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Existence Results For Some Nonlinear Degenerate Problems In The Anisotropic Spaces

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ABSTRACT: Our goal in this study is to prove the existence of solutions for the following nonlinear anisotropic degenerate elliptic problem:

$$-\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,u,\nabla u) = f - \partial_{x_i} g_i \quad \text{in } \Omega,$$

where for $i = 1,...,N$, $a_i(x,u,\nabla u)$ is allowed to degenerate with respect to the unknown u, and $H_i(x,u,\nabla u)$ is a nonlinear term without a sign condition. Under suitable conditions on a_i and H_i, we prove the existence of weak solutions.

Key Words: Degenerate Elliptic problems, Anisotropic Sobolev spaces, Weak Solutions.

Contents

1. Introduction

For the vectorial exponent $\vec{p} = (p_1,...,p_N)$ we assume that for $i = 1,...,N, 1 < p_i < \infty$.
Our aim is to prove the existence of weak solutions to the anisotropic degenerate elliptic equations

$$\begin{cases} -\partial_{x_i} a_i(x,u,\nabla u) + \displaystyle\sum_{i=1}^{N} H^i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, for $i = 1, ..., N$, $a_i(x, u, \nabla u)$ is a Carathéodory function and there exists a continuous and bounded function ν: $[0, +\infty) \to [0, +\infty)$ such that $\nu(0) = 0$ and $\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \sum_{i=1}^{N} \nu(|s|)|\xi|^{p_i}$ for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and a.e x in Ω, and $H(x, u, \nabla u)$ is a nonlinear term has a growth condition, and without a sign condition, the source data f and $g = (g_1, ..., g_N)$ belonging a suitable Lebesgue spaces (see assumptions $\mathbf{A_6}$)).
In problem (1.1), when the norm $\|b\|_{L^r(\Omega)}$ in the growth of $H^i(x, u, \nabla u)$, is not small enough, the operator becomes non-coercive, moreover, the problem (1.1) is degenerate since its modulus of ellipticity vanishes when either the solution u or its gradient ∇u vanishes [22,25].
Anisotropic operators involve today in various domains of applied Sciences, they provide models for the study of physical and mechanical processus in anisotropic continuous medium ([11,24]).
Existence to problems like (1.1) is very well understood, in the isotropic case, $\nu(s) = const > 0$ in particular, there is vast literature for analysis of the case involving the p-Laplacian operator and problems stated in the Lebesgue space setting, in the elliptic setting the foundation of the branch where laid by Boccardo et al. [10], Dall'Aglio [14] and Murat [20], we motions the work of Porretta [21] for the lower term H without sign condition.
For the anisotropic elliptic equations with $\nu(s) = const > 0$, we started by the work of Bendahmane generalizing the work H. Brésis and F. Browder [13], to the anisotropic space $W^{1,\overrightarrow{p}}(\Omega)$ using the Hedberg-type approximations. For more works in the anisotropic spaces we refer the reader to ([4,7,11,16,17,18] and [24]).
Our main contribution is to prove the existence of weak solutions of the nonlinear anisotropic equation with degenerate ellipticity $\mathbf{A_1}$), here $\nu(.)$ is non negative function on s with $\nu(0) = 0$.
There exist two main difficulties in dealing with this problem, which are related to the fact the equation is degenerate in the anisotropic case, namely in the set $B = \{x \in \Omega : u(x) = 0\}$ the degenerate function $\nu(|u|) = 0$, to overcame this obstacle we use instead a_i and H_i the positively homogenous function of degree $(p_i - 1)$ with respect to the gradient (see $\mathbf{A_1}$) and $\mathbf{A_5}$)).
The second main difficulty in lack of coerciveness for the lower order which does not allow to use the classical methods to prove the existence of a weak solution to Problem (1.1), to get the a priori estimate we need the smallness of the norm $\|b\|_{L^r(\Omega)}$, to avoid this assumption we adapt the method introduced in [12], which consists in splitting the domain Ω in q finite number of small domain Ω_i (see proposition 4.2).
This article is organized as follows: In section 2, we give some preliminaries and useful lemmas. In section 3, we give the basic assumptions. In section 4, we establish the existence result of the weak solution (see Theorem (4.4)).

