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On Connection Between the Order of a Stationary One-Dimensional Dispersive Equation and the Growth of its Convective Term

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ABSTRACT: A boundary value problem for a stationary nonlinear dispersive equation of order $2l+1$, $l \in \mathbb{N}$ with a convective term in the form $u^k u_x$, $k \in \mathbb{N}$ was considered on an interval $(0, L)$. The existence, uniqueness and continuous dependence of a regular solution as well as a relation between l and critical values of k have been established.

Key Words: Dispersive equations, Regular solutions, Existence, Uniqueness.

Contents

1. Introduction

This work concerns the existence, uniqueness and continuous dependence of regular solutions to a boundary value problem for one class of nonlinear stationary dispersive equations posed on bounded intervals

$$au + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u + u^k u_x = f(x), \quad l, k \in \mathbb{N}, \tag{1.1}$$

where a is a real positive number. This class of stationary equations appears naturally while one wants to solve the corresponding evolution equation

$$u_t + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u + u^k u_x = 0, \quad l, k \in \mathbb{N} \tag{1.2}$$

making use of an implicit semi-discretization scheme:

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$$\frac{u^n - u^{n-1}}{h} + \sum_{j=1}^{l}(-1)^{j+1}D_x^{2j+1}u^n + (u^n)^k u_x^n = 0, \quad l,k \in \mathbb{N}, \tag{1.3}$$

where $h > 0$, [37]. Comparing (1.3) with (1.1), it is clear that $a = \frac{1}{h} > 0$ and $f(x) = \frac{u^{n-1}}{h}$. The case $k = 1$ has been studied in [27].

For $l = 1$, we have the well-known generalized Korteweg-de Vries (KdV) equation which has been studied intensively for critical and supercritical values of k. In [12,29,30,31] it was proved that a supercritical equation does not have global solutions and a critical one has a global solution for "small" initial data and the right-hand side. For $l = 2, k = 2$ the generalized Kawahara equation has been studied in [2]. Initial value problems for the Kawahara equation, $l = 2$, which had been derived in [19] as a perturbation of the KdV equation, have been considered in [3,8,12,14,16,18,20,21,34,35] and attracted attention due to various applications of those results in mechanics and physics such as dynamics of long small-amplitude waves in various media [13,15,17]. On the other hand, last years appeared publications on solvability of initial-boundary value problems for various dispersive equations (which included the KdV and Kawahara equations) in bounded and unbounded domains [2,4,5,7,11,22,23,26,27,28]. In spite of the fact that there is not some clear physical interpretation for the problems on bounded intervals, their study is motivated by numerics [6]. The KdV and Kawahara equations have been developed for unbounded regions of wave propagations, however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this case, some boundary conditions are needed to specify a solution. Therefore, precise mathematical analysis of mixed problems in bounded domains for dispersive equations is welcome and attracts attention of specialists in this area [2,4,5,7,9,11,26].

As a rule, simple boundary conditions at $x = 0$ and $x = 1$ such as $u = u_x = 0|_{x=0}$, $u = u_x = u_{xx} = 0|_{x=1}$ for the Kawahara equation were imposed. Different kind of boundary conditions was considered in [7,25]. Obviously, boundary conditions for (1.1) are the same as for (1.2). Because of that, study of boundary value problems for (1.1) helps to understand solvability of initial- boundary value problems for (1.2).

Last years, publications on dispersive equations of higher orders appeared [11, 14,20,21,36]. Here, we propose (1.1) as a stationary analog of (1.2) because the last equation includes classical models such as the generalized KdV and Kawahara equations.

The goal of our work is to formulate a correct boundary value problem for (1.1) and to prove the existence, uniqueness and continuous dependence on perturbations of $f(x)$ for regular solutions as well as to study a relation between the order of the equation and the critical values of k.

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem and main results of the article. In Section 3 we give some useful facts. In Section 4 the existence of a regular solutions for the problem is proved. Here, a connection between the order of the equation and the growth of its convective term is established. Finally, in Section 5 uniqueness is proved provided certain restriction on f as well as continuous dependence of solutions.

2. Formulation of the Problem and Main Results

For real $a > 0$, consider the following one-dimensional stationary higher order equation:

$$au + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u + u^k Du = f(x) \quad \text{in } (0, L) \tag{2.1}$$

subject to boundary conditions:

$$D^i u(0) = D^i u(L) = D^l u(L) = 0, \; i = 0, \dots, l-1, \tag{2.2}$$

where $0 < L < \infty$, $l, k \in \mathbb{N}$ with $k \leq 4l$, $D^i = d^i/dx^i$, $D^1 \equiv D$ are the derivatives of order $i \in \mathbb{N}$, and f is a given function.

Throughout this paper we adopt the usual notation $(\cdot, \cdot)$ for the inner product in $L^2(0, L)$ and $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_{H^i}$, $i \in \mathbb{N}$ for the norm in $L^2(0, L)$, $L^\infty(0, L)$ and $H^i(0, L)$, respectively [1]. Symbols C_*, C_0, C_i, K_i, $i \in \mathbb{N}$, mean positive constants appearing during the text.

