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A Characterization of Ordered Γ-semigroups by Ordered (m, n)-Γ-ideals

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ABSTRACT: In this paper, we study (m, n)-regular ordered Γ-semigroups through ordered (m, n)-Γ-ideals. It is shown that if $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup; m, n are non-negative integers and $\mathcal{A}_{(m,n)}$ is the set of all ordered (m, n)-Γ-ideals of S. Then, S is (m, n)-regular$\Longleftrightarrow \forall A \in \mathcal{A}_{(m,n)}$, $A = (A^m \Gamma S \Gamma A^n]$. It is also proved that if $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup; m, n are nonnegative integers; $\mathcal{R}_{(m,0)}$; $\mathcal{L}_{(0,n)}$ is the set of all $(m, 0)$-Γ-ideals and $(0, n)$-Γ-ideals of S, respectively. Then, S is (m, n)-regular ordered Γ-semigroup $\Longleftrightarrow \forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m \Gamma L \cap R \Gamma L^n]$.

Key Words: Ordered Γ-semigroup, Regular ordered Γ-semigroup, Ordered bi-Γ-ideal, Ordered (m, n)-Γ-ideal.

Contents

1. Introduction

The notion of Γ-semigroups was given by Sen [17]. Let S and Γ be two nonempty sets. Then a system $(S, \Gamma, \cdot)$ is called a Γ-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Let A be a nonempty subset of $(S, \Gamma, \cdot)$. Then A is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is called commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$ for all $a, b \in S$ and $\gamma \in \Gamma$.

The concept of ordered Γ-semigroups was given by Kwon [15]. An ordered Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$ for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$. For other basic definitions and properties, we refer [1], [8], [2], [3], [11], [13], [14], [16].

Notation 1: For subsets A, B of an ordered Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot b = \{a \cdot \gamma \cdot b \mid a \in A, b \in B$ and $\gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as

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$A^2 = A \cdot A = A \cdot \Gamma \cdot A$.

Notation 2: For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m$ for some $m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Notation 3: Let $B \subseteq S$. Then for a non-negative integer m, the power of $B^m = B\Gamma B\Gamma B\Gamma M \cdots$, where B occurs m times. Note that the power is suppressed when $m = 0$. So $B^0\Gamma S = S = S\Gamma B^0$.

In what follows we denote the ordered Γ-semigroup $(S, \Gamma, \cdot, \leq)$ by S unless otherwise specified. Throughout the paper, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$. We now construct the following examples:

Example 1.1. *[2]Let S be the set of all $m \times n$ matrices with entries from a field, where m, n are positive integers. Let $P(S)$ be the power set of S. Then it is easy to see that $P(S)$ is not a semigroup under multiplication of matrices because for$A, B \in P(S)$, the product AB is not defined. Let Γ be the set of $n \times m$ matrices with entries from the same field. Then for$A, B, C \in P(S)$ and $P, Q \in \Gamma$, we have$APB \in P(S)$, $AQB \in P(S)$ and since the matrix multiplication is associative, we get that S is a γ-semigroup. Furthermore, define $A \leq B$ if and only if $A \subseteq B$ for all $A, B \in P(S)$, then $P(S)$ is an ordered Γ-semigroup.*

The above examples show that every ordered Γ-semigroup is not a semigroup and thus ordered Γ-semigroups constitute a more general class of semigroups. Consequently, all the corresponding results of semigroups, Γ-semigroups as well as of ordered semigroups become particular cases of the results obtained in this paper.

Let A and B be two nonempty subsets of S. Then we have the following [13].

