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Dynamics and Bifurcations of a Ratio-dependent Predator-prey Model

P. Azizi and R. Khoshsiar Ghaziani

ABSTRACT: In this paper, we study a ratio-dependent predator-prey model with modified Holling-Tanner formalism, by using dynamical techniques and numerical continuation algorithms implemented in Matcont. We determine codim-1 and 2 bifurcation points and their corresponding normal form coefficients. We also compute a curve of limit cycles of the system emanating from a Hopf point.

Key Words: Hopf bifurcation, Bogdanov-Takens bifurcation, Dynamical behaviour, Limit cycle.

Contents

1. Introduction

In recent decades, ecological systems gained a significant amount of attention which is the interactions between individuals and their surrounding environment. Movement rates of interacting individuals can dramatically affect population stability [3], [4] and the composition of communities. A convenient approach to describe the spatio-temporal population dynamics is the reaction-diffusion framework [5]. Reaction-diffusion systems were first used to explain the formation of spatial patterns in ecological systems by Segel and Jackson [6], inspired by the seminal work of Turing [8]. Baurmann et al. [2] investigated different instability mechanisms behind the emergence of spatial patterns in an ecological prey–predator system.

Turing instability and Turing-Hopf bifurcation are two well known mechanisms behind the formation of spatial pattern [1], [2], [7]. The concept of Turing instability is that the locally stable homogeneous steady state becomes unstable due to small amplitude heterogeneous perturbation around the homogeneous steady states, leading to the formation of spatially heterogeneous distribution of population over their habitats [6], [8]. The spatial patterns generated can be obtained by numerical simulations, and the conditions for Turing and Hopf bifurcations can be reached by means of local stability analysis around the suitable homogeneous steady state [2], [9].

In paper we presented a ratio-dependent predator-prey model and determine all codim 1 and codim 2 bifurcation points of a the system. We further numerically compute critical normal form coefficients corresponding to each bifurcation by Matcont.

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2. The system, its equilibria and their stability analysis

Ratio-dependent prey-predator model with modified Holling-Tanner functional response is formulated as follows [10]:

$$\begin{cases} \dfrac{du}{dt} = u(\alpha_1 - \beta_1 u - \dfrac{\gamma_1 v}{m_1 v + u}), \\ \dfrac{dv}{dt} = v(\alpha_2 - \dfrac{\gamma_2 v}{m_2 + u}), \end{cases} \tag{2.1}$$

where all the parameters are positive. For details of the model and parameters, see table [1].

Table 1: Definition of the parameter used in model (1).

Parameter	Definition
α_1	intrinsic growth rates of prey
α_2	intrinsic growth rates of predator
β_1	intra-specific competition rate of prey
γ_1	rate of capture of prey by the predator
m_1	dimensionless quantity and inversely proportional to the handling time
γ_2	strength of intra-species competition for the predator
m_2	measure of the environmental carrying capacity for the predator

In this section, we analyse the equilibria of (1) and their stability, as well as possible bifurcations of these equilibria.
$E_0 = (0,0)$ is a trivial equilibrium at the system (1). Suppose the following conditions hold:

$$a = -\beta_1$$
$$b = \alpha_1 - \beta_1 m_1 v,$$
$$c = \alpha_1 m_1 v - \gamma_1 v.$$

Case A: If $\alpha_1 m_1 = \gamma_1$, we have two further positive equilibria:

$$E_1 = \left(0, \frac{\alpha_2(u_1 + m_2)}{\gamma_2}\right), \qquad E_2 = \left(\frac{\alpha_1}{\beta_1} - m_1 v, \frac{\alpha_2(u_2 + m_2)}{\gamma_2}\right).$$

Case B: If condition $\alpha_1 m_1 > \gamma_1$ holds then we have following equilibrium:

$$E_3 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{\alpha_2(u_3 + m_2)}{\gamma_2}\right).$$

Case C: If conditions $\alpha_1 m_1 < \gamma_1$ and $b^2 \geq 4ac$ hold then, we have two further positive equilibria:

$$E_{4,5} = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \frac{\alpha_2(u_{4,5} + m_2)}{\gamma_2}\right).$$

We examine the stability of the equilibria of the system (1) by examining Jacobian matrix of system at each equilibrium. For an arbitrary equilibrium point of E, the Jacobian matrix is given by:

$$\mathbf{J}(E) = \begin{pmatrix} -2\beta_1 u + \alpha_1 - \dfrac{\gamma_1 m_1 v^2}{(m_1 v + u)^2} & -\dfrac{\gamma_1 u^2}{(m_1 v + u)^2} \\ \dfrac{\gamma_2 v^2}{(m_2 + u)^2} & \alpha_2 - 2\dfrac{\gamma_2 v}{(m_2 + u)} \end{pmatrix}.$$

The characteristic equation is given by:

$$\lambda^2 - \lambda tr(J_E) + det(J_E) = 0,$$

where

$$tr(J_E) = -2\beta_1 u + \alpha_1 - \frac{\gamma_1 m_1 v^2}{(m_1 v + u)^2} + \alpha_2$$

and

$$det(J_E) = (-2\beta_1 u + \alpha_1 - \frac{\gamma_1 m_1 v^2}{(m_1 v + u)^2})\alpha_2 + (\frac{\alpha_2^2}{\gamma_2})(\frac{\gamma_1 u^2}{(m_1 v + u)^2}).$$

Therefore, the eigenvalues of $J(E)$ are given by:

$$\lambda_{1,2} = \frac{1}{2} tr(J_E) \pm \sqrt{tr(J_E)^2 - 4det(J_E)}.$$

We define Routh-Hurwitz's table as follows:

Table 2: Definition of the parameter used in model (1).

λ^2	1	$det(J_E)$
λ^1	$-tr(J_E)$	0
λ^0	$det(J_E)$	0

Theorem 2.1. *The equilibrium E_0 is always unstable.*

Proof: At E_0, the characteristic polynomial is given by:

$$\mathbf{J}(E_0) = \begin{pmatrix} \alpha_1 & 0 \\ \dfrac{\alpha_2^2}{\gamma_2} & \alpha_2 \end{pmatrix}$$

$$det(J(E_0) - \lambda I) = \lambda^2 + \lambda(-\alpha_1 - \alpha_2) + \alpha_1\alpha_2$$

which has the solutions:

$$\lambda_1 = \alpha_1, \qquad \lambda_2 = \alpha_2$$

Since α_1 and α_1 are positive parameters, then E_0 is unstable. □

Theorem 2.2. *The equilibrium E_1 is always unstable.*

Proof: In accordance with the table 2, a simple calculation shows that

$$tr(J(E_1)) = \alpha_2$$

This implies $tr(J(E_1) > 0$, therefore the equilibrium point E_1 is unstable. □

Theorem 2.3. *The equilibrium E_2:*
(i) is asymptotically stable if $\alpha_1 \in \left(\frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta}), +\infty\right)$.
(ii) is unstable if $\alpha_1 \in \left(0, \frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta})\right)$.
(iii) Undergoes a Hopf bifurcation if $\alpha_1 = \frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta})$ *and* $m_1\alpha_2 > \gamma_2$.