Definition 2.1. *For a fixed $l \in \mathbb{N}$, equation (2.1) is a regular one for $k < 4l$ and is critical when $k = 4l$.*

The main results of this article is the following theorem:

Theorem 2.1. *Let $f \in L^2(0, L)$, then in the regular case, $1 \leq k < 4l$, problem (2.1)-(2.2) admits at least one regular solution $u \in H^{2l+1}(0, L)$ such that*

$$\|u\|_{H^{2l+1}} \leq \mathcal{C}((1 + x), f^2)^{\frac{1}{2}} \tag{2.3}$$

with the constant $\mathcal{C}$ depending only on L, l, k, a and $((1 + x), f^2)$.
In the critical case, $k = 4l$, let f be such that

$$\|f\| < \frac{[(2l + 1)(4l + 2)]^{\frac{1}{4l}} a}{2^{\frac{1}{4l}} C_*} \tag{2.4}$$

with C_* an absolute constant. Then problem (2.1)-(2.2) admits at least one regular solution $u \in H^{2l+1}(0, L)$ such that

$$\|u\|_{H^{2l+1}} \leq \mathcal{C}'((1 + x), f^2)^{\frac{1}{2}} \tag{2.5}$$

with the constant $\mathcal{C}'$ depending only on L, l, a and $((1 + x), f^2)$.

Theorem 2.2. *Let l, $k \in \mathbb{N}$ $1 \leq k \leq 4l$ and let $((1 + x), f^2)$ be sufficiently small. Then the solution from Theorem 2.1 is unique and continuously depends on perturbations of f.*

3. Preliminary Results

Lemma 3.1. *For all $u \in H^1(0,L)$ such that $u(x_0) = 0$ for some $x_0 \in [0,L]$*

$$\sup_{x\in(0,L)} |u(x)| \leq \sqrt{2}\|u\|^{\frac{1}{2}}\|Du\|^{\frac{1}{2}}. \tag{3.1}$$

Proof: Let $x_0 \in [0,L]$ be such that $u(x_0) = 0$. Then for any $x \in (0,L)$

$$u^2(x) = \int_{x_0}^{x} D[u^2(\xi)]d\xi \leq 2\int_{x_0}^{x} |u(\xi)||D(\xi)|d\xi \leq 2\int_0^L |u(x)||Du(x)|dx$$

$$\leq 2\|u\|\|Du\|.$$

From this, (3.1) follows immediately. □

We will use the following versions of the Gagliardo-Nirenberg's inequality, [24, 32,33].

Theorem 3.1. *Let u belong to $H_0^l(0,L)$, then the following inequality holds:*

$$\|u\|_\infty \leq C_*\|D^l u\|^{\frac{1}{2l}}\|u\|^{1-\frac{1}{2l}} \tag{3.2}$$

with C_* an absolute constant.

Theorem 3.2. *Suppose u and $D^{2l+1}u$ belong to $L^2(0,L)$. Then for the derivatives $D^i u$, $0 \leq i < 2l+1$ the following inequalities hold:*

$$\|D^i u\|_{L^p} \leq K_1\|D^{2l+1}u\|^\theta\|u\|^{1-\theta} + K_2\|u\|, \tag{3.3}$$

where

$$\frac{1}{p} = i - \theta(2l+1) + \frac{1}{2},$$

for all $\theta \in [\frac{i}{2l+1}, 1]$. (The constants K_1, K_2 depend only on L, l, i).

We will use the following fixed point theorem, [10].

Theorem 3.3. *(Schaefer's Fixed Point Theorem) Let X a real Banach Space. Suppose $B : X \to X$ is a compact and continuous mapping. Assume further that the set*

$$\{u \in X \mid u = \lambda Bu \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then B has a fixed point.

4. Existence

Proof: (of Theorem 2.1).

We start with the linearized version of (2.1)

$$Au \equiv au + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u = f \quad \text{in } (0,L) \tag{4.1}$$

subject to boundary conditions (2.2).

Theorem 4.1. *(See [27], Theorem 5). Let $F \in L^2(0,L)$. Then the problem (4.1),(2.2) admits a unique regular solution $u \in H^{2l+1}(0,L)$ such that*

$$\|u\|_{H^{2l+1}} \le C_0\|F\| \tag{4.2}$$

with the constant C_0 depending only on L, l and a.