(1) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(2)$A \subseteq B \Rightarrow (A] \subseteq (B]$;

The notion of (m, n)-ideals was given by Lajos [16]. Let S be a semigroup and A be a nonempty subset of S, then A is called a generalized (m, n)-ideal of S if $A^m S A^n \subseteq A$, where m, n are arbitrary non-negative integers. Note that if A is a subsemigroup of S, then A is called an (m, n)-ideal of S. Thereafter, many authors studied various aspects of (m, n)-ideals in different algebraic structures [4], [5], [6], [7], [9], [10], [12], [18], [19], [20], [21], [22]. We now need the following definition:

Definition 1.2. *Suppose B is a sub-Γ-semigroup(sub-Γ-semigroup) (resp. nonempty subset) of an ordered Γ-semigroup S. Then B is called an (resp. generalized) (m, n)-Γ-ideal of S if (i) $B^m\Gamma S\Gamma B^n \subseteq B$ and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.*

Example 1.3. *Let $(S, \cdot)$ be an ordered semigroup, where $S = \{1, 2, 3, 4\}$ such that*

$a\gamma b = a \cdot b$ for all $a, b \in S$ and $\gamma \in \Gamma$ with multiplication and order $\leq$ defined below:

$\cdot$	1	2	3	4
1	2	2	4	4
2	2	2	4	4
3	4	4	3	4
4	4	2	4	4

$$\leq = \{(1,1), (2,2), (3,3), (4,4), (1,2), (4,2), (4,3)\}$$

Let $A = \{1, 2\}$. Then A will be an (m, n)-Γ-ideal of S for all $m, n \geq 2$. But it is not a bi-Γ-ideal of S.

Note that in the above Definition 1.3, if we set $m = n = 1$, then B is called a (generalized) bi-Γ-ideal of S. We now introduce the following definition:

Definition 1.4. *Suppose $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup and m, n are non-negative integers. Then S is called (m, n)-regular if for any $s \in S$ there exists $x \in S$ such that $s \leq s^m\gamma_1 x\gamma_2 s^n$ for $\gamma_1, \gamma_2 \in \Gamma$. Equivalently: $(S, \Gamma, \cdot, \leq)$ is (m, n)-regular if $s \in (s^m\Gamma S\Gamma s^n]$ for all $s \in S$.*

2. Main Results

We start by the following:

Lemma 2.1. *Suppose $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup and $s \in S$. Let m, n be non-negative integers. Then the intersection of all ordered (generalized) (m, n)-Γ-ideals of S containing s, denoted by $[s]_{m,n}$, is an ordered (generalized) (m, n)-Γ-ideal of S containing s.*

Proof. *Let $\{A_i : i \in I\}$ be the set of all ordered (generalized) (m, n)-Γ-ideals of S containing s. Clearly, $\bigcap_{i\in I} A_i$ is a sub-Γ-semigroup of S containing s. Let $j \in I$. As $\bigcap_{i\in I} A_i \subseteq A_j$, we obtain*

$$(\bigcap_{i\in I} A_i)^m\Gamma S\Gamma(\bigcap_{i\in I} A_i)^n \subseteq A_j^m\Gamma S\Gamma A_j^n \subseteq A_j.$$

Therefore, $(\bigcap_{i\in I} A_i)^m\Gamma S\Gamma(\bigcap_{i\in I} A_i)^n \subseteq \bigcap_{i\in I} A_i$. Let $a \in \bigcap_{i\in I} A_i$ and $b \in S$ so that $b \leq a$. So $b \in \bigcap_{i\in I} A_i$. Hence $\bigcap_{i\in I} A_i$ is an ordered (generalized) (m, n)-Γ-ideal of S containing s.

Theorem 2.2. *Suppose $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup and $s \in S$. Then:*

(i) *$[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$ for any positive integers m, n.*

(ii) *$[s]_{m,0} = (\bigcup_{i=1}^{m} s^i \cup s^m\Gamma S]$ for any positive integer m.*

(iii) *$[s]_{0,n} = (\bigcup_{i=1}^{n} s^i \cup s^n]$ for any positive integer n.*