2. Preliminieries and Useful Lemmas

2.1. Anisotropic Sobolev spaces

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$) with boundary $\partial\Omega$.
Let $p_1, p_2 \dots, p_N$ be N exponents, with $1 < p_i < \infty$ for $i = 1, 2, \dots, N$. We denote $\vec{p} = (p_1, \dots, p_N)$.
We set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{2.1}$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows

$$W^{1,\vec{p}}(\Omega) = \{u \in W^{1,1}(\Omega) : \quad \partial_{x_i} u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\}$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}. \tag{2.2}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [23], [18]).

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty\ (\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2.2).

We denote by $\bar{p}$ the harmonic mean, i.e. $\dfrac{1}{\bar{p}} = \displaystyle\sum_{i=1}^{N} \dfrac{1}{p_i}$.

Proposition 2.1. *We denote the dual of the anisotropic Sobolev space* $W_0^{1,\vec{p}}(\Omega)$ *by* $W^{-1,\vec{p}'}(\Omega)$, *where* $\vec{p}' = (p'_1, \dots, p'_N)$ *and* $\frac{1}{p'_i} + \frac{1}{p_i} = 1$.
For each $F \in W^{-1,\vec{p}'}(\Omega)$ *there exists* $F_i \in L^{p'_i}(\Omega)$ *for* $i = 0, 1, \dots, N$, *such that* $F = \displaystyle\sum_{i=1}^{N} \partial_{x_i} F_i$. *Moreover, for all* $u \in W_0^{1,\vec{p}}(\Omega)$ *we have*

$$\langle F, u\rangle = \sum_{i=1}^{N} \int_\Omega F_i\, \partial_{x_i} u\, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'} = \inf \sum_{i=1}^{N} \|F_i\|_{p'_i}.$$

2.2. Useful lemmas

Lemma 2.2. *(see [23], [18]) Suppose that* $u \in W_0^{1,\vec{p}}(\Omega)$, *then we have the following inequalities*

1. $\|u\|_{L^{p_i}(\Omega)} \leq c_p \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}$ *for* $i = 1, \dots, N$.

2. $\|u\|_{L^q(\Omega)} \le c_s \prod_{i=1}^{N} (\|\partial_{x_i} u\|_{L^{p_i}(\Omega)})^{\frac{1}{N}}$

where

$$q = \begin{cases} \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}} & \textit{if} \quad \bar{p} < N \\ q \in [1, \infty[& \textit{if} \quad \bar{p} \ge N. \end{cases}$$

Lemma 2.3. *Let Ω be a bounded open set in $\mathbb{R}^N$, then the following embedding are compact*

1. *If $\overline{p} < N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ $\quad \forall q \in [1, \overline{p}^*[$, where $\frac{1}{\overline{p}^*} = \frac{1}{\overline{p}} - \frac{1}{N}$,*
2. *If $\overline{p} \ge N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{p^+}(\Omega)$ where $p^+ = \max\{p_1, ..., p_N\}$*

If we denote by $p_\infty = \max\{\overline{p}^*, p^+\}$, we have the continuous embedding
$\forall q \in [1, p_\infty]$ $W_0^{1,\vec{p}}(\Omega) \subset L^q(\Omega)$.

Let $a_1, ..., a_N$ be positive numbers, we have

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \le \frac{1}{N} \sum_{i=1}^{N} N a_i. \tag{2.3}$$

3. Basic Assumptions

We assume that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that

$\mathbf{A_1}$)

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \ge \sum_{i=1}^{N} \nu(|s|)|\xi|^{p_i} \quad \forall s \in \mathbb{R}, \xi \in \mathbb{R}^N \text{and a.e in } \Omega,$$

with ν is a bounded continuous function such that $\nu(0) = 0$

$$a_i(x, s, \xi) = \nu(|s|)\bar{a}_i(x, s, \xi).$$

$\mathbf{A_2}$)

$$\int_0^{+\infty} \nu(t)^{\frac{1}{p_i - 1}} dt = +\infty, \quad \text{for} \quad i = 1, ..., N$$

$\mathbf{A_3}$)

$$|\bar{a}_i(x, s, \xi)| \le \gamma[|s|^{\frac{p_\infty}{p_i'}} + |\xi_i|^{p_i - 1}],$$

$\mathbf{A_4}$)

$$[\bar{a}_i(x, s, \xi) - \bar{a}_i(x, s, \xi')][\xi_i - \xi_i'] > 0 \quad \text{for} \quad \xi_i \ne \xi_i',$$