Proof:

$$\begin{aligned}tr(J(E_1)) &= -\alpha_1 + 2\beta_1 m_1 v_2 - \frac{(m_1 v_2 \beta_1)^2}{\alpha_1} + \alpha_2 \\ &= \frac{1}{\alpha_1}(-\alpha_1^2 + 2\beta_1 \alpha_1 m_1 v_2 - (m_1 v_2 \beta_1)^2 \alpha_1 + \alpha_2).\end{aligned}$$

The equilibrium point E_2 is locally asymptotically stable if $tr(J(E_1)) < 0$, then

$$\alpha_1 \in \left[\frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta}), +\infty\right).$$

We have a Hopf bifurcation if $tr(J(E_1)) = 0$, so that $\alpha_1 = \frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta})$. Then the equilibrium point E_2 is unstable if $tr(J(E_1)) = 0$ and $det(J(E_1)) > 0$. So,

$$tr(J(E_1)) = \alpha_1 \in \left(0, \frac{1}{2}(2\beta_1 m_1 v_2 + \alpha_2 + \sqrt{\Delta})\right)$$

and

$$det(J(E_1)) = \frac{\alpha_2}{\alpha_1}(\alpha_1 - \beta_1 m_1 v)^2 (\frac{m_1 \alpha_2}{\gamma_2} - 1),$$

which implies $m_1 \alpha_2 > \gamma_2$. □

Theorem 2.4. *The equilibrium E_i, $i = 3, 4, 5$ undergoes:*
1) a Hopf bifurcation if $-2\beta_1 u_i + \alpha_1 - \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2} = -\alpha_2$.
2) E_2 is asymptotically stable if $\alpha_2 < 2\beta_1 u_i - \alpha_1 + \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2}$.
3) E_2 is unstable if $\alpha_2 > 2\beta_1 u_i - \alpha_1 + \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2}$.

Proof: The Jacobian matrix for system (1) at E_i is given by:

$$\mathbf{J}(E_i) = \begin{pmatrix} -2\beta_1 u_i + \alpha_1 - \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2} & \frac{\gamma_1 u_i^2}{(m_1 v_i + u_i)^2} \\ \frac{\alpha_2^2}{\gamma_2} & \alpha_2 \end{pmatrix}.$$

According to the table 2, if $tr(J_{E_i}) < 0$ the equilibrium point E_i is locally asymptotically stable, therefore:

$$\alpha_2 < 2\beta_1 u_i - \alpha_1 + \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2}$$

and unstable if:

$$\alpha_2 > 2\beta_1 u_i - \alpha_1 + \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2}$$

Then system (1) undergoes a Hopf bifurcation if $tr(J_{E_i}) = 0$. Hence:

$$-2\beta_1 u_i + \alpha_1 - \frac{\gamma_1 m_1 v_i^2}{(m_1 v_i + u_i)^2} = -\alpha_2$$

and

$$\lambda_{1,2} = \pm i\sqrt{det(J_{E_i})}.$$

□

3. Bifurcations

In this section, we shall study the stability and bifurcations of positive equilibria. We first discuss the existence of a Hopf bifurcation and a Bogdanov-Takens bifurcation of the system (1).

3.1. The Bogdanov-Takens (or double zero) bifurcation analysis

We first prove the unique positive equilibrium of system (1) is a Bogdanov-Takens of codimension 2. If $tr(J|_{E_i}) = 0, det(J|_{E_i}) = 0$, then the Jacobian matrix at E_i has double zero eigenvalues. We obtain:

Theorem 3.1. *If we choose α_1 and α_2 as bifurcation parameters, then system (1) undergoes a Bogdanov-Takens bifurcation in a small neighbourhood of E_3.*

Proof: Consider the following system:

$$\begin{cases} \dfrac{du}{dt} = u((\alpha_1+\lambda_1) - \beta_1 u - \dfrac{\gamma_1 v}{(m_1)v+u}) = f_1(u,v), \\ \dfrac{dv}{dt} = v((\alpha_2+\lambda_2) - \dfrac{\gamma_2 v}{m_2+u}) = f_2(u,v), \end{cases} \tag{3.1}$$

where (λ_1, λ_2) is a parameter vector in a small neighbourhood of $(0;0)$. In this case, with the help of the transformation $u = u_1+u_*$, $v = v_1+v_*$, $\alpha_1 = \alpha_1^*+\lambda_1$ and $\alpha_2 = \alpha_2^*+\lambda_2$, system (2) can be written as:

$$\begin{cases} \dfrac{dx_1}{dt} = & p_0(\lambda) + a_2(\lambda)x_1 + b_2(\lambda)x_2 + p'_{11}(\lambda)x_1^2 \\ & +p'_{12}(\lambda)x_1x_2 + p'_{22}(\lambda)x_2^2 + O(\| x \|^3) \\ \dfrac{dx_2}{dt} = & q_0(\lambda) + c_2(\lambda)x_1 + d_2(\lambda)x_2 + q'_{11}(\lambda)x_1^2 \\ & +q'_{12}(\lambda)x_1x_2 + q'_{22}(\lambda)x_2^2 + O(\| x \|^3) \end{cases} \tag{3.2}$$

where

$$\begin{aligned} &a_2(\lambda) \mid_{(u^*,v^*)} = \frac{\partial F_1}{\partial u},\ b_2(\lambda) \mid_{(u^*,v^*)} = \frac{\partial F_1}{\partial v},\ p'_{12}(\lambda) \mid_{(u^*,v^*)} = \frac{\partial^2 F_1}{\partial u \partial v}, \\ &p'_{11}(\lambda) \mid_{(u^*,v^*)} = \frac{1}{2}\frac{\partial^2 F_1}{\partial u \partial u},\ p'_{22}(\lambda) \mid_{(u^*,v^*)} = \frac{1}{2}\frac{\partial^2 F_1}{\partial v \partial v},\ c_2(\lambda) \mid_{(u^*,v^*)} = \frac{\partial F_2}{\partial u}, \\ &d_2(\lambda) \mid_{(u^*,v^*)} = \frac{\partial F_2}{\partial v},\ q'_{12}(\lambda) \mid_{(u^*,v^*)} = \frac{\partial^2 F_2}{\partial u \partial v},\ q'_{11}(\lambda) \mid_{(u^*,v^*)} = \frac{1}{2}\frac{\partial^2 F_2}{\partial u \partial u}, \\ &q'_{22}(\lambda) \mid_{(u^*,v^*)} = \frac{1}{2}\frac{\partial^2 F_2}{\partial v \partial v} \end{aligned}$$

So that:

$$\begin{aligned} &a_2(\lambda) = (\alpha_1+\lambda_1) - 2\beta_1 u_* + \frac{\gamma_1 v_*^2 m_1}{(m_1 v_* + u_*)^2},\ b_2(\lambda) = \frac{-\gamma_1 u_* v_*}{(m_1 v_* + u_*)^3} \\ &p'_{11}(\lambda) = -\beta_1 + \frac{\gamma_1 v^2 m_1}{(m_1 v_* + u_*)^3},\ p'_{22}(\lambda) = \frac{u_*^2 m_1}{(m_1 v_* + u_*)^3} \\ &p'_{12}(\lambda) = \frac{-2\gamma_1 u_* v_* m_1}{(m_1 v_* + u_*)^3},\ p_0 = -\lambda_1 u_* v_* \\ &c_2(\lambda) = \frac{\gamma_2 v_*^2}{(m_2+u_*)^3},\ d_2(\lambda) = (\alpha_2+\lambda_2) - \frac{2\gamma_2 v_*}{(m_2+u_*)} \\ &q'_{11}(\lambda) = \frac{\gamma_2 v_*^2}{(m_2+u_*)^3},\ q'_{22}(\lambda) = \frac{-\gamma_2}{(m_2+u_*)} \\ &q'_{12}(\lambda) = \frac{2\gamma_2 v_*}{(m_2+u_*)^2},\ q_0 = -\lambda_2 u_* v_* \end{aligned}$$