Proof. *(i)* $(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n] \neq \emptyset$. *Suppose* $a, b \in (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$ *is such that* $a \leq x$ *and* $b \leq y$ *for some* $x, y \in \bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$. *If* $x, y \in s^m\Gamma S\Gamma s^n$ *or* $x \in \bigcup_{i=1}^{m+n} s^i$, $y \in s^m\Gamma S\Gamma s^n$ *or* $x \in s^m\Gamma S\Gamma s^n$, $y \in \bigcup_{i=1}^{m+n} s^i$, *then* $x\gamma y \in s^m\Gamma S\Gamma s^n$, *and therefore* $x\gamma y \in \bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n$ *for* $\gamma \in \Gamma$. *It follows that* $a\gamma b \in (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$. *Let* $x, y \in \bigcup_{i=1}^{m+n} s^i$. *Then* $x = s^p, y = s^q$ *for some* $1 \leq p, q \leq m+n$. *Now two cases arise: If* $1 \leq p+q \leq m+n$, *then* $x\gamma y \in \bigcup_{i=1}^{m+n} s^i$. *If* $m+n < p+q$, *then* $x\gamma y \in s^m\Gamma S\Gamma s^n$. *So,* $x\gamma y \in (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$. *This implies that* $(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$ *is a sub-*Γ*-semigroup of* S. *Moreover, we have*

$$\begin{aligned}
(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^m\Gamma S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]\Gamma S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i\Gamma S \cup s^m\Gamma S\Gamma S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-1}\Gamma(s\Gamma S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-2}\Gamma(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]\Gamma(s\Gamma S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-2}\Gamma(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma(s\Gamma S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S]^{m-2}\Gamma(s^2\Gamma S] \\
&\;\;\vdots \\
&\subseteq (s^m\Gamma S].
\end{aligned}$$

In a similar fashion, $S\Gamma(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]^n \subseteq (S\Gamma s^n]$. *Therefore,* $(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]^m\Gamma S\Gamma(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]^n \subseteq (s^m\Gamma S\Gamma s^n] \subseteq (\bigcup_{i=1}^{m+n}\{s^i \cup s^m\Gamma S\Gamma s^n]$. *So,* $(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$ *is an* (m, n)*-*Γ*-ideal of* S *containing* s*; hence* $[s]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$. *For the reverse inclusion, suppose* $a \in (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$ *is such that* $a \leq t$ *for some* $t \in (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n$. *If* $t = s^j$ *for some* $1 \leq j \leq m+n$, *then* $t \in [s]_{m,n}$, *therefore,* $a \in [s]_{m,n}$. *If* $t \in s^m\Gamma S\Gamma s^n$, *by*

$$s^m\Gamma S\Gamma s^n \subseteq ([s]_{m,n})^m\Gamma S\Gamma([s]_{m,n})^n \subseteq [s]_{m,n},$$

then $t \in [s]_{m,n}$*; hence* $a \in [s]_{m,n}$. *This implies that* $(\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n] \subseteq [s]_{m,n}$. *Hence,* $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m\Gamma S\Gamma s^n]$. *(ii) and (iii) can be proved in a similar fashion.*

Lemma 2.3. *Suppose $(S,\Gamma,\cdot,\leq)$ is an ordered Γ-semigroup and $s\in S$. Suppose m,n are positive integers. Then:*

(i) $([s]_{m,0})^m\Gamma S\subseteq(s^m\Gamma S]$.

(ii) $S\Gamma([s]_{0,n})^n\subseteq(S\Gamma s^n]$.

(iii) $([s]_{m,n})^m\Gamma S\Gamma([s]_{m,n})^n\subseteq(s^m\Gamma S\Gamma s^n]$.

Proof. *(i)Applying Theorem 2.2, we have*

$$
\begin{aligned}
([s]_{m,0})^m\Gamma S &= (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S]^m\Gamma S\\
&= (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S]\Gamma S\\
&\subseteq (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i\Gamma S\cup s^m\Gamma S\Gamma S]\\
&\subseteq (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S]^{m-1}\Gamma(s\Gamma S]\\
&.\\
&.\\
&.\\
&\subseteq (s^m\Gamma S].
\end{aligned}
$$