$\mathbf{A_5}$)

$$|\widehat{H}_i(x, \xi)| \le b_i(x)|\xi_i|^{p_i - 1},$$
$$H_i(x, s, \xi) = \nu(|s|)\widehat{H}_i(x, \xi).$$

A₆)

$$f \in L^{p'_\infty}(\Omega) \;\; and \;\; g_i \in L^{p'_i}(\Omega) \;\; \text{for} \;\; i = 1, ..., N$$

A₇) The function $\bar{a}_i$ and $\widehat{H}_i$ are positively homogeneous of degree $(p_i - 1)$ with respect to the variable ξ, i.e

$$\bar{a}_i(x, s, t\xi) = t^{p_i - 1}\bar{a}_i(x, s, \xi), \quad \widehat{H}_i(x, t\xi) = t^{p_i - 1}\widehat{H}_i(x, \xi), \quad \forall t \geq 0.$$

where b_i belong to the space $L^{r_i}(\Omega)$ with $\frac{1}{r_i} = \frac{1}{p_i} - \frac{1}{p_\infty}$ for $i = 1, ..., N$, and γ positive constant.

4. Pincipal Results

We denote by $\tilde{v}$ the function

$$\tilde{v}(s) = \int_0^s \nu(|t|)^{\frac{1}{p_i - 1}} dt$$

and

$$V = \{u \;\; \text{is measurable function in} \;\; \Omega : \tilde{v}(u) \in W_0^{1,\vec{p}}(\Omega)\}$$

4.1. Definition of the weak solution

Definition 4.1. *A function u in V is a weak solution to problem (P) if $a_i(., u, \nabla u)$, $H_i(., u, \nabla u) \in L^{p'_i}(\Omega)$ and*

$$\sum_{i=1}^N \int_\Omega [a_i(x, u, \nabla u)\partial_{x_i}\varphi + H_i(x, u, \nabla u)\varphi] = \int_\Omega [f\varphi + \sum_{i=1}^N g_i \partial_{x_i}\varphi], \quad \forall \varphi \in W_0^{1,\vec{p}}(\Omega).$$

To avoid the smallness of the norm of b_i, splitting the domain Ω in a finite number of small domains Ω_s, by adopting the technique introduced in [12] for the linear case, and [15] for the nonlinear case.

Proposition 4.2. *Let $A \in \mathbb{R}^+$ and $u \in V$, (i.e. $\tilde{v}(u) \in W_0^{1,\vec{p}}(\Omega)$). Then there exists t measurable subsets $\Omega_1, ..., \Omega_t$ of Ω and t functions $\tilde{v}(u)_1, ..., \tilde{v}(u)_t$ such that $\Omega_i \cap \Omega_j = \emptyset$*
for $i \neq j, |\Omega_t| \leq A$ and $|\Omega_s| = A$ for $s \in \{1, ..., t-1\}$,

$$\{x \in \Omega : |\partial_{x_i}\tilde{v}(u)_s| \neq 0 \;\; for \;\; i = 1, ..., N\} \subset \Omega_s, \partial_{x_i}\tilde{v}(u) = \partial_{x_i}\tilde{v}(u)_s \;\; a.e. \;\; in \;\; \Omega_s, \tag{4.1}$$

$$\partial_{x_i}([\tilde{v}(u)]_1 + ... + [\tilde{v}(u)]_s)[\tilde{v}(u)]_s = (\partial_{x_i}\tilde{v}(u))[\tilde{v}(u)]_s, [\tilde{v}(u)]_1 + ... + [\tilde{v}(u)]_s = \tilde{v}(u) \tag{4.2}$$

and $sign(\tilde{v}(u)) = sign([\tilde{v}(u)]_s)$ if $[\tilde{v}(u)]_s \neq 0$ for $s \in \{1, ..., t\}$ and $i \in \{1, ..., N\}$.

Proof:
Let $0 \leq k < h \leq +\infty$, define $S_{h,k}(s) = T_k(s) - T_h(s)$, where

$$T_k(s) = sign(s)(min(|s|, k))$$

the truncation of hight k. We put $\Omega_{h,k}=\{x\in\Omega:|\partial_{x_i}S_{h,k}(\tilde{v}(u))|\neq 0,\ \text{for } i=1,..;N\}$, we have $\partial_{x_i}S_{h,k}(\tilde{v}(u))=\partial_{x_i}\tilde{v}(u)$ a.e. in $\Omega(h,k)$ for $i=1,...,N$.

