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On the Solution of an f(R) Theory of Gravity Motivated by Gravitational Waves in Spherically Symmetric Spacetime

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ABSTRACT: There are many modification of Einstein theory have been established which explain the behavior of universe realistically or hypo-theoretically. Out of those, an f(R) theory of gravity based on non-conformal invariance of gravitational waves has been developed. We attempted to find the solution of this theory for spherically symmetric spacetime in vacuum and compared its result to Einstein theory. We have concluded that solution is consistent with Newtonian limit at large distance from source. Solution predicts two horizon in the spacetime, none of them coincides with Schwarzschild counterpart. However, as this f(R) theory converges to Einstein theory, these horizons coincide.

Key Words: Gravitational Waves, Singularity, Black-hole horizon, Cosmological horizon.

Contents

1. Introduction

$f(R)$ theories of gravity play crucial role to answer problems in relativity that cannot be answered by Einstein field Equations [1,2]. Today observed acceleration expansion of Hubble flow (dark energy) and missing matter of astrophysical large scale structure are primarily enclosed with $f(R)$ theory [3,4] and provide alternative model of widely accepted λCDM model of the universe [5]. $f(R)$ theory also consistent with classical theory of gravitation as gravitational potential of the theory equals Newtonian potential but corrected by Yukawa like term in weak field limit and $f(R) = R$ provide standard Newtonian potential while many theories of gravity failed to recover it [6]. In order to test correctness of any theory of GR

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(General Relativity), its spherically symmetric solution is inevitable. These solutions have been obtained by using different approaches like Ricci scalar curvature [7] and Noether symmetry approach [8]. Also solutions of $f(R)$ theory have been studied for spherically symmetric spacetime in matric and Palatini [8,9] and ADM formalism [10].

On the other hand, non conformal invariance of gravitational waves which are inevitable consequence of Einstein theory in weak field limit [11,12] motivated to modify the Einstein theory by choosing $f(R)$ as a polynomial of R of a finite number of terms without associating any other field expect gravitation [13,14,15]. Therefore, Lagrangian in the form [16],

$$\mathfrak{L} = R + \sum_{n=2}^{N} c_n \frac{(l^2 R)^n}{6l^2} \tag{1.1}$$

Or, equivalently

$$\mathfrak{L} = R + \sum_{n=2}^{N} a_n R^n \tag{1.2}$$

where l is the characteristic length, c_n are dimensionless coefficients corresponding to n introduced to nullify the manifestation of gravitation [17]. The choice of $f(R)$ should not be disturbing the observe fact that universe is asymptotically flat. Higher order terms in $f(R)$ (like R^2) advocate early time inflation, avoidance of instabilities and linear growth of gravitational forces [18] in small curvature scale. As a result we generalize the action

$$A = \int (\mathfrak{L}/\kappa + \mathfrak{L}_s) d^4x \tag{1.3}$$

with $\mathfrak{L}_s$ stands for source Lagrangian density. The classical field equation have been derived by (1.3) are

$$\begin{aligned} G_{ij} - \sum_{n=2}^{N} \frac{nC_n}{6} (l^2 R)^{n-1} [R_{ij} - \frac{1}{2n} g_{ij} R - \frac{n-1}{R}(R_{;i;j} - g_{ij}\Box R) \\ - \frac{(n-1)(n-2)}{R^2}(R_{;i}R_{;j} - g_{ij}R_{;\alpha}R^{;\alpha})] = \kappa T_{ij} \end{aligned} \tag{1.4}$$

here κ is coupling constant between energy and matter and T_{ij} is energy momentum tensor which provides a further generalization of $f(R)$ theories to $f(R,T)$ theories [19] with T as trance of energy momentum tensor.

This article is organized as follows: In section 2, we discussed solution of an $f(R)$ theory of gravity based on non-conformal invariance of gravitational waves in spherically symmetric spacetime. In section 3, we represented singular behavior of this solution in detail. Trajectories of a particle and a photon and their respective travel time in this spacetime have been discussed in next section. Section 5 devoted in searching a good spacetime representation of the solution which shows only real singularities. We conclude the results in the last section.

2. Solution of Field Equation

The trace of (1.4) relates R with T diffentially not algebraically as in case of Einstein field equation, where $R = \kappa T$. This shows that the field equation (1.4) will admit a large variety of solutions than Einstein's theory [20]. Considering Birkhoff theorem, as an example, it can be said that this f(R) theory $T = 0$ no longer implies $R = 0$, or is even constant. In vacuum, equation (1.4) reduces

$$G_{ij} - \sum_{n=2}^{N} \frac{nC_n}{6}(l^2R)^{n-1}[R_{ij} - \frac{1}{2n}g_{ij}R - \frac{n-1}{R}(R_{;i;j} - g_{ij}\Box R)$$
$$-\frac{(n-1)(n-2)}{R^2}(R_{;i}R_{;j} - g_{ij}R_{;\alpha}R^{;\alpha})] = 0 \tag{2.1}$$