Making the affine transformation

$$y_1 = x_1,\ y_2 = a_2x_1 + b_2x_2.$$

we have

$$\begin{cases} \dfrac{dy_1}{dt} = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}(\lambda)y_2^2 + O(\|y\|), \\ \dfrac{dy_2}{dt} = q\prime_0(\lambda) + c_3(\lambda)y_1 + d_3(\lambda)y_2 + \beta_{11}(\lambda)y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + O(\|y\|), \end{cases} \tag{3.3}$$

where

$$\begin{aligned} q_0\prime(\lambda) =& p_0a_2 + b_2q_0,\ c_3 = b_2c_2 - a_2d_2,\ d_3 = a_2 + d_2, \\ \alpha_{11} =& \frac{p'_{22}a_2^2}{b_2} - \frac{p'_{12}a_2}{b_2} + p'_{11},\ \alpha_{12} = -\frac{2p'_{22}a_2}{b_2^2} + \frac{p'_{12}}{b_2},\ \alpha_{22} = \frac{p'_{22}}{b_2^2}, \\ \beta_{11} =& b_2q'_{11} + a_2(p'_{11} - q'_{12}) - \frac{a_2^2(p'_{12} - q'_{22})}{b_2} + \frac{p'_{22}a_2^3}{b_2^2}, \\ \beta_{12} =& -(2\frac{p'_{22}a_2^2}{b_2^2} - \frac{a_2(p'_{12} - q'_{22})}{b_2} - q'_{12}),\ \beta_{22} = \frac{p'_{22}a_2}{b_2^2} + \frac{q'_{22}}{b_2}. \end{aligned}$$

The functions $q_0\prime(\lambda)$, α_{kl}, β_{kl}, are smooth functions of λ. We have $q\prime_0(\lambda^*) = c_3(\lambda^*) = d_3(\lambda^*) = 0$ and

$$\begin{aligned} \alpha_{11} =& -\frac{m_1}{(m_1v+u)}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2}) \\ & - \gamma_1uv(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2} - 2) \\ & + \frac{\gamma_1m_1v^2}{(m_1v+u)^3} - \beta_1, \end{aligned}$$

$$\alpha_{12} = 2m_1\left[\frac{1}{\gamma_1^2v^2}(-(m_1u)(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2}))\right],\ \alpha_{22} = \frac{m_1v+u}{m_1\gamma_1^2v^2},$$

$$\begin{aligned} \beta_{11} =& \frac{\gamma_1\gamma_2uv^3}{(m_1v+u)^3(m_2+u)^3} \\ & + (\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2})(-\beta_1\frac{\gamma_1v^2m_1}{(m_1v+u)^3} - \frac{2\gamma_2v}{(m_2+u)^2}) \\ & + \frac{(m_1+u)^2}{-\gamma_1uv}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2})\frac{-2u\gamma_1m_1v}{(m_1v+u)^3} + \frac{\gamma_2}{m_2+u} \\ & - \frac{m_1u}{(m_1v+u)(-\gamma_1v)}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2}), \end{aligned}$$

$$\begin{aligned} \beta_{12} =& -\frac{m_1+u}{\gamma_1v}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2})\frac{2}{m_1\gamma_1v}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2}) \\ & + \frac{m_1v+u}{u}(\frac{-2uv\gamma_1m_1}{(m_1v+u)^3}) + \frac{\gamma_2}{m_2+u} + \frac{2\gamma_2v}{(m_2+u)^2}, \end{aligned}$$

$$\beta_{22} = \frac{m_1v+u}{\gamma_1v}\frac{1}{m_1\gamma_1v}(\alpha_1 - 2\beta_1u - \frac{\gamma_1m_1v^2}{(m_1v+u)^2}) + \frac{\gamma_2(m_1v+u)}{(m_2+u)u}.$$

Now, we consider the following transformation:

$$z_1 = y_1,\ z_2 = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}y_2^2 + O(\|y\|).$$

This transformation brings (3.3) into the following from

$$\begin{cases} \dfrac{dz_1}{dt} = & z_2, \\ \dfrac{dz_2}{dt} = & g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{01}(\lambda)z_2 \\ & +g_{20}(\lambda)z_1^2 + g_{11}(\lambda)z_1z_2 + g_{02}(\lambda)z_2^2 + A, \end{cases} \tag{3.4}$$

where $A = O(\|z\|^3)$, $g_{00}(0) = 0$, $g_{10}(0) = 0$, $g_{01}(0) = 0$, and $z = (z_1, z_2)$. Furthermore, we also have

$$\begin{aligned} g_{00}(\lambda) =& q_0\prime(\lambda) - p_0(\lambda)d_3(\lambda) + ..., \\ g_{10}(\lambda) =& c_3(\lambda) + \alpha_{12}(\lambda)q_0\prime(\lambda) - \beta_{12}(\lambda)p_0(\lambda) + ..., \\ g_{01}(\lambda) =& d_3(\lambda) + 2\alpha_{22}(\lambda)q_0\prime(\lambda) - \alpha_{12}(\lambda)p_0(\lambda) - 2\beta_{22}(\lambda)p_0(\lambda), \\ g_{20}(\lambda) =& \beta_{11}(\lambda) - \alpha_{11}(\lambda)d_3(\lambda) + c_3(\lambda)\alpha_{12}(\lambda) + ..., \\ g_{02}(\lambda) =& \alpha_{12}(\lambda) + \beta_{22}(\lambda) - \alpha_{22}(\lambda)d_2(\lambda) + ..., \\ g_{11}(\lambda) =& \beta_{12}(\lambda) + 2\alpha_{11}(\lambda) + 2\alpha_{22}(\lambda)c_3(\lambda) - \alpha_{12}(\lambda)d_3(\lambda) + ..., \end{aligned}$$

Correspondingly,

$$\begin{aligned} g_{00}(\lambda^*) =& 0,\ g_{10}(\lambda^*) = 0,\ g_{01}(\lambda^*) = 0,\ g_{20}(\lambda^*) = \beta_{11}(\lambda^*), \\ g_{02}(\lambda^*) =& \alpha_{11}(\lambda^*) + \beta_{22},\ g_{11}(\lambda^*) = \beta_{12}(\lambda^*) + 2\alpha_{11}(\lambda^*). \end{aligned}$$

Again, we can write (3.4) as the following form:

$$\begin{cases} \dfrac{dz_1}{dt} = & z_2, \\ \dfrac{dz_2}{dt} = & (g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{20}(\lambda)z_1^2 + O(\|z\|^3)) \\ & +(g_{01}(\lambda)z_2 + (g_{11}(\lambda)z_1 + O(\|z\|^2))z_2 + (g_{02}(\lambda) + O(\|z\|))z_2^2 \\ = & \mu(z_1, \lambda) + \nu(z_1, \lambda)z_2 + \Phi(z, \lambda)z_2^2, \end{cases} \tag{3.5}$$

where μ, ν, Φ are smooth functions and satisfy the following conditions

$$\begin{aligned} & \mu(0, \lambda^*) = g_{00}(\lambda^*) = 0,\ \frac{\partial\mu}{\partial z_1}|_{(0,\lambda^*)} = g_{10}(\lambda^*) = 0, \\ & \frac{\partial^2\mu}{\partial z_1^2}|_{(0,\lambda^*)} = g_{20}(\lambda^*) = \beta_{11}(\lambda^*) = \rho_1 \neq 0, \\ & \nu(0, \lambda^*) = g_{01}(\lambda^*) = 0, \frac{\partial\nu}{\partial z_1}|_{(0,\lambda^*)} = g_{11}(\lambda^*) = \beta_{11}(\lambda^*) + 2\alpha_{11}(\lambda^*) = \rho_2 \neq 0. \end{aligned}$$