Construction of the subset Ω_s and the function $(\tilde{v}(u))_s$: the idea of our approach is inspired from the paper of Del Vecchio et al.(see appendix, [15]). Let $\{k_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^+$ a decreasing sequence constructed in the following way: If $|\Omega|\leq A$ then $k_1=0$, else k_1 is chosen such that $|\{x\in\Omega:|\tilde{v}(u(x))|\geq k_1|=A$. For $j>2$, if $|\{x\in\Omega: 0\leq|\tilde{v}(u(x))|\geq k_{j-1}|\leq A$ then $k_j=0$ else k_j is chosen such that $0<k_j<k_{j-1}$ and $|\{x\in\Omega:k_j\leq|\tilde{v}(u(x))|\geq k_{j-1}|=A$. Let t be the first index such that $k_t=0$, then we put $\Omega_1=\Omega(k_1,+\infty)$; $\Omega_s=\Omega(k_s,k_{s-1})$ for $s=2,..,t$ and $u_1=S_{k_1,\infty}(u)$, $u_s=S_{k_s,k_{s-1}}(u)$ for $s=2,...,t$. □

4.2. Some a priori estimates

We consider a sequence of regularized problems:

$$(P_n)\begin{cases}-\partial_{x_i}a_i(x,u_n,\nabla u_n)+\displaystyle\sum_{i=1}^{N}H_n^i(x,u_n,\nabla u_n)=f-\partial_{x_i}g_i & \text{in}\quad\Omega\\ u_n=0 & \text{on}\quad\partial\Omega,\end{cases}$$

where $H_n^i(x,u,\nabla u)=T_n(H_i(x,u,\nabla u))$.
It is well known, for the functions b_i with norms small enough, the problems (P_n) has at least a weak solution $u_n\in W_0^{1,\vec{p}}(\Omega)$, we refer to ([19]).

Proposition 4.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in V such that $\tilde{v}(u_n)\rightharpoonup\tilde{v}(u)$ in $W_0^{1,\vec{p}}(\Omega)$ and*

$$\sum_{i=1}^{N}\int_\Omega[\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))](\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u_n))\to 0,$$

then $\partial_{x_i}\tilde{v}(u_n)\to\partial_{x_i}\tilde{v}(u)$ a.e. in Ω and for $i=1,...,N$.

Proof:
Let $D_n^i=[\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))](\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u_n))$, $D_n^i\geq 0$ and $D_n^i\to 0$ in $L^1(\Omega)$. Extracting a subsequence $\tilde{v}(u_n)$, by Lemma 2.3 we have $u_n\to u$ *a.e in* $\Omega, D_n^i\to 0$ *a.e in* Ω, which implies that $u_n\to u$ a.e. in Ω.
Let $A\subset\Omega$, such that $|A|=0$, we have $u_n\to u$ and $D_n^i\to 0$ pointwise in $\Omega\backslash A$. For all $x\in\Omega\backslash A$, denoting $\tilde{\xi}_n=\partial_{x_i}\tilde{v}(u_n(x))$, $\tilde{\xi}=\partial_{x_i}\tilde{v}(u(x))$, $\xi_n=\partial_{x_i}u_n(x)$ and $\xi=\partial_{x_i}u(x)$.
Then,

$$\begin{aligned}D_n^i=\ &\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u_n)+\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u)\\ &-\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u)-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u_n)\\ &=a_i(x,u_n,\nabla u_n)\nu(|u_n|)^{\frac{1}{p_i-1}}\partial_{x_i}u_n+a_i(x,u_n,\nabla u)\nu(|u_n|)^{\frac{1}{p_i-1}}\partial_{x_i}u\\ &-\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u_n)-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u_n)\\ &\geq\nu(|u_n|)^{\frac{p_i}{p_i-1}}|\xi_n^i|^{p_i}+\nu(|u|)^{\frac{p_i}{p_i-1}}|\xi^i|^{p_i}-\gamma[|u_n|^{\frac{p_\infty}{p_i'}}+|\tilde{\xi}^i|^{p_i-1}]|\tilde{\xi}_n^i|\\ &-\gamma[|u_n|^{\frac{p_\infty}{p_i'}}+|\tilde{\xi}_n^i|^{p_i-1}]|\tilde{\xi}^i|\end{aligned}\tag{4.3}$$

by the fact that ν is real positive bounded function we conclude that

$$D_n^i \geq c_1\big(|\tilde{\xi}_n^i|^{p_i} - c_x(1 + |\tilde{\xi}_n^i|^{p_i-1} + |\tilde{\xi}_n^i|)\big). \tag{4.4}$$