Given $u \in H_0^l(0,L)$, set $F := f - u^k Du$. By (3.2), we get

$$\begin{aligned}\|F\| &\le \|f\| + \|u^k Du\| \le \|f\| + \|u\|_\infty^k \|Du\| \\ &\le \|f\| + C_*^k \|u\|^{(1-\frac{1}{2l})k}\|D^l u\|^{\frac{k}{2l}}\|Du\| \\ &\le \|f\| + C_*^k \|u\|_{H_0^l}^{(1-\frac{1}{2l})k}\|u\|_{H_0^l}^{\frac{k}{2l}}\|u\|_{H_0^l} \\ &\le \|f\| + C_*^k\|u\|_{H_0^l}^{k+1}. \end{aligned} \tag{4.3}$$

By Theorem 4.1, let $w \in H^{2l+1}(0,L)$ be a unique solution of the linear equation

$$aw + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}w = F \quad \text{in } (0,L) \tag{4.4}$$

subject to boundary conditions (2.2). By (4.2)-(4.3),

$$\|w\|_{H^{2l+1}} \le C_0\|F\| \le C_0(\|f\| + C_*^k\|u\|_{H_0^l}^{k+1}). \tag{4.5}$$

We will write henceforth $Bu = w$ whenever w is derived from u via (4.4),(2.2), that is, $Bu \equiv A^{-1}(F(u))$, where A is defined by (4.1).

Lemma 4.1. *The mapping $B : H_0^l(0,L) \to H_0^l(0,L)$ is compact and continuous.*

Proof: Indeed, if $\{u_n\}$ is a bounded sequence in $H_0^l(0,L)$, then in view of estimate (4.5), the sequence $\{w_n\}$, where $w_n = Bu_n$, $n \in \mathbb{N}$ is bounded in $H^{2l+1}(0,L)$. Since $H^{2l+1}(0,L)$ is compactly embedded in $H_0^l(0,L)$, there exists a convergent in $H_0^l(0,L)$ subsequence $\{Bu_{n_m}\}_{m=1}^\infty$, therefore B is compact.

To prove continuity of the mapping B, let $\{u_n\}$ be a sequence such that $u_n \to u$ in $H_0^l(0,L)$. Then the difference $v_n = w_n - w$, where $w_n = Bu_n$, $n \in \mathbb{N}$ and $w = Bu$ satisfies

$$av_n + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}v_n = u^k D(u - u_n) + (u^k - u_n^k)Du_n \tag{4.6}$$

and the boundary conditions (2.2).

Multiplying (4.6) by v_n and integrating by parts over $(0, L)$, we obtain

$$a\|v_n\|^2 + \frac{1}{2}(D^l v_n(0))^2 = (u^k D(u-u_n) + (u^k - u_n^k)Du_n, v_n),$$

whence

$$a\|v_n\| \leq \|u^k D(u-u_n)\| + \|(u^k - u_n^k)Du_n\|. \tag{4.7}$$

According to (3.1),

$$\begin{aligned} \|u^k D(u-u_n)\| &\leq \left(\sup_{x\in(0,L)} |u(x)|^{2k}\right)^{\frac{1}{2}} \|D(u_n - u)\| \\ &\leq 2^{\frac{k}{2}}\|u\|^{\frac{k}{2}}\|Du\|^{\frac{k}{2}}\|u_n - u\|_{H_0^l} \\ &\leq 2^{\frac{k}{2}}\|u\|^k_{H_0^l}\|u_n - u\|_{H_0^l} \to 0 \end{aligned}$$

because $u_n \to u$ in $H_0^l(0, L)$. On the other hand, let $g \in C^1(\mathbb{R})$ be such that $g(y) = y^k$. By the Mean Value Theorem, for arbitrary $y, z \in \mathbb{R}$ there is $\xi \in (y, z)$ such that

$$|y^k - z^k| = k\xi^{k-1}|y - z|.$$

Since $\xi \in (y, z)$ we can write $\xi = (1-\tau)y + \tau z$, with $\tau \in (0, 1)$. Taking $y = u_n(x)$ and $z = u(x)$ for each $x \in (0, L)$, we obtain

$$\begin{aligned} |u_n^k(x) - u^k(x)|^2 &= k^2|(1-\tau)u_n(x) + \tau u(x)|^{2(k-1)}|u_n(x) - u(x)|^2 \\ &\leq k^2[|1-\tau||u_n(x)| + |\tau||u(x)|]^{2(k-1)}|u_n(x) - u(x)|^2 \\ &\leq k^2[|u_n(x)| + |u(x)|]^{2(k-1)}|u_n(x) - u(x)|^2 \\ &\leq k^2 2^{2(k-1)}|u_n(x)|^{2(k-1)}|u_n(x) - u(x)|^2 \\ &+ k^2 2^{2(k-1)}|u(x)|^{2(k-1)}|u_n(x) - u(x)|^2. \end{aligned} \tag{4.8}$$

By (3.1),

$$\sup_{x\in(0,L)} |u_n(x)|^{2(k-1)} \leq 2^{k-1}\|u_n\|^{k-1}\|Du_n\|^{k-1} \leq 2^{k-1}\|u_n\|^{2(k-1)}_{H_0^l},$$

$$\sup_{x\in(0,L)} |u(x)|^{2(k-1)} \leq 2^{k-1}\|u\|^{k-1}\|Du\|^{k-1} \leq 2^{k-1}\|u\|^{2(k-1)}_{H_0^l}$$

and

$$\sup_{x\in(0,L)} |u_n(x) - u(x)|^2 \leq 2\|u_n - u\|\|D(u_n - u)\| \leq 2\|u_n - u\|^2_{H_0^l}.$$