Hence, $([s]_{m,0})^m\Gamma S\subseteq(s^m\Gamma S]$. (ii) can be proved similarly as (i).
(iii) Applying Theorem 2.2, we have

$$
\begin{aligned}
([s]_{m,n})^m\Gamma S &= (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S\Gamma s^n]^m\Gamma S\\
&= (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S\Gamma s^n]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S\Gamma s^n]\Gamma S\\
&\subseteq (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S\Gamma s^n]^{m-1}\Gamma(\bigcup_{i=1}^{m+n} s^i\Gamma S\cup s^m\Gamma S\Gamma s^n\Gamma S]\\
&= (\bigcup_{i=1}^{m+n} s^i\cup s^m\Gamma S\Gamma s^n]^{m-1}\Gamma(s\Gamma S]\\
&.\\
&.\\
&.\\
&= (s^m\Gamma S].
\end{aligned}
$$

Therefore, $([s]_{m,n})^m\Gamma S \subseteq (s^m\Gamma S]$. *In a similar fashion,* $S\Gamma([s]_{m,n})^n \subseteq (S\Gamma s^n]$. *Therefore,*

$$\begin{aligned}([s]_{m,n})^m\Gamma S\Gamma([s]_{m,n})^n &\subseteq (s^m\Gamma S]\Gamma([s]_{m,n})^n\\ &\subseteq (s^m\Gamma(S\Gamma([s]_{m,n})^n)]\\ &\subseteq (s^m\Gamma(S\Gamma s^n]]\\ &\subseteq (s^m\Gamma S\Gamma s^n].\end{aligned}$$

Hence, (iii) holds.

Theorem 2.4. *Suppose* $(S, \Gamma, \cdot, \leq)$ *is an ordered* Γ*-semigroup and* m, n *are positive integers. Let* $\mathcal{R}_{(m,0)}$ *and* $\mathcal{L}_{(0,n)}$ *be the set of all ordered* $(m, 0)$-Γ*-ideals and the set of all ordered* $(0, n)$-Γ*-ideals of* S*, respectively. Then:*

(i) S *is* $(m, 0)$*-regular if and only if for all* $R \in \mathcal{R}_{(m,0)}, R = (R^m\Gamma S]$.

(ii) S *is* $(0, n)$*-regular if and only if for all* $L \in \mathcal{R}_{(0,n)}, L = (S\Gamma L^n]$.

Proof. *(i) Suppose* S *is* $(m, 0)$*-regular. Then*

$$\forall s \in S, s \in (s^m\Gamma S]. \tag{2.1}$$

Suppose $R \in \mathcal{R}_{(m,0)}$. *As* $R^m\Gamma S \subseteq R$ *and* $R = (R]$, *we have* $(R^m\Gamma S] \subseteq R$. *If* $s \in R$, *by (2.1), we obtain* $s \in (s^m\Gamma S] \subseteq (R^m\Gamma S]$, *therefore* $R \subseteq (R^m\Gamma S]$. *So* $(R^m\Gamma S] = R$.
Conversely, suppose

$$\forall R \in \mathcal{R}_{(m,0)}, R = (R^m\Gamma S]. \tag{2.2}$$

Suppose $s \in S$. *Therefore,* $[s]_{m,0} \in \mathcal{R}_{(m,0)}$. *By (2), we get*

$$[s]_{m,o} = (([s]_{m,0})^m\Gamma S].$$

Applying Lemma 2.1, we obtain

$$[s]_{m,o} \subseteq (s^m\Gamma S],$$

Therefore, $s \in (s^m\Gamma S]$. *Hence,* S *is* $(m, 0)$*-regular.*
(ii) It can be proved analogously.

Theorem 2.5. *Suppose* $(S, \Gamma, \cdot, \leq)$ *is an ordered* Γ*-semigroup and* m, n *are non-negative integers. Suppose* $\mathcal{A}_{(m,n)}$ *is the set of all ordered* (m, n)-Γ*-ideals of* S*. Then:*

$$S \ is \ (m, n) - regular \Longleftrightarrow \forall A \in \mathcal{A}_{(m,n)}, A = (A^m\Gamma S\Gamma A^n] \tag{2.3}$$

Proof. *Consider the following four cases:*
Case (i): $m = 0$ *and* $n = 0$. *Then (3) implies*
S is (0, 0)-regular $\Longleftrightarrow \forall A \in \mathcal{A}_{(0,0)}, A = S$ *because* $\mathcal{A}_{(0,0)} = \{S\}$ *and S is (0, 0)-regular.*