Since u_n is bounded in $\Omega\backslash A$. Then, $|\xi_n^i|$ is bounded uniformly with respect to n, indeed (4.4) becomes

$$D_n^i \geq c_2|\tilde{\xi}_n^i|^{p_i}\big(1 - \frac{c_x}{|\tilde{\xi}_n^i|^{p_i}} - \frac{c_x}{|\tilde{\xi}_n^i|^{p_i-1}} - \frac{c_x}{|\tilde{\xi}_n^i|}\big)).$$

If $\tilde{\xi}_n^i \to \infty$, (for a subsequence) implies that $D_n^i(x) \to \infty$, which gives a contradiction.

Denoting by $\tilde{\xi}^*$ the limit of subsequence of $\tilde{\xi}_n^i$, for i=1,...,N. Applying the continuity of $\bar{a}$ with respect to the two last variables we obtain

$$(\bar{a}_i(x, u, \tilde{\xi}^*) - \bar{a}_i(x, u, \tilde{\xi}))(\tilde{\xi}_i^* - \tilde{\xi}_i) = 0.$$

By using A_4 we get $\tilde{\xi}_n^i \to \tilde{\xi}^i$, a.e. in Ω for $i = 1, ..., N$.

□

Proposition 4.4. *Let $u_n \in V$ be a solution of the approximate problem (P_n). Then we have*

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i}\tilde{v}(u_n)|^{p_i} \leq C, \tag{4.5}$$

where $C = C\big(N, \Omega, \gamma, \|\beta\|_\infty, \|f\|_{p'_\infty}, \|g_i\|_{p'_i}\big) > 0,$ for $i = 1, ..., N$.

Proof: Let $u_n \in V$, then $\tilde{v}(u_n) \in W_0^{1,\vec{p}}(\Omega)$, and $[\tilde{v}(u_n)]_s$ defined as in Lemma 4.2, we have

$$\begin{aligned}\int_\Omega a_i(x, u_n, \nabla u_n)\partial_{x_i}[\tilde{v}(u_n)]_s &= \int_{\Omega_s} a_i(x, u_n, \nabla u_n)\partial_{x_i} u_n \nu(|u_n|)^{\frac{1}{p_i-1}} \\ &\geq \int_{\Omega_s} \nu(|u_n|)^{\frac{p_i}{p_i-1}}|\partial_{x_i}u_n|^{p_i} \\ &\geq \int_\Omega |\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}.\end{aligned} \tag{4.6}$$

We choose $[\tilde{v}(u_n)]_s$ as test function in the approximate problems (P_n). By (4.6), Young, Hölder inequalities, and embedding results in Lemma 2.3, we have

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega |\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} &\leq c_1\bigg(\|f\|_{p'_\infty} d_s^{\frac{1}{N}} \\ &+ \sum_{i=1}^{N} \int_\Omega |H_i^n(x, u_n, \nabla u_n)||[\tilde{v}(u_n)]_s| + \sum_{i=1}^{N} \|g_i\|_{p'_i}^{p'_i}\bigg)\end{aligned} \tag{4.7}$$

where $d_s = \prod_{j=1}^{N} \left(\int_\Omega |\partial_{x_j} \tilde{v}(u_n)_s|^{p_j} \right)^{\frac{1}{p_j}}$.

