

$$\mathfrak{u}(t) = \mathfrak{A}_{\mathcal{Z}} + \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds.$$

By integration of the inequality (2.5) and using Remark 2.10, we obtain

$$\begin{aligned}
&\left|\mathcal{Z}(t) - \mathfrak{A}_{\mathcal{Z}} - \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \mathscr{H}(s,\mathcal{Z}(s),\mathcal{Z}(\eta s))ds\right| \\
&\leq \epsilon \frac{(\psi(T))^{\omega}}{\Gamma(\omega+1)} \left(1 + \frac{|b|}{|a+b|}\right).
\end{aligned}$$

We have for any $t \in \mathscr{I}$

$$\begin{aligned}
&|\mathcal{Z}(t) - \mathfrak{u}(t)| \\
&\leq \left|\mathcal{Z}(t) - \mathfrak{A}_{\mathcal{Z}} - \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \mathscr{H}(s,\mathcal{Z}(s),\mathcal{Z}(\eta s))ds\right| \\
&+ \frac{1}{\Gamma(\omega)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\omega-1} \left|\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s)) - \mathscr{H}(s,\mathcal{Z}(s),\mathcal{Z}(\eta s))\right| ds \\
&\leq \epsilon \frac{(\psi(T))^{\omega}}{\Gamma(\omega+1)} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{2\mathfrak{K}}{\Gamma(\omega)} \int_0^t \psi^{'}(t)(\psi(t)-\psi(s))^{\omega-1} \left|\mathcal{Z}(s) - \mathfrak{u}(s)\right| ds
\end{aligned}$$

Using Grönwall inequality, Lemma 2.12 and Remark 2.13, we obtain

$$|\mathcal{Z}(t) - \mathfrak{u}(t)| \leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{\epsilon(\psi(T))^{\omega}}{\Gamma(\omega+1)} E_{\omega;\psi}\left(2\mathfrak{K}(\psi(T))^{\omega}\right).$$

Thus, the problem (1.1)-(1.2) is UH stable. □

Theorem 4.2. *Assume* $(A1) - (A2)$ *together with inequality* (3.2) *and*

(A4) there exists an increasing function $\varphi \in \Phi$ *and* $\lambda_{\varphi} > 0$ *such that*

$$I^{\omega;\psi}\varphi(t) \leq \lambda_{\varphi}\varphi(t), \quad \textit{for every} \quad t \in \mathscr{I}$$

hold. Then the proposed problem (1.1)-(1.2) is UHR stable.

Proof: Let $\epsilon > 0$ and let $\mathfrak{Z} \in \Phi$ be a function which satisfies inequality (2.6) and let $\mathfrak{u} \in \Phi$ be the unique solution for the given BVP

$$\begin{aligned}&{}^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \mathscr{H}(t, \mathfrak{u}(t), \mathfrak{u}(\eta t)),\ \eta \in (0,1), \quad t \in \mathscr{I},\ \omega \in (0,1),\\ &\mathfrak{u}(0) = \mathfrak{Z}(0),\ \mathfrak{u}(T) = \mathfrak{Z}(T).\end{aligned}$$

Applying Lemma 3.4, we get

$$\mathfrak{u}(t) = \mathfrak{A}_{\mathfrak{u}} + \frac{1}{\Gamma(\omega)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds.$$

We know that, $\mathfrak{A}_u = \mathfrak{A}_{\mathfrak{Z}}$. We have

$$\mathfrak{u}(t) = \mathfrak{A}_{\mathfrak{Z}} + \frac{1}{\Gamma(\omega)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s))ds.$$

By integration of the inequality (2.6), we obtain

$$\begin{aligned}&\left|\mathfrak{Z}(t) - \mathfrak{A}_{\mathfrak{Z}} - \frac{1}{\Gamma(\omega)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{Z}(s),\mathfrak{Z}(\eta s))ds\right|\\ &\leq \epsilon\lambda_{\varphi}\varphi(t)\left(1+\frac{|b|}{|a+b|}\right).\end{aligned}$$