Case (ii): $m = 0$ and $n \neq 0$. Therefore (3) implies
S is (0, n)-regular$\iff \forall A \in \mathcal{A}_{(0,n)}, A = (S\Gamma A^n]$. This follows by Theorem 2.4(ii).
Case (iii): $m \neq 0$ and $n = 0$. This can be proved applying Theorem 2.4(i).
Case (iv): $m \neq 0$ and $n \neq 0$. Suppose S is (m, n)-regular. Therefore,

$$\forall s \in S, s \in (s^m \Gamma S \Gamma s^n]. \tag{2.4}$$

Let $A \in \mathcal{A}_{(m,n)}$. As $A^m\Gamma S\Gamma A^n \subseteq A$ and $A = (A]$, we obtain $(A^m\Gamma S\Gamma A^n] \subseteq A$. Suppose $s \in A$. Applying (4), $s \in (s^m\Gamma S\Gamma s^n] \subseteq (A^m\Gamma S\Gamma A^n]$. Therefore, $A \subseteq (A^m\Gamma S\Gamma A^n]$. Hence $A = (A^m\Gamma S\Gamma A^n]$.
Conversely, suppose $A = (A^m\Gamma S\Gamma A^n]$ for all $A \in \mathcal{A}_{(m,n)}$. Suppose $s \in S$. As $[s]_{m,n} \in \mathcal{A}_{(m,n)}$, we have

$$[s]_{m,n} = (([s]_{m,n})^m \Gamma S \Gamma ([s]_{m,n})^n].$$

Applying Lemma 2.3(iii), we obtain $[s]_{m,n} \subseteq (s^m\Gamma S\Gamma s^n]$, therefore, $s \in (s^m\Gamma S\Gamma s^n]$. Hence S is (m, n)-regular.

Theorem 2.6. *Suppose $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup and m, n are non-negative integers. Suppose $\mathcal{R}_{(m,0)}$ and $\mathcal{L}_{(0,n)}$ is the set of all $(m, 0)$-Γ-ideals and $(0, n)$-Γ-ideals of S, respectively. Then:*

$$\begin{aligned} S \textit{ is } (m,n)\text{−regular ordered } \Gamma-\text{semigroup} \iff & \forall \mathrm{R} \in \mathcal{R}_{(\mathrm{m},0)} \forall \mathrm{L} \in \mathcal{L}_{(0,\mathrm{n})}, \\ & R \cap L = (R^m\Gamma L \cap R\Gamma L^n]. \end{aligned} \tag{2.5}$$

Proof. *Consider the following four cases:*
Case (i): $m = 0$ and $n = 0$. Therefore, (5) implies
S is (0, 0)-regular $\iff \forall R \in \mathcal{R}_{(0,0)} \forall L \in \mathcal{L}_{(0,0)}, R \cap L = (L \cap R]$ because $\mathcal{R}_{(0,0)} = \mathcal{L}_{(0,0)} = \{S\}$ and S is $(0, 0)$-regular.
Case (ii): $m = 0$ and $n = 0$. Therefore (5) implies
S is $(0, n)$-regular $\iff \forall R \in \mathcal{R}_{(0,n)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (L \cap R\Gamma L^n]$. Suppose S is $(0, n)$-regular. Suppose $R \in \mathcal{R}_{(0,0)}$ and $L \in \mathcal{L}_{(0,n)}$. By Theorem 2.4(ii), $L = (S\Gamma L^n]$. As $R \in \mathcal{R}_{(0,0)}$, we have $R = S$, therefore $R \cap L = L$. Therefore,

$$(L \cap R\Gamma L^n] = (L \cap S\Gamma L^n] = ((S\Gamma L^n] \cap S\Gamma L^n] = (S\Gamma L^n] = L = R \cap L.$$

Conversely, suppose

$$\forall R \in \mathcal{R}_{(0,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (L \cap R\Gamma L^n]. \tag{2.6}$$