Here and in what follows, the constants depend on the data but not on u. Using conditions A_5), (4.1), (4.2) Young, Hölder inequalities, the embedding $W_0^{1,\vec{p}}(\Omega) \subset L^{p_\infty}(\Omega)$, and $H^i(x, u_n, \nabla u_n) = \widehat{H}^i(x, u_n, \nabla \tilde{v}(u_n))$ we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |H^i(x, u_n, \nabla u_n)||[\tilde{v}(u_n)]_s| \le \sum_{i=1}^{N} \int_\Omega |b_i(x)||\partial_{x_i} \tilde{v}(u_n)|^{p_i-1}|[\tilde{v}(u_n)]_s| \\
&\le \sum_{i=1}^{N} \sum_{\sigma=1}^{s} \int_{\Omega_\sigma} |b_i(x)||\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i-1}|[\tilde{v}(u_n)]_s| \\
&\le \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} \sum_{\sigma=1}^{s} \left[\int_{\Omega_\sigma} |\partial_{x_i}[\tilde{v}(u_n)]_s|^{(p_i-1)r_i'} |[\tilde{v}(u_n)]_s|^{r_i'} \right]^{\frac{1}{r_i'}} \\
&\le \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} \sum_{\sigma=1}^{s} \left(\int_{\Omega_\sigma} |\partial_{x_i}[\tilde{v}(u_n)]_s|^{(p_i-1)r_i't_i} \right)^{\frac{1}{r_i't_i}} \left(\int_{\Omega_\sigma} |[\tilde{v}(u_n)]_s|^{r_i't_i'} \right)^{\frac{1}{r_i't_i'}} \\
&\le c_2 \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} \sum_{\sigma=1}^{s} \int_{\Omega_\sigma} \left[|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} + d_s^{\frac{p_i}{N}} \right] \\
&\le c_2 \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} \left[\int_{\Omega_s} |\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} + \sum_{\sigma=1}^{s-1} \int_{\Omega_\sigma} |\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i} + d_s^{\frac{p_i}{N}} \right].
\end{aligned} \tag{4.8}$$

where Ω' and t_i is such that

$$\begin{cases} \|b_i\|_{L^{r_i}(\Omega')} &= \max\{\|b_i\|_{L^{r_i}(\Omega_\sigma)};\ \sigma = 1, .., s\}, \\ (p_i - 1)r_i' t_i &= p_i, \\ r_i' t_i' &= p_\infty. \end{cases}$$

Replacing the inequality (4.8) in (4.7) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} \le c_1 \Big\{ \|f\|_{p_\infty'} d_s^{\frac{1}{N}} + \sum_{i=1}^{N} \|g_i\|_{p_i'}^{p_i'} \\
&+ \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} \big[\int_{\Omega_s} |\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} + \sum_{\sigma=1}^{s-1} \int_{\Omega_\sigma} |\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i} + \sum_{i=1}^{N} d_s^{\frac{p_i}{N}} \big] \Big\}.
\end{aligned} \tag{4.9}$$

Choosing Ω' such that

$$1 - c_1 \sum_{i=1}^{N} \|b_i\|_{L^{r_i}(\Omega')} > 0, \tag{4.10}$$

(4.9) becomes

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} &\leq c_2\{\|f\|_{p'_\infty} d_s^{\frac{1}{N}} + \textstyle\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}\\ &+(\sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}})(\sum_{\sigma=1}^{s-1}\sum_{j=1}^{N}\int_{\Omega_\sigma}|\partial_{x_j}[\tilde{v}(u_n)]_\sigma|^{p_j})\\ &+\textstyle\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')} d_s^{\frac{p_i}{N}}\}.\end{aligned}$$

for some constant $c_3 > 0$. For $s = 1$ we get

$$\begin{aligned}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i} &\leq \sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i}\\ &\leq c_2\left[\|f\|_{p'_\infty} d_1^{\frac{1}{N}} + \sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i} + \sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')} d_1^{\frac{p_i}{N}}\right].\end{aligned} \tag{4.11}$$

Thanks to Proposition 4.3 in [16], by choosing Ω' such that (4.10) and

$$1 - c_2\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')} > 0.$$

We get

$$d_1 \leq c_3\left[(\|f\|_{p'_\infty}^{\frac{N}{\bar{p}}} + \|\gamma\|_{p'_\infty}^{\frac{N}{\bar{p}}}) d_1^{\frac{1}{\bar{p}}} + \sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}\right].$$

Then there exists a constant $c_4 > 0$ such that $d_1 \leq c_4$, and by (4.11) we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i} \leq c_5 \tag{4.12}$$

for some constant $c_5 > 0$. Moreover, using (4.12) in (4.11) and iterating on s, we have