Since $\mu(0, \lambda^*) = 0$, $\frac{\partial\nu}{\partial z_1}|_{(0,\lambda^*)} = \rho_2 \neq 0$ (due to the nondegeneracy assumption), it follows from the Implicit function theorem that there exists a C^∞ function z_1 defined in a small neighbourhood of $\lambda = \lambda^*$ such that $\phi(\lambda^*) = 0$, $\nu(\phi, \lambda) = 0$ for any $\lambda \in N(\lambda^*)$. We now use a parameter-dependent shift of co-ordinates in the z_1-direction to annihilate the z_2 term on the RHS of the second equation of (3.5)

$$z_1 = u_1 + \phi(\lambda),\ z_2 = u_2.$$

The above transformation brings the system (3.5) to the following system

$$\begin{cases} \dfrac{du_1}{dt} = & u_2, \\ \dfrac{du_2}{dt} = & (h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 + O(\|u_1\|^3)) \\ & +(h_{01}(\lambda)u_2 + (h_{11}(\lambda)u_1 + O(\|u\|^2))u_2 + (h_{02}(\lambda) + O(\|u\|))u_2^2 \\ = & \overline{\mu}(u_1, \lambda) + \overline{\nu}(u_1, \lambda)u_2 + \overline{\Phi}(u, \lambda)u_2^2, \end{cases} \tag{3.6}$$

where $u = (u_1, u_2)$,

$$\begin{aligned} h_{00} = g_{00} + g_{10}\phi + ..., \ h_{10} = g_{10} + 2g_{20}\phi + ..., \\ h_{20} = g_{20} + ..., \ h_{01} = g_{01} + g_{11}\phi + ..., \\ h_{11} = g_{11} + ..., \ h_{02} = g_{02} + ... \end{aligned}$$

The coefficient of u_2 term on the RHS of the second equations of (3.6) is given by

$$\begin{aligned} h_{01} = \overline{\nu}(0,\lambda) = g_{01} + g_{11}\phi + O(\|\phi\|^2) =& [d_2 + 2\alpha_{22}q_0\prime - \alpha_{12}p_0 - 2\beta_{22}p_0 + ...] \\ &+ [\beta_{12} + 2\alpha_{11} + \alpha_{22}c_2 - \alpha_{12}d_2 + ...]\phi. \end{aligned}$$

Thus we have the following

$$h_{01}(0,\lambda) = g_{01}(\lambda^*) = 0, \ \frac{\partial h_{01}}{\partial \phi}|_{(0,\lambda^*)} = \beta_{12}(\lambda^*) + 2\alpha_{11}(\lambda^*) = \rho_2 \neq 0.$$

Let for $\lambda \in N(\lambda^*)$, $\phi(\lambda) \in M$. Then in the region M, $\phi(\lambda)$ can be approximated by

$$\phi(\lambda) \approx -\frac{g_{01}(\lambda)}{\rho_2}.$$

Thus, (3.6) reduces to the following

$$\left\{ \begin{array}{rcl} \dfrac{du_1}{dt} &=& u_2, \\ \dfrac{du_2}{dt} &=& h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 \\ && + h_{11}(\lambda)u_1u_2 + h_{02}(\lambda)u_2^2 + O(\|u\|)^3. \end{array} \right. \tag{3.7}$$

we now introduce a new time scale, defined by $dt = (1 + \psi u_1)d\tau$, where $\psi = \psi(\lambda)$ is a smooth function to be defined later. with this transformation, (3.7) reduces to

$$\left\{ \begin{array}{rcl} \dfrac{du_1}{d\tau} &=& u_2(1 + \psi u_1), \\ \dfrac{du_2}{d\tau} &=& h_{00} + (h_{10} + h_{00}\psi)u_1 + (h_{20} + h_{10}\psi)u_1^2 \\ && + h_{11}u_1u_2 + h_{02}u_2^2 + O(\|u\|)^3, \end{array} \right. \tag{3.8}$$

assume

$$\nu_1 = u_1, \ \nu_2 = u_2(1 + \psi u_1).$$

then we obtain,

$$\left\{ \begin{array}{rcl} \dfrac{d\nu_1}{d\tau} &=& \nu_2, \\ \dfrac{d\nu_2}{d\tau} &=& l_{00}(\lambda) + l_{10}(\lambda)\nu_1 + l_{20}(\lambda)\nu_1^2 \\ && + l_{11}(\lambda)\nu_1\nu_2 + l_{02}(\lambda)\nu_2^2 + O(\|\nu\|)^3, \end{array} \right. \tag{3.9}$$

where

$$\begin{aligned} l_{00}(\lambda) =& h_{00}, \ l_{10}(\lambda) = h_{10} + 2h_{00}\psi(\lambda), \\ l_{20} =& h_{20} + 2h_{00}\psi(\lambda) + h_{00}(\lambda)\psi(\lambda)^2, \\ l_{11}(\lambda) =& h_{11}(\lambda), \ l_{02}(\lambda) = h_{02} + \psi(\lambda). \end{aligned}$$

Now, we take $\psi(\lambda) = -h_{02}(\lambda)$ in order to get rid of ν_2^2-term. We then have

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = \nu_2, \\ \dfrac{d\nu_2}{d\tau} = \beta_1(\lambda) + \beta_2(\lambda)\nu_1 + \eta(\lambda)\nu_1^2 + \zeta(\lambda)\nu_1\nu_2 + O(\|\nu\|)^3, \end{cases} \tag{3.10}$$

where $v = (v_1, v_2)$,

$$\begin{aligned} \beta_1(\lambda) =& h_{00}(\lambda),\ \beta_2(\lambda) = h_{10}(\lambda) - 2h_{00}(\lambda)h_{02}(\lambda), \\ \eta(\lambda) =& h_{20}(\lambda) - 2h_{10}(\lambda)h_{02}(\lambda) + h_{02}^2(\lambda)h_{00}(\lambda) \neq 0, \\ \zeta(\lambda) =& h_{11}(\lambda) \neq 0. \end{aligned}$$

we now introduce a new time scale given by

$$t = |\frac{\eta(\lambda)}{\zeta(\lambda)}|\tau.$$

with the new stable variables $\xi_1 = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\nu_1$ and $\xi_2 = \frac{\eta^2(\lambda)}{\zeta^3(\lambda)}\nu_2$ such that $s = \text{sign}\frac{\eta(\lambda)}{\zeta(\lambda)} = \text{sign}\frac{\eta(\lambda^*)}{\zeta(\lambda^*)} = \frac{\rho_2}{g_{20}(\lambda^*)} = \pm 1$. (3.10) reduces to

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2 + O(\|\xi\|)^3, \end{cases} \tag{3.11}$$

where

$$\mu_1(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_1(\lambda),\ \mu_2(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_2(\lambda).$$

The system (3.11) is locally topologically equivalent near the origin for small $\|\mu\|$ to the system

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2, \end{cases} \tag{3.12}$$

where $s = \pm 1$. We have obtained the generic normal from of the Bogdanov-Takens bifurcation for the system(3.12)

$$\text{rank}(\frac{\partial(\mu_1, \mu_2)}{\partial\lambda})_{\lambda=\lambda^*} = 2,$$

$$J = \begin{vmatrix} \frac{\partial\mu_1}{\partial\lambda_2} & \frac{\partial\mu_1}{\partial\lambda_1} \\ \frac{\partial\mu_2}{\partial\lambda_2} & \frac{\partial\mu_2}{\partial\lambda_1} \end{vmatrix} \neq 0.$$

□

4. Numerical simulation

In this section, we report the results of numerical continuation method on system (1). By numerical continuation, we compute codim1 and 2 bifurcation and then compute a family of limit cycles which bifurcation from the Hopf point. This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm.

