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On The New Type Of Solutions To Benney-Luke Equation

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abstract: In this study, the Benney-Luke equation is considered. In order to derive new type of solutions, the sn-ns method is applied to this equation. Then, we introduce trigonometric and elliptic functions solutions in addition to the hyperbolic ones which are gained by tanh-coth. Three types of solutions are derived at the same time with the help of this method. Therefore, it can be said that this method is convenient to obtain more solutions for many kinds of nonlinear partial differential equations.

Key Words: Benney-Luke equation (BL), The sn-ns method, tanh-coth method, Elliptic function solution, Trigonometric solution.

Contents

1. Introduction

Obtaining solutions of nonlinear partial differential equations (NLPDEs) has a significant role in applied sciences. It is due to the fact that scientists come across the nonlinear phenomena in most disciplines of applied sciences. Therefore, gaining the exact solutions to NLPDEs takes large place during the past decades. In this regard, plenty of theoretical and computational methods have been established and used for this purpose. Some of the powerful methods related to this study are; the generalized tanh method [1], the tanh-coth method (extended tanh method) [2], sine-cosine method [3], the tanh-sech method [4,5], the modified simple equation method [6], the exp-function method [7], the projective Ricatti equations method [8], the generalized projective Ricatti equations method [9], the (G'/G)-expansion method [10], the Jacobi elliptic-function method [11].
This work is about Benney-Luke equation given by

$$u_{tt} - u_{xx} + \mathrm{au}_{\mathrm{xxxx}} - \mathrm{bu}_{\mathrm{xxtt}} + nu_t(u_x)^{n-1}u_{xx} + 2\,(u_x)^n u_{tx} = 0, n \in N) \tag{1.1}$$

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where $u(x,t)$ is the solution of equation (1.1) and considered as a class of non-periodic functions defined on $-\infty \leq x \leq \infty$ and $t \geq 0$. And also, equation (1.1) is called the general form of Benney-Luke equation. This eqaution is derived by Benney and Luke [12].
In our work, the case of $n = 1$ has been studied and the equation (1.1) takes the following form

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{1.2}$$

The equation (1.2) is called Benney-Luke equation. Here, a and b are positive numbers, such that $a - b = \sigma - \frac{1}{3}$ and σ is named the Bond number, which captures the effects of surface tension and gravity force. It is formally valid approximation for describing two-way water wave propagation in the presence of surface tension[13]. Pego and Quintero studied the propagation of long water waves with small amplitude in [14]. They showed that in the presence of a surface tension, the propagation of such waves governed by equation (2), originally derived by Benney and Luke [12]. There are many studies concerning with this equation. Among these studies, the stability analysis [15], Cauchy problem [16,18], existence and analytics of solutions [19], and traveling wave solutions [20] can be mentioned here. The aim of this work is to get exact solutions of Benney-Luke equation given in (2) by using the sn-ns method. Then, we have compared the solutions of the same equation found by tanh-coth method in [21].
This study is organized as follow. Firstly, we have mentioned about the method in Section 2. After that, we have applied the method to equation (1.2) and introduced new traveling wave solutions in Section 3. In Section 4, the solutions are compared to ones gained in [21]. Finally, some conclusions have been given.

2. Outline of the Method

This method has been introduced by H. S. Alvaro [11] in detail. In this method, we have searched for the traveling wave solutions to nonlinear partial differential equation of the form

$$P\left(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, u_{xxt}, \ldots\right) = 0. \tag{2.1}$$

Let us consider the wave transformation

$$u(x,t) = v(\theta(\xi)), \xi = x + \lambda \mathrm{t} + \xi_0 \tag{2.2}$$

where λ is a constant and ξ_0 is an arbitrary constant. And, $\theta = \theta(\xi)$ is a suitable function that makes the transformation simple. For this purpose, we take $\theta(\xi)$ as the identity function, i.e. $\theta(\xi) = \xi$.
Using the transformation (2.2) and putting ordinary derivative of $v(\xi)$ instead of the partial derivatives of $u(x,t)$, (3.1) converts to an ordinary differential equation (ODE) with respect to the function $v(\xi)$