Thus

$$\begin{aligned} \|(u^k - u_n^k)Du_n\| &\leq \left(\sup_{x\in(0,L)} |u_n^k(x) - u^k(x)|^2\right)^{\frac{1}{2}} \|Du_n\| \\ &\leq k2^{\frac{3k-2}{2}}(\|u_n\|^{k-1}_{H_0^l} + \|u\|^{k-1}_{H_0^l})^{\frac{1}{2}}\|u_n - u\|_{H_0^l} \to 0 \end{aligned}$$

because the sequence $\{u_n\}$ is bounded in $H_0^l(0,L)$ and $u_n \to u$ in $H_0^l(0,L)$. From (4.7), we conclude that $\|v_n\| \to 0$.

Multiplying (4.6) by $(1+x)v_n$ and integrating over $(0,L)$, we obtain

$$a(v_n,(1+x)v_n) + \sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}v_n,(1+x)v_n)$$
$$= (u^k D(u-u_n) + (u^k - u_n^k)Du_n,(1+x)v_n).$$

Integrating by parts and using (2.2) it follow that

$$a\|v_n\|^2 + \sum_{j=1}^{l}\left(\frac{2j+1}{2}\right)\|D^j v_n\|^2 + \frac{1}{2}(D^l v_n(0))^2$$
$$\leq (\|u^k D(u-u_n)\| + \|(u^k - u_n^k)Du_n\|)\|(1+x)v_n\|.$$

Since $\|u^k D(u-u_n)\|, \|(u^k-u_n^k)Du_n\|, \|v_n\| \to 0$, we get $\|v_n\|_{H_0^l} \to 0$, that is, $w_n \to w$ in $H_0^l(0,L)$. Hence, $u_n \to u$ in $H_0^l(0,L)$ implies $Bu_n \to Bu$ in $H_0^l(0,L)$. This proves that B is continuous. □

Lemma 4.2. *The set*

$$\{u \in H_0^l(0,L) \mid u = \lambda Bu \textit{ for some } 0 \leq \lambda \leq 1\}$$

is bounded in $H_0^l(0,L) \cap H^{2l+1}(0,L)$.

Proof: Assume $u \in H_0^l(0,L)$ such that

$$u = \lambda Bu \text{ for some } 0 < \lambda \leq 1,$$

then

$$a\left(\frac{u}{\lambda}\right) + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}\left(\frac{u}{\lambda}\right) = f - u^k Du \quad \text{in } (0,L)$$

and

$$D^i\left(\frac{u}{\lambda}\right)(0) = D^i\left(\frac{u}{\lambda}\right)(L) = D^l\left(\frac{u}{\lambda}\right)(L) = 0,\ i = 0,\dots,l-1,$$

that is

$$au + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u + \lambda u^k Du = \lambda f \quad \text{in } (0,L) \tag{4.9}$$

and u satisfies the boundary conditions (2.2).

To prove this Lemma, we need some a priori estimates:

Estimate I:

Multiplying (4.9) by u and integrating over $(0, L)$, we obtain

$$a\|u\|^2 + \sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, u) + \lambda(u^k Du, u) = (\lambda f, u). \tag{4.10}$$

Integrating by parts and using (2.2), we get

$$\lambda(u^k Du, u) = 0$$

and

$$\sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, u) = \frac{1}{2}(D^l u(0))^2.$$

Thus (4.10) becomes

$$a\|u\|^2 + \frac{1}{2}(D^l u(0))^2 = (\lambda f, u)$$

and

$$\|u\| \leq \frac{1}{a}\|f\|. \tag{4.11}$$

Estimate II:

Multiplying (4.9) by $(1+x)u$ and integrating over $(0, L)$, we obtain

$$\begin{aligned} a(u, (1+x)u) &+ \sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, (1+x)u) \\ &+\lambda(u^k Du, (1+x)u) = (\lambda f, (1+x)u). \end{aligned} \tag{4.12}$$

Since

$$\sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, (1+x)u) = \sum_{j=1}^{l}\left(\frac{2j+1}{2}\right)\|D^j u\|^2 + \frac{1}{2}(D^l u(0))^2,$$

integrating by parts and using (2.2),(3.2), we get

$$\begin{aligned} \lambda(u^k Du, (1+x)u) &= \lambda(u^k Du, xu) = \frac{\lambda}{k+2}\int_0^L xD[u^{k+2}]dx \\ &= -\frac{\lambda}{k+2}\int_0^L u^{k+2}dx \leq \frac{1}{k+2}\|u\|_\infty^k\|u\|^2 \\ &\leq \underbrace{\frac{C_*^k}{k+2}\|u\|^{2+(\frac{2l-1}{2l})k}\|D^l u\|^{\frac{k}{2l}}}_{I}. \end{aligned} \tag{4.13}$$

Regular case $1 \le k < 4l$.