4.1. Continuation Curve of Equilibrium Point (one-parameter bifurcation diagram)

We consider the fixed parameters

$$\alpha_1 = 2,\ \beta_1 = 0.8,\ \gamma_2 = 0.45,\ m_1 = 0.2,\ m_2 = 0.1,$$

and starting from the initial point for first stage

$$(u(0), v(0)) = (1.5, 2),\ \alpha_2 = 0.5,\ \gamma_1 = 1.4,$$

the second stage $(u(0), v(0)) = (2.5, 3)$ and $\alpha_2 = 1.2$, $\gamma_1 = 0.8$, third stage $(u(0), v(0)) = (1, 2)$ and $\alpha_2 = 0.6$, $\gamma_1 = 1.3$ and fourth stage $(u(0), v(0)) = (0.3825144, 0.1286705)$ where $\alpha_2 = 0.12$, $\gamma_1 = 5$ is considered. We examine stability and unstability at equilibria is α_2 and γ_1 free parameter witch by changing the two points in four steps it will be obtained codim 1 and codim 2 bifurcation points, family of limit cycles and their corresponding critical normal form coefficients in Matlab software. Due to the negative coordinates or parameter for the equilibria point $E_i, i = 4, 5$, Hopf bifurcations are not acceptable. therefore, I analyse Hopf bifurcations in E_3 and E_2, so, in the curve the equilibrium E_3 is:

Table 3: Routh-Hurwitz's table in E_3.

lable	u value	v value
The first stage attract point	1.451373386	1.723678536
The second stage attract point	1.307287195	3.752470941
The third stage attract point	1.319658552	1.892649549
The third stage attract point	5.16987045$e_0$08	0.02666683107

Considering the initial points, the time of the graph is:

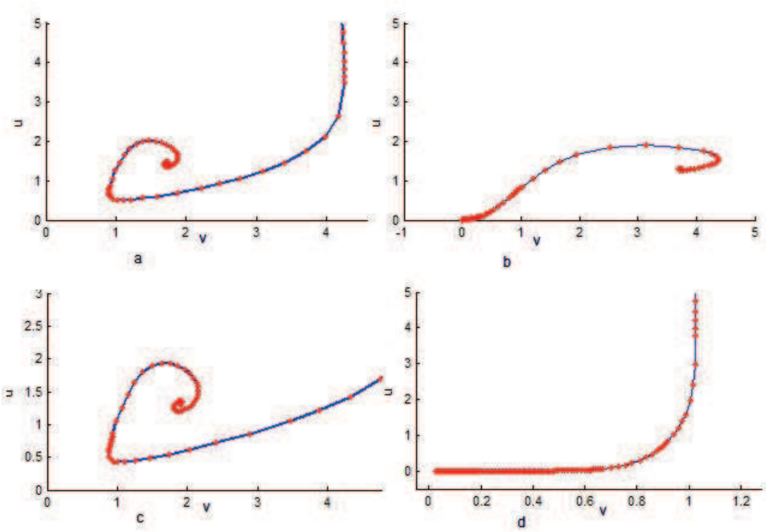

Figure 1: (a) curve diagram for first stage, (b)curve diagram for second stage, (b)curve diagram for third stag (d)curve diagram for fourth stage.

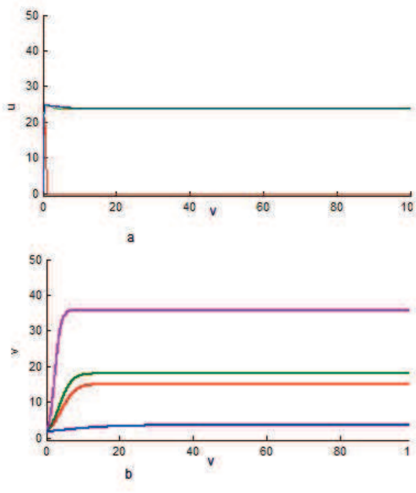

Figure 2: Time diagram for u and v.

4.2. Codim 1 and Hopf bifurcation and cycle continuation starting from the Hopf-point for E_3

By stating from initial point and fixed parameters and changing the parameters γ_1 and α_2 in four section we will had:
Codim 1 bifurcation in first stage under condition $\alpha_1 m_1 > \gamma_1$:

for free parameter α_2:
label = H, x = (0.773598, 0.950753, 0.489743)
First Lyapunov coefficient = 2.973372e-002
label = LP, x = (0.355455, 0.577015, 0.570104)
a=6.067227e-001

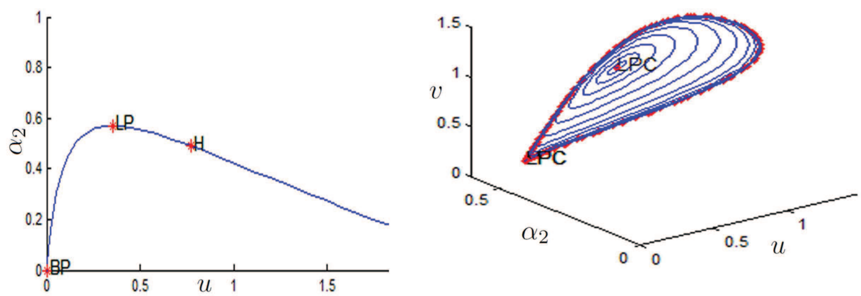

Figure 3: Limit cycle and Hopf curve at E_3 for first stage

Under condition $\alpha_1 m_1 > \gamma_1$, codim 1 bifurcation not exit in second stage.
Codim 1 bifurcation in third stage under condition $\alpha_1 m_1 > \gamma_1$:

for free parameter β_1:
label = H , x = (1.096141, 1.594855, 0.487965)
First Lyapunov coefficient = -8.717216e-003
label = LP, x = (0.310181, 0.546908, 0.984666)
a=9.556130e-001

for free parameter γ_1:
label = H , x = (0.658663, 1.011551, 1.253791)
First Lyapunov coefficient = 6.508452e-002
label = LP, x = (0.353060, 0.604080, 1.347349)
a=6.842578e-001

for free parameter α_2:
label = BP, x = (2.500000, 0.000000, 0.000000)
label = H , x = (0.700753, 0.996583, 0.560051)
First Lyapunov coefficient = 4.828517e-002
label = LP, x = (0.350716, 0.630709, 0.629708)
a=7.781495e-001

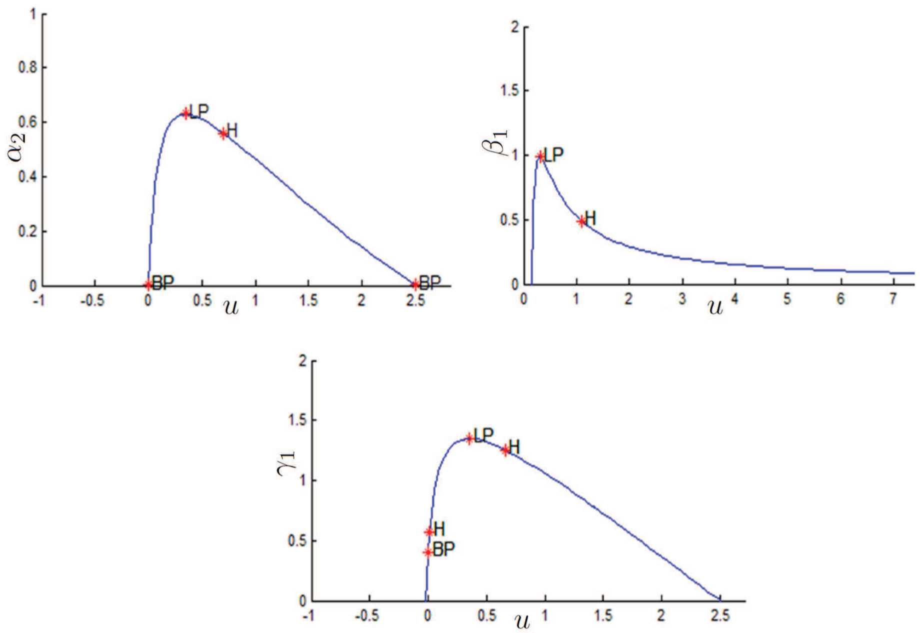

Figure 4: Hopf curve at E_3 for third stage.