In order to solve equation (2.1), we recall some properties of Ricci tensor. The components of the tensor R_{ik} satisfy a differential identity obtained by contracting the Bianchi identity;

$$R^l_{m;l} = \frac{1}{2}\frac{\partial\ R}{\partial\ x^m} \tag{2.2}$$

It provides;

$$(R^1_i - \frac{1}{2}\delta^1_i R)_{;1} = -(R^a_i - \frac{1}{2}\delta^a_i R)_{;a} \tag{2.3}$$

An application of equation (2.2) on (2.3) yields

$$R^1_1 - \frac{1}{2}R = R^2_2 - \frac{1}{2}R = \psi(r) \tag{2.4}$$

here ψ is an arbitrary function of r. Consider a metric with spherically symmetry

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.5}$$

where $A(r)$ and $B(r)$ are arbitrary functions of r. Using Ricci components we get $B(r) = [A(r)]^{-1}$ and hence,

$$A(r) = 1 - \frac{2GM}{r} + \frac{1}{r}\int r^2\psi(r)dr \tag{2.6}$$

where $-2GM$ is a constant which is determined from the fact that at when $r \to \infty$, multiple coefficient of dt^2 must be consistent with Newtonian theory. M is the central mass of the body and G is universal constant of gravitation. Thus an $f(R)$ theory of gravity motivated by Gravitational waves provides space-time

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{1}{r}\int r^2\psi(r)dr\right)dt^2 - \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{1}{r}\int r^2\psi(r)dr)}$$
$$-r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.7}$$

for spherically symmetric bodies in vacuum. Since $\psi(r)$ is a regular function of r and hence can be expanded in a power series of r within radius of convergence;

$$\psi(r) = b_0 + b_1 r + b_2 r^2 + ... \tag{2.8}$$

leads to

$$A(r) = [B(r)]^{-1} = 1 - \frac{2MG}{r} + \frac{b_0}{3}r^2 + \frac{b_1}{4}r^3 + \frac{b_2}{5}r^4 + ... \tag{2.9}$$

If all of b's are zero in equation (2.9), we get spherically symmetric solution of Einstein field equation i.e. Schwarzschild solution. So, these terms appear due to the correction in the Hilbert Lagrangian, R and hence the contribution of each successive term becomes smaller and smaller as we go to higher powers of r in equation (2.8). We restrict ourselves only first term of equation (2.8) for the discussion in next section. Thus space-time becomes

$$ds^2 = (1 - \frac{2GM}{r} + \frac{b_0 r^2}{3})dt^2 - \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3})} - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.10}$$

It is important to note that the appearance of term $b_0 r^2/3$ in the solution of an $f(R)$ theory of gravity does not allow it to be asymptotically flat when r tends to infinity. Therefore, the term behaves like a repulsive cosmological constant, i.e.,

$$R_{ij} - \frac{1}{2}g_{ij}R = \lambda g_{ij} \tag{2.11}$$

where λ is a cosmological constant.

3. Singularity

Rewriting the metric (2.10) in the form,

$$ds^2 = -\Phi(r)dt^2 + \Phi^{-1}(r)dr^2 + r^2(d\theta^2 + Sin^2\theta d\phi^2) \tag{3.1}$$

where

$$\Phi(r) = \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right) = \frac{(r - r_+)(r_{++} - r)(r + r_+ + r_{++})}{r(r_+^2 + r_+ r_{++} + r_{++}^2)} \tag{3.2}$$

The parameters, r_+, r_{++}, with $0 < r_+ < r_{++}$, are two real positive roots of the equation $\Phi(r) = 0$ with $G = 1$ and the third root r_- is negative. Roots can be parameterized as,

$$r_+ = \frac{2}{\sqrt{\lambda}}Cos(\frac{\alpha}{3} + \frac{4\pi}{3}) \tag{3.3}$$

$$r_{++} = \frac{2}{\sqrt{\lambda}}Cos(\frac{\alpha}{3}) \tag{3.4}$$

$$r_- = \frac{2}{\sqrt{\lambda}}Cos(\frac{\alpha}{3} + \frac{2\pi}{3}) \tag{3.5}$$

Where $\lambda = -b_0$ and $cos(\alpha) = -3M/\sqrt{\lambda}$.The physical aspect of these parameters can be seen from equation (3.2). r_+ describes the black-hole horizon and r_{++} is the cosmological horizon whereas $r = 0$ is essential or real singularity of the space-time.