By the Young inequality, with $p = \frac{4l}{k}$, $q = \frac{4l}{4l-k}$ and arbitrary $\epsilon_1 > 0$,

$$I \le \epsilon_1 \frac{k}{4l}\|D^l u\|^2 + \frac{1}{\epsilon_1^{\frac{k}{4l-k}}}\left(\frac{4l-k}{4l}\right)\left(\frac{C_*^k}{k+2}\right)^{\frac{4l}{4l-k}}\|u\|^{\frac{8l+(4l-2)k}{4l-k}}.$$

Again by the Young inequality with arbitrary $\epsilon_2 > 0$,

$$(f,(1+x)u) \le \frac{\epsilon_2}{2}((1+x),u^2) + \frac{1}{2\epsilon_2}((1+x),f^2).$$

Therefore, (4.12) reduces to the inequality

$$\begin{aligned}&\left(a - \tfrac{\epsilon_2}{2}\right)((1+x),u^2) + \textstyle\sum_{j=1}^{l-1}\left(\frac{2j+1}{2}\right)\|D^j u\|^2 + \left(\frac{2l+1}{2} - \epsilon_1\frac{k}{4l}\right)\|D^l u\|^2\\ &\le \tfrac{1}{\epsilon_1^{\frac{k}{4l-k}}}\left(\tfrac{4l-k}{4l}\right)\left(\tfrac{C_*^k}{k+2}\right)^{\frac{4l}{4l-k}}\|u\|^{\frac{8l+(4l-2)k}{4l-k}} + \tfrac{1}{2\epsilon_2}((1+x),f^2).\end{aligned}$$

Taking $\epsilon_1 = \frac{4l(2l-1)}{2k} > 0$ and $\epsilon_2 = a > 0$, we get

$$\begin{aligned}&\frac{a}{2}((1+x),u^2) + \sum_{j=1}^{l-1}\left(\frac{2j+1}{2}\right)\|D^j u\|^2 + \|D^l u\|^2\\ &\qquad \le C_1\|u\|^{\frac{8l+(4l-2)k}{4l-k}} + \frac{1}{2a}((1+x),f^2),\end{aligned} \tag{4.14}$$

where

$$C_1 = \left(\frac{2k}{4l(2l-1)}\right)^{\frac{k}{4l-k}}\left(\frac{4l-k}{4l}\right)\left(\frac{C_*^k}{k+2}\right)^{\frac{4l}{4l-k}}.$$

Since

$$((1+x),f^2) = \|f\|^2 + (x,f^2) \ge \|f\|^2,$$

it follows from (4.11) that

$$\|u\|^{\frac{8l+(4l-2)k}{4l-k}} \le \left(\frac{1}{a}\right)^{\frac{8l+(4l-2)k}{4l-k}}((1+x),f^2)^{\frac{4l+(2l-1)k}{4l-k}}$$

and (4.14) implies

$$\|u\|_{H_0^l} \le C_2((1+x),f^2)^{\frac{1}{2}}, \tag{4.15}$$

where

$$C_2 = \frac{1}{\sqrt{\beta}}\left[C_3((1+x),f^2)^{\frac{2lk}{4l-k}} + \frac{1}{2a}\right]^{\frac{1}{2}}$$

with $\beta = \min\{\frac{a}{2},1\}$ and $C_3 = C_1 a^{-\frac{8l+(4l-2)k}{4l-k}}$.

Rewriting (4.9) in the form

$$(-1)^{l+1}D^{2l+1}u = \lambda f - au - \sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u - \lambda u^k Du,$$

we estimate

$$\|D^{2l+1}u\| \leq \|f\| + a\|u\| + \sum_{j=1}^{l-1}\|D^{2j+1}u\| + \|u^k Du\|. \tag{4.16}$$

For $l=1$ we have $\sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u = 0$ and for $l \geq 2$ denote $J = \{1,\dots,l-1\}$ and

$$I_1 = \{j \in J|\ 2j+1 \leq l\}, \quad I_2 = \{j \in J|\ l < 2j+1 < 2l+1\}.$$

Hence we can write

$$\|D^{2l+1}u\| \leq \|f\| + a\|u\| + \sum_{j\in I_1}\|D^{2j+1}u\| + \sum_{j\in I_2}\|D^{2j+1}u\| + \|u^k Du\|. \tag{4.17}$$

By (4.15),

$$a\|u\| + \sum_{j\in I_1}\|D^{2j+1}u\| \leq (a+l)C_2((1+x),f^2)^{\frac{1}{2}} \tag{4.18}$$

and by (3.2),(4.15),

$$\|u^k Du\| \leq \|u\|_\infty^k \|Du\| \leq C_*^k \|u\|_{H_0^l}^{k+1} \leq C_*^k C_2^{k+1}((1+x),f^2)^{\frac{k+1}{2}}. \tag{4.19}$$