Codim 1 bifurcation in four stage under condition $\alpha_1 m_1 > \gamma_1$:

for free parameter α_2:
label = BP, x = (2.500000, 0.000000, 0.000000)
label = H , x = (1.163570, 0.259920, 0.092566)
First Lyapunov coefficient = 6.997416e-002
label = LP, x = (0.395824, 0.142882, 0.129677)
a=8.751007e-002

for free parameter m_1:
label = H , x = (1.030719, 0.301525, 0.835429)
First Lyapunov coefficient = -6.885476e-002
label = LP, x = (0.411561, 0.136416, -0.024284)
a=-7.522370e-002
label = BP, x = (0.000000, 0.026667, 2.500000)

for free parameter γ_1:
label = H , x = (1.137644, 0.330038, 3.974816)
First Lyapunov coefficient = 6.231720e-002
label = LP, x = (0.396826, 0.132487, 5.376064)
a=8.035877e-002

for free parameter α_1:
label = H , x = (1.444925, 0.411980, 2.504642)
First Lyapunov coefficient = 2.972985e-002
label = LP, x = (0.382513, 0.128670, 1.881896)
a=8.714273e-002

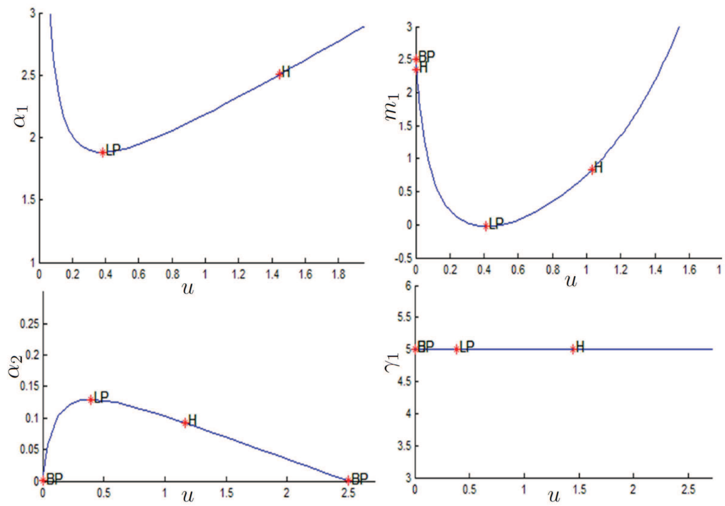

Figure 5: Hopf curve at E_3 for fourth stage.

4.3. Limit cycle starting from the Hopf-point for E_3

By starting Hopf point in the one-parameter bifurcation in four stage, we plot family of limit cycles. Bifurcation of limit cycle in first stage for free parameter α_2:

Limit point cycle (period = 1.280044e+001, parameter = 4.897432e-001)
Normal form coefficient = 3.333236e-002
Limit point cycle (period = 1.233198e+002, parameter = 4.812819e-001)
Normal form coefficient = 1.505258e+000
Limit point cycle (period = 1.708022e+002, parameter = 4.812818e-001)
Normal form coefficient = 1.429389e+000
Limit point cycle (period = 5.454173e+002, parameter = 4.812818e-001)
Normal form coefficient = 1.328374e+000
Limit point cycle (period = 7.801135e+002, parameter = 4.812818e-001)
Normal form coefficient = 1.742707e+000
Limit point cycle (period = 7.861108e+002, parameter = 4.812817e-001)
Normal form coefficient = -1.170958e+000
Limit point cycle (period = 9.241083e+002, parameter = 4.812819e-001)
Normal form coefficient = 1.462724e+000
Limit point cycle (period = 1.026107e+003, parameter = 4.812814e-001)
Normal form coefficient = -9.665144e-001

Bifurication of limit cycle in third stage:
for free parameter β_1:

Limit point cycle (period = 1.296772e+001, parameter = 4.912518e-001) Normal form coefficient = 2.177984e-002

for free parameter γ_1:

Limit point cycle (period = 6.752531e+001, parameter = 1.239319e+000)
Normal form coefficient = -3.450739e-001
Limit point cycle (period = 7.201274e+001, parameter = 1.239319e+000)
Normal form coefficient = 2.738747e-001

for free parameter α_2:

Limit point cycle (period = 1.294512e+001, parameter = 5.600522e-001)
Normal form coefficient = 5.773644e-002

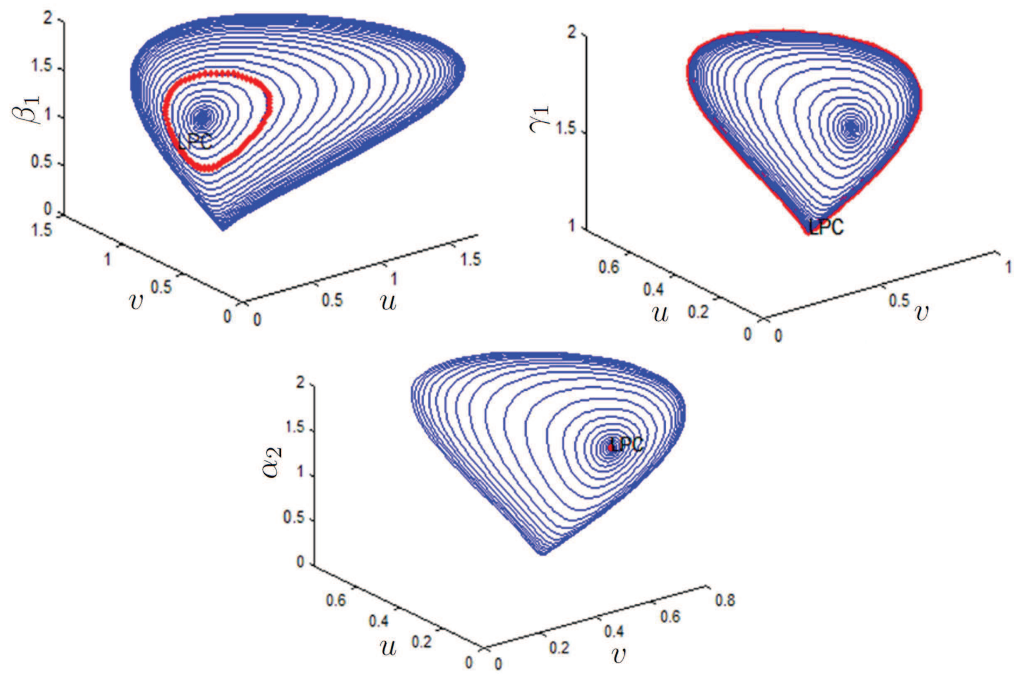

Figure 6: Family of limit cycles in interior equilibrium point E_3 for third stage.