$$Q\left(v, v', v'', \ldots\right) = 0 \tag{2.3}$$

with Q being a polynomial with respect to variables $v, v', v'', ...$
In this method, we seek the traveling wave solutions to (2.3) in the form
$v(\xi) = a_0 + \sum_{j=1}^{n} \left[a_j.\left(\mathrm{sn}^j(k\xi)\right) + b_j.\left(\mathrm{ns}^j(k\xi)\right)\right].$
To determine the coefficients (a_0, a_j, b_j), the balancing constant (n), λ and k, the same process above can be followed.
Generally, we find $n = 1, 2$ and then $v(\xi)$ takes the forms

$$v(\xi) = a_0 + a_1.\mathrm{sn}(k\xi) + b_1.\mathrm{ns}(k\xi) \tag{2.4}$$

and

$$v(\xi) = a_0 + a_1.\mathrm{sn}(k\xi)) + b_1.\mathrm{ns}(k\xi) + a_2.\mathrm{sn}^2(k\xi) + b_2.\mathrm{ns}^2(k\xi) \tag{2.5}$$

respectively.
For a special values of $m = 1$, we have $sn(k\xi, 1) = tanh(k\xi)$ and $ns(k\xi, 1) = coth(k\xi)$.

3. Application of the Method

Let us consider the B-L equation given in (1.2). After using the wave transformation, equation (1.2) converts to the following ordinary differential equation with respect to ξ:

$$(V^2 - 1)v'' + (a - bV^2)v^{\mathrm{IV}} - 3Vv'v'' = 0. \tag{3.1}$$

We can integrate both sides of (8) since each term contains derivative with respect to ξ. Then, we get

$$(V^2 - 1)v' + (a - bV^2)v''' - \frac{3V}{2}(v')^2 = 0. \tag{3.2}$$

Now, let us consider the transformation, given by $v' = w$. Then, the equation (3.2) converts into

$$(V^2 - 1)w + (a - bV^2)w'' - \frac{3V}{2}w^2 = 0. \tag{3.3}$$

This equation can be written as

$$w'' = \frac{3V}{2(a - bV^2)}w^2 - \frac{V^2 - 1}{a - bV^2}w. \tag{3.4}$$

Equation (13) has form of Quadratic Duffing Equation, described by

$$v'' = pv^2 + qv + r \tag{3.5}$$

where p, q and r are constants and $p \neq 0$. In (3.5),

$$p = \frac{3V}{2(a - bV^2)}, q = -\frac{V^2 - 1}{a - bV^2}, r = 0. \tag{3.6}$$

To determine balancing constant M, we use the transformation $w = \phi^M$, $w' = M\phi^{M+1}$, $w'' = M(M+1)\phi^{M+2}, \ldots$ and $w^2 = \phi^{2M}, \ldots, w^k = \phi^{kM}$. Putting these into the equation (3.4) and balancing the power of w^2 with w'' gives $M = 2$. So, the sn-ns method admits the use of the finite expansion

$$w(\xi) = a_0 + a_1.\mathrm{sn}(k\xi)) + b_1.\mathrm{ns}(k\xi) + a_2.\mathrm{sn}^2(k\xi) + b_2.\mathrm{ns}^2(k\xi). \tag{3.7}$$

Inserting this into (3.4) gives the following algebraic system:

$a_1\mu^2m^2 - a_1a_2p = 0,$

$6a_2\mu^2m^2 - a_2^2p = 0,$

$4a_2\mu^2 + 4a_2\mu^2m^2 + a_1^2p + 2a_0a_2p + a_2q = 0,$

$2a_2b_1p + 2a_0a_1p + a_1\mu^2 + a_1\mu^2m^2 + a_1q = 0,$

$b_1\mu^2 - b_1b_2p = 0,$

$6b_2\mu^2 - b_2^2p = 0,$

$2a_0b_2p + b_1^2p + 4b_2\mu^2 + 4b_2\mu^2m^2 + b_2q = 0,$

$2a_0b_1p + 2a_1b_2p + b_1\mu^2 + b_1\mu^2m^2 + b_1q = 0,$

$2a_1b_1p + 2a_2b_2p + a_0^2p - 2a_2\mu^2 + a_0q - 2b_2\mu^2m^2 = 0.$

Solving this system, we obtain the following six sets of solutions of the parameters, a_0, a_1, b_1, a_2, b_2 and μ :