On the other hand, $l < 2j+1 < 2l+1$ for all $j \in I_2$. Hence, by (3.3), there are K_1^j, K_2^j, depending only on L and l, such that

$$\|D^{2j+1}u\| \leq K_1^j \|D^{2l+1}u\|^{\theta^j}\|u\|^{1-\theta^j} + K_2^j\|u\| \quad \text{with} \quad \theta^j = \frac{2j+1}{2l+1}.$$

Making use of Young's inequality with $p^j = \frac{1}{\theta^j}$, $q^j = \frac{1}{1-\theta^j}$ and arbitrary $\epsilon > 0$, we get

$$\|D^{2j+1}u\| \leq \epsilon\|D^{2l+1}u\| + C_4^j(\epsilon)\|u\| + K_2^j\|u\|,$$

where $C_4^j(\epsilon) = \left[q^j\left(\frac{p^j\epsilon}{(K_1^j)^{p^j}}\right)^{\frac{q^j}{p^j}}\right]^{-1}$. Summing over $j \in I_2$ and making use of (4.11), we find

$$\sum_{j\in I_2}\|D^{2j+1}u\| \leq l\epsilon\|D^{2l+1}u\| + \left(\frac{1}{a}\sum_{j\in I_2}(C_4^j(\epsilon)+K_2^j)\right)\|f\|. \tag{4.20}$$

Substituing (4.18),(4.19) and (4.20) into (4.17), we obtain

$$\begin{aligned}\|D^{2l+1}u\| \quad &\leq \quad l\epsilon\|D^{2l+1}u\| + \left(\frac{1}{a}\sum_{j\in I_2}(C_4^j(\epsilon)+K_2^j)\right)((1+x),f^2)^{\frac{1}{2}} \\ &+ \quad \left(1+(a+l)C_2 + C_*^k C_2^{k+1}((1+x),f^2)^{\frac{k}{2}}\right)((1+x),f^2)^{\frac{1}{2}}.\end{aligned}$$

Taking $\epsilon = \frac{1}{2l}$, we conclude

$$\|D^{2l+1}u\| \le C_5((1+x), f^2)^{\frac{1}{2}}, \tag{4.21}$$

where C_5 depends only on L, l, k, a and $((1+x), f^2)$.

Again by (3.3), for all $i = l+1, \ldots, 2l$, there are K_1^i, K_2^i depending only on L and l such that

$$\|D^i u\| \le K_1^i \|D^{2l+1}u\|^{\theta^i} \|u\|^{1-\theta^i} + K_2^i \|u\| \quad \text{with} \quad \theta^i = \frac{i}{2l+1}.$$

Making use of (4.11) and (4.21), we get

$$\|D^i u\| \le \left(\frac{K_1^i C_5^{\theta^i}}{a^{1-\theta^i}} + \frac{K_2^i}{a} \right) ((1+x), f^2)^{\frac{1}{2}}, \quad i = l+1, \ldots, 2l. \tag{4.22}$$

Taking into account (4.15), (4.21) and (4.22), we obtain (2.3), that is

$$\|u\|_{H^{2l+1}} \le \mathcal{C}((1+x), f^2)^{\frac{1}{2}}$$

with $\mathcal{C}$ depending only on L, l, k, a and $((1+x), f^2)$.

Critical case $\mathbf{k = 4l}$.

Returning to (4.13), we find

$$I = \frac{C_*^{4l}}{4l+2} \|u\|^{4l} \|D^l u\|^2 \le \frac{C_*^{4l}}{(4l+2)a^{4l}} \|f\|^{4l} \|D^l u\|^2.$$

Since

$$(f, (1+x)u) \le \frac{a}{2}((1+x), u^2) + \frac{1}{2a}((1+x), f^2),$$

we transform (4.12) as follows

$$\frac{a}{2}\|u\|^2 + \sum_{j=1}^{l-1} \left(\frac{2j+1}{2} \right) \|D^j u\|^2 + \left(\frac{2l+1}{2} - \frac{C_*^{4l}}{(4l+2)a^{4l}} \|f\|^{4l} \right) \|D^l u\|^2$$
$$+ \frac{1}{2}(D^l u(0))^2 \le \frac{1}{2a}((1+x), f^2).$$

For fixed l, a and $f \in L^2(0, L)$ such that

$$\|f\| < \frac{[(2l+1)(4l+2)]^{\frac{1}{4l}} a}{2^{\frac{1}{4l}} C_*},$$

we obtain

$$\frac{2l+1}{2} - \frac{C_*^{4l}}{(4l+2)a^{4l}} \|f\|^{4l} > 0.$$

Therefore

$$\|u\|_{H_0^l} \leq \frac{1}{\sqrt{2a\gamma_l}}((1+x), f^2)^{\frac{1}{2}} \tag{4.23}$$

with $\gamma_l = \min\{\frac{a}{2}, \frac{3}{2}, \frac{2l+1}{2} - \frac{C_*^{4l}}{(4l+2)a^{4l}}\|f\|^{4l}\}$. Retunrning to (4.9) and acting as in the regular case with (4.23), we conclude (2.5), that is

$$\|u\|_{H^{2l+1}} \leq \mathcal{C}'((1+x), f^2)^{\frac{1}{2}}$$

with $\mathcal{C}'$ depending only on L, l, a and $((1+x), f^2)$. □

Applying Theorem 3.3, we complete the proof of the Theorem 2.1. □

5. Uniqueness and Continuous Dependence

Proof: (of Theorem 2.2).