Bifurcation of limit cycle in fourth stage:
for free parameter α_2:

Limit point cycle (period = 6.779822e+001, parameter = 9.233150e-002)
Normal form coefficient = -2.159508e-001
Limit point cycle (period = 1.046472e+002, parameter = 9.233185e-002)
Normal form coefficient = -1.554724e+000
Limit point cycle (period = 1.365217e+002, parameter = 9.233109e-002)
Normal form coefficient = -2.632481e+001
Limit point cycle (period = 1.597130e+002, parameter = 9.233234e-002)

Normal form coefficient = -1.516112e+001
Period Doubling (period = 1.576785e+002, parameter = 9.233231e-002)
Normal form coefficient = 3.209311e-009

for free parameter m_1:

Limit point cycle (period = 3.669945e+001, parameter = 8.281309e-001)
Normal form coefficient = 4.119744e-002

for free parameter α_1:

Limit point cycle (period = 4.038269e+001, parameter = 2.508117e+000)
Normal form coefficient = -3.181254e-001
Limit point cycle (period = 7.179243e+001, parameter = 2.508115e+000)
Normal form coefficient = -1.013944e+000
Limit point cycle (period = 8.428238e+001, parameter = 2.508119e+000)
Normal form coefficient = -2.729010e+000
Limit point cycle (period = 1.013298e+002, parameter = 2.508111e+000)
Normal form coefficient = -4.745843e+000
Limit point cycle (period = 1.233206e+002, parameter = 2.508123e+000)
Normal form coefficient = 2.742166e+000

for free parameter γ_1:

Limit point cycle (period = 7.025772e+001, parameter = 3.963935e+000)
Normal form coefficient = -1.241419e+000
Limit point cycle (period = 8.045373e+001, parameter = 3.963937e+000)
Normal form coefficient = -3.139889e+000
Period Doubling (period = 1.190927e+002, parameter = 3.963935e+000)
Normal form coefficient = -4.032949e-009
Limit point cycle (period = 1.206995e+002, parameter = 3.963935e+000)
Normal form coefficient = 2.797562e+001
Period Doubling (period = 1.190928e+002, parameter = 3.963935e+000)
Normal form coefficient = -4.030002e-009

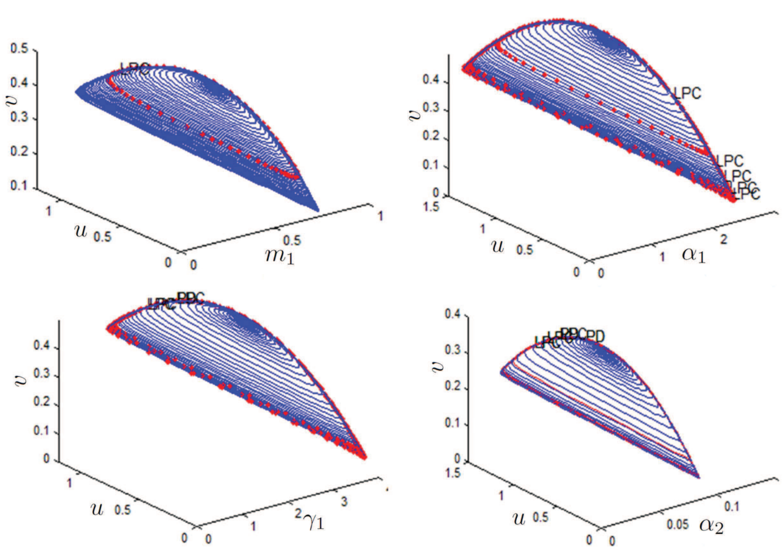

Figure 7: Family of limit cycles in interior equilibrium point E_3 for fourth stage.

4.4. Codim 2 bifurcation for E_3

By selecting Hopf point in the one-parameter bifurcation diagram of the equilibrium as initial point.

Codim 2 bifurcation in first stage:

for free parameter α_1:

label = BT , x = (0.416185, 1.589486, 3.364326, 1.385683)
(a,b)=(-3.136720e-001, -2.801808e-001)

for free parameter γ_1:

label = BT , x = (0.332144, 0.847537, 1.026511, 0.882557)
(a,b)=(-2.956389e-001, -4.812075e-001)

for free parameter m_1:

label = BT , x = (0.306198, 0.677048, 0.345448, 0.750057)
(a,b)=(-2.757923e-001, -4.666803e-001)

for free parameter γ_2:

label = BT , x = (0.355455, 0.577015, 1.010790, 0.797847)
(a,b)=(-4.740475e-001, -7.483568e-001)

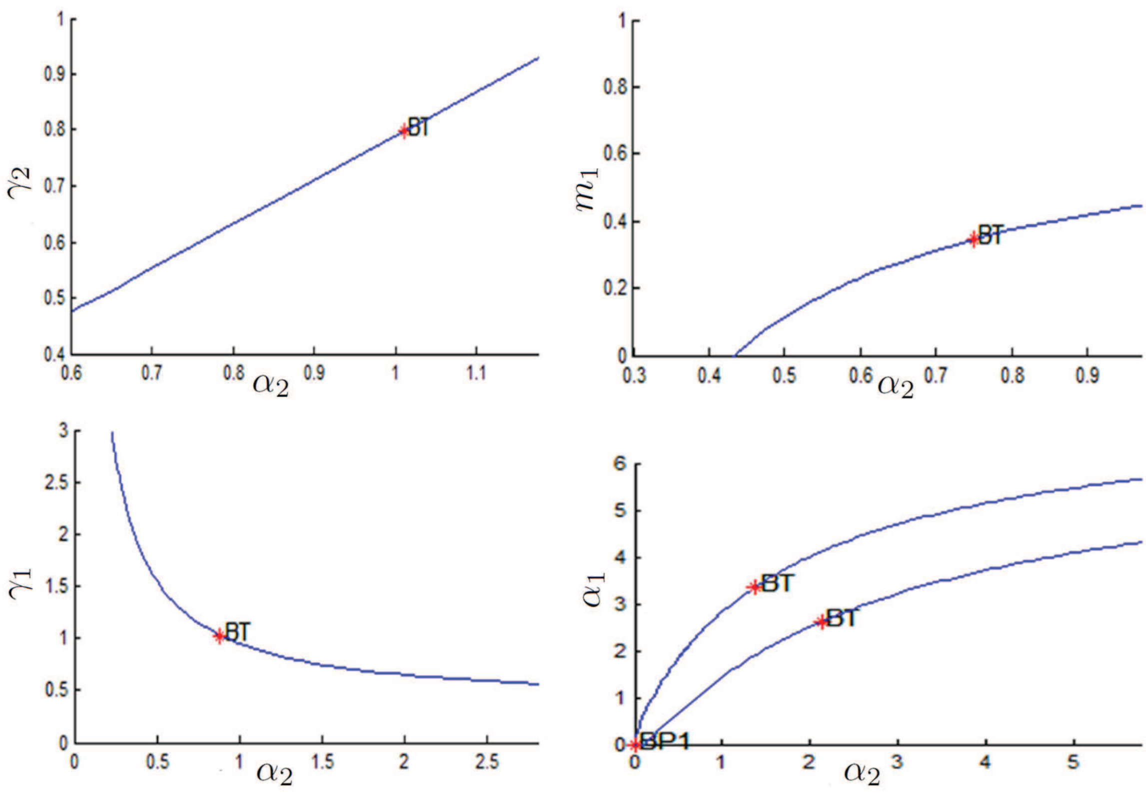

Figure 8: Two-parameter bifurcation diagram with starting from the limit-point the variation for first stage.