4. Some Aspects of the Solution

We now turn to explore the behavior of particles between distances $r = r_0$ to $2GM$ of the space-time. The geometry of a space-time can be understood by understanding its casual structure, that is, how various regions of the space-time are casually connected or disconnected with each other. Consider the radial null curves, which are basically the trajectories corresponding to photons, provided by line element;

$$ds^2 = -(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3})dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3})} \tag{4.1}$$

The Trajectories of test particles are given by

$$\frac{dr}{dt} = \pm\sqrt{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)\left(\frac{ds}{dt}\right)^2 + \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)^2} \tag{4.2}$$

For photons, $ds^2 = 0$, that is, we have

$$dr = \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right) dt \tag{4.3}$$

Thus a photon will require a time T_0 given by

$$T_0 = \int_{r_0}^{2GM} \frac{dr}{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)} \tag{4.4}$$

i.e.,

$$T_0 = \frac{r_+ ln\left(\frac{2GM - r_+}{r_0 - r_+}\right)}{(1 + b_0 r_+^2)} + \frac{r_{++} ln\left(\frac{2GM - r_{++}}{r_0 - r_{++}}\right)}{(1 + b_0 r_{++}^2)} + \frac{r_- ln\left(\frac{2GM - r_-}{r_0 - r_-}\right)}{(1 + b_0 r_-^2)} \tag{4.5}$$

for spatial displacement between $r = r_0$ to $2GM$ to cover the finite stretch L_0 given by,

$$L_0 = \int_{r_0}^{2GM} \frac{dr}{\sqrt{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)}} \tag{4.6}$$

Here it should be noted that $r = 2GM$ is a singular surface in Schwarzschild metric. Thus, a light ray that approaches $r = 2GM$ would reach there unlike the case with Schwarzschild solution. So, $r = 2GM$ will not be a horizon in this case.

Consider a particle travelling from distance between $r = r_0$ and r_{++} i.e., the length L_{++}

$$L_{++} = \int_{r_0}^{r_{++}} \frac{dr}{\sqrt{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)}} \tag{4.7}$$

Time taken by a photon is

$$T_{++} = \int_{r_0}^{r_{++}} \frac{dr}{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)} \tag{4.8}$$

Here, integral becomes improper. So, a photon will take infinite time to travel a finite distance from $r = r_0$ to r_{++}. The similar will be the case when we consider the distance from $r = r_0$ to r_+. Again, for test particle, the time taken to cover distance between $r = r_0$ to $2GM$ will be

$$T_t = \int dt = \int_{r_0}^{2GM} \frac{dr}{\sqrt{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)\left(\frac{ds}{dt}\right)^2 + \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)^2}} \tag{4.9}$$

or,

$$T_t = \int_{r_0}^{2GM} \frac{dr}{\left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right)\sqrt{1 + \frac{(ds/dt)^2}{(1 - \frac{2MG}{r} + \frac{b_0 r^2}{3})}}} \tag{4.10}$$

5. Advanced Time and Retarded Time

In this section, our aim is to find suitable coordinate frame where space-time (2.10) shows only essential singularity. The space-time is four dimensional but because of spherical symmetry, only two parameters i.e., r and t coordinates of the metric contribute for analyzing the nature of the singularity. So, transformation of coordinates must be applied in these coordinates.

We defines the transformation in retarded time coordinate,

$$u = t - \frac{r_+ ln(r - r_+)}{1 + b_0 r_+^2} - \frac{r_{++} ln(r - r_{++})}{1 + b_0 r_{++}^2} - \frac{r_- ln(r - r_-)}{1 + b_0 r_-^2} \tag{5.1}$$

and advanced time coordinate

$$v = t + \frac{r_+ ln(r - r_+)}{1 + b_0 r_+^2} + \frac{r_{++} ln(r - r_{++})}{1 + b_0 r_{++}^2} + \frac{r_- ln(r - r_-)}{1 + b_0 r_-^2} \tag{5.2}$$

where r_+,r_{++} and r_- defined by equation (3.3, 3.4 and 3.5). Under these transformation coordinates, the metric (2.10) can be express as

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right) du^2 + 2dudr - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.3}$$

and

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{b_0 r^2}{3}\right) dv^2 + 2dvdr - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.4}$$

We find that there is no singularity in equation (5.3) and (5.4) at $r = r_+$,r_{++} (or r_-) showing that these are only coordinate singularity of the original expression (2.10) for the metric. However, it is not possible to find a coordinate system in which the singularity at $r = 0$ is removed. This is because the space-time curvature, which is measured by invariants, is unbounded at r tends to infinity.

6. Conclusion

We studied some aspects of $f(R)$ theory of gravity in spherically symmetric space-time. The solution of field equation obtained by this theory for spherically symmetric case is as similar as the solution comes from solving Einstein's field equation with cosmological constant. The spherically symmetric solution of an $f(R)$ theory of gravitation provided us a great deal more than the ability to predict tests of general relativity. Sufficiently massive bodies are unable to support themselves against complete gravitational collapse. So, the spherically symmetric solution of an $f(R)$ theory of gravitation also describes the space-time geometry and singularity. The study tells the essential singularity of space-time is at $r = 0$ but it contains two horizon, namely, black-hole horizon and cosmological horizon. We have also calculated the time taken by a particle traveling a finite distance within the black-hole and we found that particle will take large enough time to travel a finite distance.

It is worth mentioning that in the slow motion, weak field limit, the predictions of general relativity reduce to those of Newtonian theory. However, the spherically symmetric solution of an $f(R)$ theory of gravitation predicts tinny departures from Newtonian theory.

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