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} \leq c_3\left[\|f\|_{p'_\infty} d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i} + 1 + \sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega_\sigma)} d_1^{\frac{p_i}{N}}\right] \tag{4.13}$$

finally by (4.13), we get

$$\begin{aligned}\|\tilde{v}(u_n)\|_{W_0^{1,\vec{p}}} &= \sum_{i=1}^{N}\left(\int_{\Omega}|\partial_{x_i}\tilde{v}(u_n)|^{p_i}\right)^{\frac{1}{p_i}}\\ &\leq \sum_{i=1}^{N}\left(\int_{\Omega}\left(\sum_{s=1}^{t}|\partial_{x_i}[\tilde{v}(u_n)]_s|\right)^{p_i}\right)^{\frac{1}{p_i}}\\ &\leq k\sum_{i=1}^{N}\left(\sum_{s=1}^{t}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}\right)^{\frac{1}{p_i}}\\ &\leq C.\end{aligned}$$

□

4.3. Existence Theorem

Theorem 4.5. *We suppose that the conditions $A_1) - A_7)$ holds true, then the problem (P) has at least one weak solution.*

Proof: By Proposition refp1, we conclude that $\partial_{x_i}\tilde{v}(u_n)$ is bounded in $L^{p_i}(\Omega)$, which gives a weakly convergence of $\partial_{x_i}\tilde{v}(u_n)$ to $\partial_{x_i}\tilde{v}(u)$ in the space $L^{p_i}(\Omega)$, for $i = 1, ..., N$, and consequently by embedding theorem we get the strongly convergence of $\tilde{v}(u_n)$ to $\tilde{v}(u)$ in $L^{\bar{p}}(\Omega)$, for some u and some subsequence, still denote by u_n.

1. **Almost everywhere convergence of the gradient**
   According to the proposition 4.3, it is enough to show that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega [\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n)) - \bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)] = 0. \quad (4.14)$$

Indeed, we can write the integral of 4.3 as follows

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega [\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n)) - \bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)]\\ &\qquad = \sum_{i=1}^{N}\int_\Omega [\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))][\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)]dx\\ &\qquad - \sum_{i=1}^{N}\int_\Omega [\bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)]dx\\ &\qquad = A_n - B_n.\end{aligned}$$

By assumption $A_1)$ and $A_7)$, we have $\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n)) = a_i(x,u_n,\nabla u_n)$. Thus, we can rewrite A_n as

$$A_n = \sum_{i=1}^{N}\int_\Omega [a_i(x,u_n,\nabla u_n)][\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)]dx.$$

We claim that A_n goes to zero as n tend to infinity. Indeed, taking $v = \tilde{v}(u_n) - \tilde{v}(u)$ as test function in the approximate problem we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)[\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u)]dx\\ &+\sum_{i=1}^{N}\int_\Omega H^i(x,u_n,\nabla u_n)(\tilde{v}(u_n) - \tilde{v}(u))dx\\ &= \int_\Omega f(\tilde{v}(u_n) - \tilde{v}(u))dx + \sum_{i=1}^{N}\int_\Omega g_i(\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u))dx.\end{aligned} \quad (4.15)$$

Since $\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)$ is bounded uniformly in $(L^{p_i}(\Omega))^N$, and $\tilde{v}(u_n)$ converge strongly to $\tilde{v}(u)$ in $L^{p_i}(\Omega)$, g_i, f belong to $L^{p'_i}(\Omega)$, we conclude that

$$\lim_{n\to\infty}\int_\Omega g_i(\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u))dx=0 \text{ and } \lim_{n\to\infty}\int_\Omega f(\tilde{v}(u_n)-\tilde{v}(u))dx=0. \quad (4.16)$$

By assumption A_3), we can show that $H^i(x,u_n,\nabla u_n)\rightharpoonup \eta$ in $L^{p'_\infty}(\Omega)$ and since $\tilde{v}(u_n)$ converge strongly to $\tilde{v}(u)$ in $L^{d_i}(\Omega)$ for all $d_i<p'_\infty$, we get

$$\lim_{n\to\infty}\int_\Omega H^i(x,u_n,\nabla u_n)(\tilde{v}(u_n)-\tilde{v}(u))dx=0. \quad (4.17)$$

By 4.16 and 4.17, we conclude that $\lim_{n\to\infty} A_n=0$. Let E be subset of Ω, we have

$$\int_E|\bar{a}_i(x,u_n,\nabla\tilde{v}(u))|^{q_i}dx\le c_1\int_E|u_n|^{\frac{p_\infty q_i}{p'_i}}dx+c_2\int_E|\partial_{x_i}\tilde{v}(u_)|^{p_i}dx, \forall q_i<p'_i. \quad (4.18)$$