If $R \in \mathcal{R}_{(0,0)}$, then $R = S$. If $L \in \mathcal{L}_{(0,n)}$, $S\Gamma L^n \subseteq L$ and $L = (L]$. Therefore, (6) implies

$$\forall L \in \mathcal{L}_{(0,n)}, L = (S\Gamma L^n].$$

Applying Theorem 2.4(ii), S is $(0, n)$-regular.
Case (iii): $m \neq 0$ and $n = 0$. This can be proved as before.
Case (iv): $m \neq 0$ and $n \neq 0$. Suppose that S is (m, n)-regular. Suppose $R \in \mathcal{R}_{(m,0)}$

and $L \in \mathcal{L}_{(0,n)}$. *To prove that* $R \cap L \subseteq (R^m \Gamma L] \cap (R\Gamma L^n]$, *suppose* $s \in R \cap L$. *We have*

$$s \in (s^m \Gamma S \Gamma s^n] \subseteq (s^m \Gamma L] \subseteq (R^m \Gamma L] \text{ and } \mathrm{s} \in (\mathrm{s^m \Gamma S \Gamma s^n}] \subseteq (\mathrm{R\Gamma s^n}] \subseteq (\mathrm{R\Gamma L^n}].$$

Hence, $R \cap L \subseteq (R^m \Gamma L] \cap (R\Gamma L^n]$. *As*

$$(R^m \Gamma L] \subseteq (R^m \Gamma S] \subseteq (R] = R \text{ and } (\mathrm{R\Gamma L^n}] \subseteq (\mathrm{S\Gamma L^n}] \subseteq (\mathrm{L}] = \mathrm{L}.$$

This implies that $(R^m \Gamma L] \cap (R\Gamma L^n] \subseteq R \cap L$, *therefore* $R \cap L = (R^m \Gamma L] \cap (R\Gamma L^n]$. *Conversely, suppose*

$$\forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m \Gamma L \cap R\Gamma L^n]. \tag{2.7}$$

Suppose $R = [s]_{m,0}$ *and* $L = S$. *Applying (2.7), we obtain* $[s]_{m,0} \subseteq (([s]_{m,0})^m \Gamma S]$. *Applying Lemma 2.2, we obtain*

$$[s]_{m,0} \subseteq (s^m \Gamma S]. \tag{2.8}$$

In a similar fashion, we obtain

$$[s]_{0,n} \subseteq (S\Gamma s^n]. \tag{2.9}$$

As $R^m \subseteq R$ *and* $L^n \subseteq L$, *by (7), we have*

$$\forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L \subseteq (R\Gamma L].$$

As $(s^m \Gamma S] \in \mathcal{R}_{(m,0)}$ *and* $(S\Gamma s^n] \in \mathcal{L}_{(0,n)}$, *we obtain*

$$(s^m \Gamma S] \cap (S\Gamma s^n] \subseteq ((s^m \Gamma S]\Gamma(S\Gamma s^n]] \subseteq (s^m \Gamma S \Gamma s^n].$$

Applying (2.8) and (2.9), we obtain

$$[s]_{m,0} \cap [s]_{0,n} \subseteq (s^m \Gamma S \Gamma s^n].$$

Hence, S *is* (m,n)*-regular.*

3. Conclusion

In the current article, we investigated (m,n)-regular ordered Γ-semigroups in terms of ordered (m,n)-Γ-ideals. We proved that if $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup; m,n are non-negative integers and $\mathcal{A}_{(m,n)}$ is the set of all ordered (m,n)-Γ-ideals of S. Then, S is (m,n)-regular$\Longleftrightarrow \forall A \in \mathcal{A}_{(m,n)}$, $A = (A^m \Gamma S \Gamma A^n]$. We also proved that if $(S, \Gamma, \cdot, \leq)$ is an ordered Γ-semigroup; m,n are nonnegative integers; $\mathcal{R}_{(m,0)}$; $\mathcal{L}_{(0,n)}$ is the set of all $(m,0)$-Γ-ideals and $(0,n)$-Γ-ideals of S, respectively. Then, S is (m,n)-regular ordered Γ-semigroup $\Longleftrightarrow \forall R \in \mathcal{R}_{(m,0)} \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m \Gamma L \cap R\Gamma L^n]$.

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