Codim 2 bifurication in second stage by start from β_1:
for free parameter α_1:

label = BT , x = (0.049638, 0.189576, 4.179896, 23.137970)
(a,b)=(-5.807615e+000, -1.625516e+001)

for free parameter γ_1:

label = BT , x = (0.049638, 0.189576, 4.179896, 23.137970)
(a,b)=(-5.807615e+000, -1.625516e+001)

for free parameter m_1:

label = BT , x = (0.158450, 0.327429, 2.270760, 0.369635)
(a,b)=(-6.676870e-001, -1.595281e+000)

for free parameter γ_2:

label = BT , x = (0.100265, 0.128865, 5.670936, 0.885979)
(a,b)=(-2.441178e+000, -7.587503e+000)

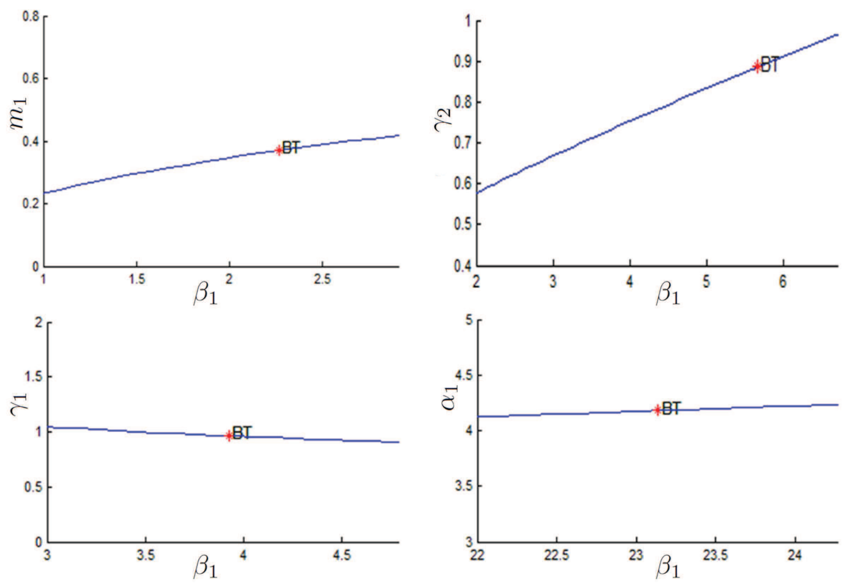

Figure 9: Two-parameter bifurcation diagram with starting from the limit-point the variation for third stage.

Codim 2 bifurication in second stage by start from α_1:
for free parameter γ_1:

label = BT , x = (0.122475, 0.059327, 0.337078, 0.541417)
(a,b)=(-8.907601e-002, -1.409431e+000)
label = BP3, x = (0.000002, 0.026667, 0.004277, 0.000855)

for free parameter m_1:

label = BT , x = (0.122473, 0.059326, 0.824574, 4.817004)
(a,b)=(-6.357250e-002, -4.637807e-001)
label = BP4, x = (0.000000, 0.026667, 0.326600, 15.309265)

for free parameter α_2:

label = BT , x = (0.761634, 8.885729, 18.109305, 4.640691)
(a,b)=(-3.292091e-001, -4.632751e-002)
label = BT , x = (0.861634, -10.052396, 16.810691, 5.939305)
(a,b)=(-3.919633e-001, -3.626396e-002)

for free parameter γ_2:

label = BT , x = (0.122475, 0.005435, 0.317894, 4.912417)
(a,b)=(-9.559190e-002, -1.585453e+000)

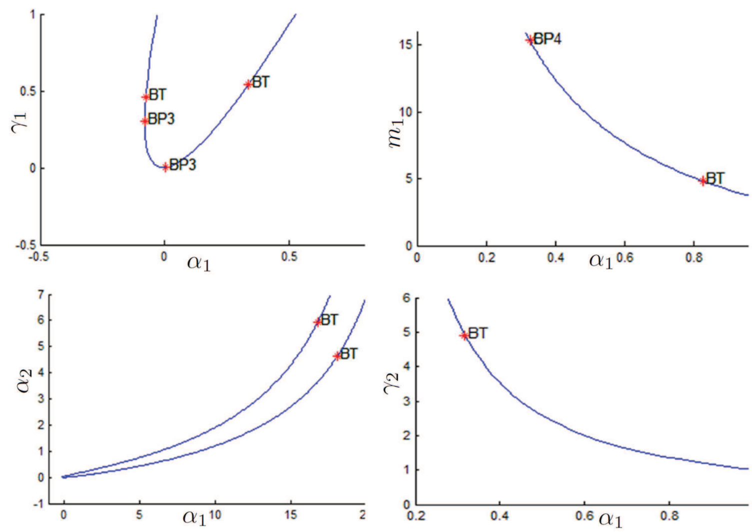

Figure 10: Two-parameter bifurcation diagram with starting from the limit- point the variation for fourth stage.

4.5. Codim 1 and Hopf bifurcation for E_2

By starting fixed parameters $\alpha_1 = 2, \beta_1 = 0.8, \gamma_2 = 0.45, m_1 = 1, m_2 = 0.1, \alpha_2 = 0.5, \gamma_1 = 1.4$, starting from the initial point $(u(0), v(0)) = (2, 3)$, free parameters γ_1 and m_1 and under condition $m_1\alpha_2 > \gamma_2$, we have:

for free parameter γ_1:
label = LP, x = (0.250938, 0.389931, 2.957147)
a=-5.880133e+01
label = H , x = (0.256836, 0.396484, 2.956999)
First Lyapunov coefficient = 1.133940e+01

for free parameter m_1:
label = H , x = (0.756201, 0.951335, 0.208672)
First Lyapunov coefficient = 2.834816e-02
label = LP, x = (0.380629, 0.534032, 0.112971)
a=-3.944763e-01

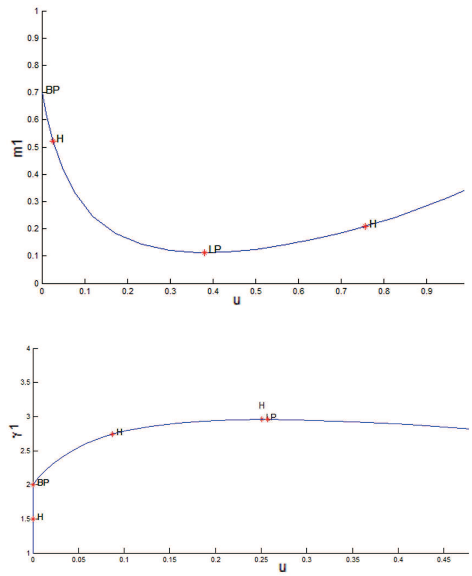

Figure 11: Codim 1 and Hopf bifurication for E_2.

5. Conclusion

In this paper, we investigate the complex dynamics of a ratio-dependent predator-prey model. We investigate different bifurcations of the system. It has been shown that the system under consideration , can have Hopf and Bogdanov-Takens bifurcations. We employed numerical continuation technique to compute several bifurcation curves, Hopf and compute the bifurcation points with their corresponding normal form coefficients. The detected bifurcations have biological implications. At a computed supercritical Hopf bifurcation a stable limit cycle is born that gives rise to periodic behaviour of the populations. In fact, if the predator death rate is smaller than the bifurcation parameter at a supercritical Hopf, both predator and prey coexist in the steady state, but if the predator death rate exceeds this value of bifurcation parameter then both predator and prey still coexist and their densities vary periodically. We have shown that the stable point equilibrium of the system under continuation theorem 5 a subcritical Hopf bifurcation. finally, we plot diagrams simulation.

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P. Azizi,
R. khoshsiar Ghaziani
Department of Applied Mathematics and Computer Sciences,
P. O. Box 115, Shahrekord University,
Shahrecord, Iran.
E-mail address: `Aziziparisa1@gmail.com.`
E-mail address: `Khoshsiar@sci.sku.ac.ir.`