We separated two cases: $l \geq 2$ and $l = 1$.

For $\mathbf{l \geq 2}$, let u_1 and u_2 be two distinct solutions of (2.1)-(2.2). Then the difference $w = u_1 - u_2$ satisfies the equation

$$aw + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}w + u_1^k Dw + (u_1^k - u_2^k)Du_2 = 0 \tag{5.1}$$

and the boundary conditions (2.2).

Multiplying (5.1) by w and integrating over $(0, L)$, we obtain

$$a\|w\|^2 + \frac{1}{2}(D^l w(0))^2 + \underbrace{(u_1^k Dw, w)}_{I_1} + \underbrace{((u_1^k - u_2^k)Du_2, w)}_{I_2} = 0. \tag{5.2}$$

Integrating by parts and using (2.2),(3.1), we get

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_0^L w^2(x)Du_1^k(x)dx \leq \frac{k}{2}\int_0^L |u_1(x)|^{k-1}|Du_1(x)||w(x)|^2 dx \\ &\leq \frac{k}{2}\sup_{x\in(0,L)}|u_1(x)|^{k-1}\sup_{x\in(0,L)}|Du_1(x)|\|w\|^2 \\ &\leq k2^{\frac{k-2}{2}}\|u_1\|_{H_0^l}^k\|w\|^2. \end{aligned}$$

By (3.1),(4.8), we have

$$\begin{aligned} |I_2| &\leq \int_0^L |u_1^k(x) - u_2^k(x)||Du_2(x)||w(x)|dx \\ &\leq k2^{k-1}\sup_{x\in(0,L)}|Du_2(x)|\int_0^L (|u_1(x)|^{k-1} + |u_2(x)|^{k-1})|w(x)|^2 dx \\ &\leq k2^{\frac{2k-1}{2}}\|u_2\|_{H_0^l}\sup_{x\in(0,L)}\{|u_1(x)|^{k-1} + |u_2(x)|^{k-1}\}\|w\|^2 \\ &\leq k2^{\frac{3k-2}{2}}\|u_2\|_{H_0^l}(\|u_1\|_{H_0^l}^{k-1} + \|u_2\|_{H_0^l}^{k-1})\|w\|^2. \end{aligned}$$

Substituting I_1, I_2 into (5.2), we reduce it to the inequality

$$\left(a - k2^{\frac{k-2}{2}}\|u_1\|^k_{H^l_0} - k2^{\frac{3k-2}{2}}\|u_2\|_{H^l_0}(\|u_1\|^{k-1}_{H^l_0} + \|u_2\|^{k-1}_{H^l_0})\right)\|w\|^2 \le 0. \quad (5.3)$$

Regular case $1 \le k < 4l$.

Making use of (4.15), we can estimate (5.3) as

$$\left(a - (2^{\frac{k-2}{2}} + 2^{\frac{3k}{2}})kC_2^k((1+x), f^2)^{\frac{k}{2}}\right)\|w\|^2 \le 0, \quad (5.4)$$

where

$$C_2 = \frac{1}{\sqrt{\beta}}\left[C_3((1+x), f^2)^{\frac{2lk}{4l-k}} + \frac{1}{2a}\right]^{\frac{1}{2}}$$

with $\beta = \min\{\frac{a}{2}, 1\}$ and C_3 depending only on l, k and a. For fixed l, k and a, assume that

$$((1+x), f^2)^{\frac{1}{2}} < \min\left\{\left(\frac{1}{2aC_3}\right)^{\frac{4l-k}{4lk}}, \frac{a^{\frac{1}{k}}}{[(2^{\frac{k-2}{2}} + 2^{\frac{3k}{2}})k]^{\frac{1}{k}}(a\beta)^{-\frac{1}{2}}}\right\}. \quad (5.5)$$

Then $C_2 < \left(\frac{1}{a\beta}\right)^{\frac{1}{2}}$ and consequently

$$\left(a - (2^{\frac{k-2}{2}} + 2^{\frac{3k}{2}})kC_2^k((1+x), f^2)^{\frac{k}{2}}\right) > 0.$$

Hence (5.4) implies $\|w\| = 0$ and uniqueness is proved for $l \ge 2$ and $1 \le k < 4l$.

Critical case $k = 4l$.