By the strongly convergence $\tilde{v}(u_n)$ to $\tilde{v}(u)$ in $L^{\frac{p_\infty q_i}{p'_i}}(\Omega)$ and since $\tilde{v}$ is bijective (i.e u_n tend to u strongly in $L^{q_i\frac{p_\infty}{p'_i}}(\Omega)$) we have the terms in the right hand side in 4.18 goes to zero as $|E|$ tend to zero, and by almost everywhere convergence of $\bar{a}_i(x,u_n,\nabla\tilde{v}(u))$ to $\bar{a}_i(x,u,\nabla\tilde{v}(u))$, we conclude by Vitali's Theorem that, $\lim_{n\to\infty} B_n=0$, according to Lemma 4.3, we conclude that

$$\partial_{x_i}\tilde{v}(u_n) \longrightarrow \partial_{x_i}\tilde{v}(u), \quad \text{a.e in } \Omega. \quad (4.19)$$

2. Passage to the limit By using the growth condition A_2), we get

$$|\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))|^{p'_i}dx\le c[|u_n|^{p_\infty}+|\partial_{x_i}\tilde{v}(u_n)|^{p_i}] \quad (4.20)$$

by the continuous embedding in anisotropic space $W_0^{1,\vec{p}}(\Omega)$ into $L^{p_\infty}(\Omega)$, Proposition refp1, (4.20) and (4.19) we conclude that

$$\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\rightharpoonup\bar{a}_i(x,u,\nabla\tilde{v}(u)) \text{ weakly in } L^{p'_i}(\Omega),$$

and since $\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))=a_i(x,u_n,\nabla u_n)$, and $\bar{a}_i(x,u,\nabla\tilde{v}(u))=a_i(x,u,\nabla u)$ we have

$$a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u) \text{ weakly in } L^{p'_i}(\Omega). \quad (4.21)$$

On the other hand, we have

$$\begin{aligned}|\widehat{H}^i(x,\nabla\tilde{v}(u_n))|^{p'_\infty} &\le |b_i(x)|^{p'_\infty}|\partial\tilde{v}(u_n)|^{(p_i-1)p'_\infty}\\ &\le c_3|b_i(x)|^{r_i}+c_4|\partial_{x_i}\tilde{v}(u_n)|^{(p_i-1)p'_\infty(\frac{r_i}{p'_\infty})'}\\ &\le c_3|b_i(x)|^{r_i}+c_4|\partial_{x_i}\tilde{v}(u_n)|^{p_i},\end{aligned} \quad (4.22)$$

with c_3 and c_4 are the positive constant. (4.22), and (4.19) gives

$$\widehat{H}_i(x,\nabla\tilde{v}(u_n))\rightharpoonup\widehat{H}_i(x,\nabla\tilde{v}(u)) \text{ weakly in } L^{p'_\infty}(\Omega). \quad (4.23)$$

And since
$\widehat{H_i}(x,\nabla\tilde{v}(u_n)) = H_i(x,u_n,\nabla u_n)$ and $\widehat{H_i}(x,\nabla\tilde{v}(u)) = H_i(x,u,\nabla u)$ we get

$$H_i(x,u_n,\nabla u_n) \rightharpoonup H_i(x,u,\nabla u) \ \text{ weakly in } \ L^{p'_\infty}(\Omega). \tag{4.24}$$

Passing to the limit in the approximate problems (4.2), and using (4.21) and (4.24) we conclude that the problem (P) has at least a weak solution in the sense of definition (4.1).

□

Remark 4.6. *The main difficulty in this kinds of problem that is studied in Theorem (4.5, is due to the fact that the operator a_i is not coercive, because of the condition $\nu(0) = 0$. To overcame this difficulty, we have assumed the boundary of ν.*
Today, we have proved only the existence result of weak solution for the problem (P). The existence of the same result without assuming the boundary of ν, is very important.

4.4. Perspective

The result of the uniqueness of the weak solution of the problem is very important (this is the object of our future paper), the problem comes from the strong monotony condition of the operator, namely:

$$[a_i(x,s,\xi) - a_i(x,s,\xi')][\xi_i - \xi'_i] \geq \nu(|s|)(\epsilon + |\xi_i| + |\xi'_i|)^{p_i-2}|\xi_i - |\xi_i|^2,$$

we have $\nu(|s|) = 0$ when s is small enough.

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