$$\begin{aligned}
a_0 &= -\frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4-m^2+1}},\\
a_1 &= 0,\\
a_2 &= \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}},\\
b_1 &= 0,\\
b_2 &= 0,\\
\mu &= \frac{1}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}},
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4-m^2+1}},\\
a_1 &= 0,\\
a_2 &= 0,\\
b_1 &= 0,\\
b_2 &= \frac{3}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}},\\
\mu &= \frac{1}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}},
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
a_1 &= 0,\\
a_2 &= \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
b_1 &= 0,\\
b_2 &= \frac{3}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
\mu &= \frac{1}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}},
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4-m^2+1}},\\
a_1 &= 0,\\
a_2 &= 0,\\
b_1 &= 0,\\
b_2 &= -\frac{3}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}},\\
\mu &= \frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}},
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4-m^2+1}},\\
a_1 &= 0,\\
a_2 &= -\frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}},\\
b_1 &= 0,\\
b_2 &= 0,\\
\mu &= \frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}},
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{q}{2p} - \frac{1}{2p}\left(m^2+1\right)\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
a_1 &= 0,\\
a_2 &= -\frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
b_1 &= 0,\\
b_2 &= -\frac{3}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}},\\
\mu &= \frac{1}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}},
\end{aligned}$$

where V is left as a free parameter.
Putting these coefficients into the expansion (3.7), we get the following solutions in terms of jacobi elliptic functions;

$$
\begin{aligned}
w_1(\xi) &= a_0 + \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}}\mathrm{sn}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}}\xi\,\middle|\, m\right),\\
w_2(\xi) &= a_0 + \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}}\mathrm{ns}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}}\xi\,\middle|\, m\right),\\
w_3(\xi) &= a_0 + \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}}\mathrm{sn}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}}\xi\,\middle|\, m\right)\\
&\quad + \frac{3}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}}\mathrm{ns}^2\left(\frac{1}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}}\xi\,\middle|\, m\right),\\
w_4(\xi) &= a_0 - \frac{3}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}}\mathrm{ns}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}}\xi\,\middle|\, m\right),\\
w_5(\xi) &= a_0 - \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4-m^2+1}}\mathrm{sn}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4-m^2+1}}\xi\,\middle|\, m\right),\\
w_6(\xi) &= a_0 - \frac{3m^2}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}}\mathrm{sn}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}}\xi\,\middle|\, m\right)\\
&\quad - \frac{3}{2p}\sqrt{\frac{q^2}{m^4+14m^2+1}}\mathrm{ns}^2\left(\frac{\sqrt{-1}}{2}\sqrt[4]{\frac{q^2}{m^4+14m^2+1}}\xi\,\middle|\, m\right).
\end{aligned}
$$

Letting $m \longrightarrow 1$, we obtain hyperbolic solutions:

$$
\begin{aligned}
w_1(\xi) &= -\frac{3q}{2p} + \frac{3q}{2p}\mathrm{Tanh}^2(\frac{1}{2}\sqrt{q}\xi),\\
w_2(\xi) &= -\frac{3q}{2p} + \frac{3q}{2p}\mathrm{Coth}^2(\frac{1}{2}\sqrt{q}\xi),\\
w_3(\xi) &= -\frac{3q}{4p} + \frac{3q}{8p}(\mathrm{Tanh}^2(\frac{1}{4}\sqrt{q}\xi) + \mathrm{Coth}^2(\frac{1}{4}\sqrt{q}\xi)).
\end{aligned}
$$

And also for $m \longrightarrow 0$, we have derived trigonometric solutions:

$$
\begin{aligned}
w_4(\xi) &= -\frac{3q}{2p} - \frac{3q}{2p}\mathrm{Cot}^2(\frac{1}{2}\sqrt{q}\xi),\\
w_5(\xi) &= -\frac{3q}{2p} - \frac{3q}{2p}\mathrm{Tan}^2(\frac{1}{2}\sqrt{q}\xi),\\
w_6(\xi) &= -\frac{3q}{4p} - \frac{3q}{8p}(\mathrm{Tan}^2(\frac{1}{4}\sqrt{q}\xi) + \mathrm{Cot}^2(\frac{1}{4}\sqrt{q}\xi)).
\end{aligned}
$$