We have for any $t \in \mathscr{I}$

$$\begin{aligned}&|\mathfrak{Z}(t)-\mathfrak{u}(t)|\\ &\leq \left|\mathfrak{Z}(t) - \mathfrak{A}_{\mathfrak{Z}} - \frac{1}{\Gamma(\omega)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\omega-1}\mathscr{H}(s,\mathfrak{Z}(s),\mathfrak{Z}(\eta s))ds\right|\\ &+ \frac{1}{\Gamma(\omega)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\omega-1}\left|\mathscr{H}(s,\mathfrak{u}(s),\mathfrak{u}(\eta s)) - \mathscr{H}(s,\mathfrak{Z}(s),\mathfrak{Z}(\eta s))\right|ds\\ &\leq \epsilon\lambda_{\varphi}\varphi(t)\left(1+\frac{|b|}{|a+b|}\right) + \frac{2\mathfrak{K}}{\Gamma(\omega)}\int_0^t \psi'(t)(\psi(t)-\psi(s))^{\omega-1}|\mathfrak{Z}(s)-\mathfrak{u}(s)|\,ds.\end{aligned}$$

Using Gronwall inequality, Lemma 2.12 and Remark 2.13, we obtain

$$|\mathfrak{Z}(t)-\mathfrak{u}(t)| \leq \left(1+\frac{|b|}{|a+b|}\right)\epsilon\lambda_{\varphi}\varphi(t)E_{\omega;\psi}\left(2\mathfrak{K}(\psi(T))^{\omega}\right),\ t \in \mathscr{I}.$$

Thus, the problem (1.1)-(1.2) is UHR stable. □

5. An example

For the demonstration of our results, we take the following ψ- FPE

$${}^{c}\mathscr{D}^{\omega;\psi}\mathfrak{u}(t) = \frac{1}{6} + \frac{1}{5}\mathfrak{u}(t) + \frac{1}{5}\sin\mathfrak{u}\left(\frac{t}{2}\right),\ t \in \mathscr{I} = [0,1],\ \omega \in (0,1], \tag{5.1}$$

$$\mathfrak{u}(0) + \mathfrak{u}(1) = 0. \tag{5.2}$$

Set

$$\mathscr{H}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{6} + \frac{1}{5}\mathbf{x}(t) + \frac{1}{5}\sin \mathbf{y}\left(\frac{t}{2}\right), \ \mathbf{x}, \mathbf{y} \in \mathbf{R}, \ t \in \mathscr{I}.$$

Let $\mathbf{x}, \overline{\mathbf{x}}, \mathbf{y}, \overline{\mathbf{y}} \in \mathbf{R}$ and $t \in \mathscr{I}$. Then, we have

$$\begin{aligned} |\mathscr{H}(t, \mathbf{x}, \mathbf{y}) - \mathscr{H}(t, \overline{\mathbf{x}}, \overline{\mathbf{y}})| &\leq \frac{1}{5}|\mathbf{x} - \overline{\mathbf{x}}| + \frac{1}{5}|\mathbf{y} - \overline{\mathbf{y}}| \\ &\leq \frac{1}{5}\left(|\mathbf{x} - \overline{\mathbf{x}}| + |\mathbf{y} - \overline{\mathbf{y}}|\right). \end{aligned}$$

Therefore the condition (A2) holds with $\mathfrak{K} = \frac{1}{5}$. We shall verify that condition (3.2) is satisfied for suitable values of $\omega = \frac{1}{5}$, $\eta = \frac{1}{2}$, $a = b = T = 1$. In fact,

$$\frac{2\mathfrak{K}(\psi(T))^{\omega}}{\Gamma(\omega + 1)}\left(1 + \frac{|b|}{|a + b|}\right) \approx 0.6535 < 1.$$

Thank to Theorem 3.6, the problem (5.1)-(5.2) has a unique solution on $\mathscr{I}$. Also due to Theorem (4.1), the solution of the problem (5.1)-(5.2) is UH stable and consequently is UHR stable.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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