Since $v' = w$, it is needed to integrate the above solutions with respect to ξ. Then, we have the hyperbolic and trigonometric solutions:

$$
\begin{aligned}
v_1(\xi) &= -\frac{3\sqrt{q}}{p}\mathrm{Tanh}(\pm\frac{1}{2}\sqrt{q}\xi),\\
v_2(\xi) &= -\frac{3\sqrt{q}}{p}\mathrm{Coth}(\pm\frac{1}{2}\sqrt{q}\xi),\\
v_3(\xi) &= -\frac{3\sqrt{q}}{2p}(\mathrm{Tanh}(\pm\frac{1}{4}\sqrt{q}\xi)+\mathrm{Coth}(\pm\frac{1}{4}\sqrt{q}\xi)),\\
v_4(\xi) &= \frac{3\sqrt{q}}{p}\mathrm{Cot}(\pm\frac{1}{2}\sqrt{q}\xi),\\
v_5(\xi) &= -\frac{3\sqrt{q}}{p}\mathrm{Tan}(\pm\frac{1}{2}\sqrt{q}\xi),\\
v_6(\xi) &= -\frac{3\sqrt{q}}{2p}(\mathrm{Tan}(\pm\frac{1}{4}\sqrt{q}\xi)+\mathrm{Cot}(\pm\frac{1}{4}\sqrt{q}\xi)),
\end{aligned}
$$

where $p = \frac{3V}{2(a-bV^2)}$, $q = -\frac{V^2-1}{a-bV^2}$ and $u(x,t) = v(\xi)$, $\xi = x - Vt$.

Putting these in the above solutions gives the solutions depended on V as follows:

$$
\begin{aligned}
u_1(\xi) &= -\left(\frac{2\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)\mathrm{Tanh}(\pm\frac{1}{2}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt)),\\
u_2(\xi) &= -\left(\frac{2\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)\mathrm{Coth}(\pm\frac{1}{2}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt)),\\
u_3(\xi) &= -\left(\frac{\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)(\mathrm{Tanh}(\pm\frac{1}{4}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt))\\
&\quad +\mathrm{Coth}(\pm\frac{1}{4}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt))),\\
u_4(\xi) &= \left(\frac{2\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)\mathrm{Cot}(\pm\frac{1}{2}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt)),\\
u_5(\xi) &= -\left(\frac{2\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)\mathrm{Tan}(\pm\frac{1}{2}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt)),\\
u_6(\xi) &= -\left(\frac{\sqrt{1-v^2}\sqrt{a-bv^2}}{v}\right)(\mathrm{Tan}(\pm\frac{1}{4}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt))\\
&\quad +\mathrm{Cot}(\pm\frac{1}{4}\sqrt{\frac{1-V^2}{a-bV^2}}(x-Vt))).
\end{aligned}
$$

4. Comparison

In this section, we have compared the solutions gained by the sn-ns method in this work with those obtained by the tanh-coth method in [21]. In that work, three hyperbolic solutions are found by tanh-coth method. The same solutions can be attained easily by sn-ns method just letting $m \longrightarrow 1$. On the other hand, we acquired trigonometric and elliptic function solutions by the sn-ns method.

5. Conclusion

In this paper, we have found exact solutions of Benney-Luke equation given in (1.2) by the sn-ns method. We obtained the same hyperbolic solutions found by tanh-coth method in [21]. Even if we have the same solutions in terms of hyperbolic functions, the sn-ns method usually gives more solutions than the tanh-coth method. Clearly, it is seen that the sn-ns method gives trigonometric and elliptic function solutions in addition to hyperbolic ones. It can be said that the sn-ns mehtod is more comprehensive than tanh-coth method. Therefore, the sn-ns method is useful and trustworthy method to get more solutions to NLPDEs.

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