Rewrite (5.3) in the form:

$$\left(a - l2^{2l+1}\|u_1\|^{4l}_{H^l_0} - l2^{6l+1}\|u_2\|_{H^l_0}(\|u_1\|^{4l-1}_{H^l_0} + \|u_2\|^{4l-1}_{H^l_0})\right)\|w\|^2 \le 0.$$

Making use of (4.23), we obtain

$$\left(a - l(2^{2l+1} + 2^{6l+2})\left(\frac{1}{2a\gamma_l}\right)^{2l}((1+x), f^2)^{2l}\right)\|w\|^2 \le 0,$$

where

$$\gamma_l = \min\left\{\frac{a}{2}, \frac{3}{2}, \frac{2l+1}{2} - \frac{C_*^{4l}}{(4l+2)a^{4l}}\|f\|^{4l}\right\}.$$

For fixed l and a, suppose that

$$((1+x), f^2)^{\frac{1}{2}} < \min\left\{\frac{[(2l+1)(4l+2)]^{\frac{1}{4l}}a}{2^{\frac{1}{4l}}C_*}, \left(\frac{a}{\eta}\right)^{\frac{1}{4l}}\right\}, \quad (5.6)$$

where $\eta = l(2^{2l+1} + 2^{6l+2})(2a\gamma_l)^{-2l}$. Since $\|f\| \le ((1+x), f^2)^{\frac{1}{2}}$, it follows that (2.4) is satisfied and

$$\left(a - l(2^{2l+1} + 2^{6l+2})\left(\frac{1}{2a\gamma_l}\right)^{2l}((1+x), f^2)^{2l}\right) > 0.$$

Thus $\|w\| = 0$ and uniqueness is proved for $l \ge 2$ and $k = 4l$.

The case l = 1.

The problem (2.1)-(2.2) becomes:

$$au + D^3u + u^kDu = f \quad \text{in } (0, L), \tag{5.7}$$

$$u(0) = u(L) = Du(L) = 0. \tag{5.8}$$

Let u_1 and u_2 be two distinct solutions of (5.7)-(5.8). Then the difference $w = u_1 - u_2$ satisfies the equation

$$aw + D^3w + u_1^kDw + (u_1^k - u_2^k)Du_2 = 0 \tag{5.9}$$

and the boundary conditions (5.8).

Multiplying (5.9) by w and integrating over $(0, L)$, we obtain

$$a\|w\|^2 + \frac{1}{2}(Dw(0))^2 + \underbrace{(u_1^kDw, w)}_{I_1} + \underbrace{((u_1^k - u_2^k)Du_2, w)}_{I_2} = 0. \tag{5.10}$$

Integrating by parts and using (3.1),(5.8), we get

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_0^L Du_1^k(x)w^2(x)dx \le \frac{k}{2}\int_0^L |u_1(x)|^{k-1}|Du_1(x)||w(x)|^2dx \\ &\le \frac{k}{2}\sup_{x\in(0,L)}|u_1(x)|^{k-1}\sup_{x\in(0,L)}|Du_1(x)|\|w\|^2 \\ &\le k2^{\frac{k-3}{2}}\|u_1\|_{H_0^1}^{k-1}\sup_{x\in(0,L)}|Du_1(x)|\|w\|^2. \end{aligned}$$

By (3.1),(4.8), it follows that

$$\begin{aligned} |I_2| &\le \int_0^L |u_1^k(x) - u_2^k(x)||Du_2(x)||w(x)|dx \\ &\le k2^{k-1}\sup_{x\in(0,L)}\{|u_1(x)|^{k-1} + |u_2(x)|^{k-1}\}\sup_{x\in(0,L)}|Du_2(x)|\|w\|^2 \\ &\le k2^{\frac{3(k-1)}{2}}(\|u_1\|_{H_0^1}^{k-1} + \|u_2\|_{H_0^1}^{k-1})\sup_{x\in(0,L)}|Du_2(x)|\|w\|^2. \end{aligned}$$

Substituting I_1, I_2 into (5.10), we get

$$\begin{aligned} &a\|w\|^2 - k2^{\frac{k-3}{2}}\|u_1\|_{H_0^1}^{k-1}\sup_{x\in(0,L)}|Du_1(x)|\|w\|^2 \\ &-k2^{\frac{3(k-1)}{2}}(\|u_1\|_{H_0^1}^{k-1} + \|u_2\|_{H_0^1}^{k-1})\sup_{x\in(0,L)}|Du_2(x)|\|w\|^2 \le 0. \end{aligned} \tag{5.11}$$

Regular case $\mathbf{1 \leq k < 4}$.

By (4.11),(4.19),

$$\|D^3u_i\| \leq 2\|f\| + C_*^k C_2^{k+1}((1+x), f^2)^{\frac{k+1}{2}}, \ i = 1, 2. \tag{5.12}$$

Making use of (3.3),(4.11) and (5.12), we estimate

$$\begin{aligned}
\sup_{x\in(0,L)} |Du_i(x)| &\leq K_1\|D^3u_i\|^{\frac{1}{2}}\|u_i\|^{\frac{1}{2}} + K_2\|u_i\| \\
&\leq \frac{K_1}{2}\|D^3u_i\| + \left(\frac{K_1}{2} + K_2\right)\|u_i\| \\
&\leq \frac{K_1}{2}C_*^k C_2^{k+1}((1+x), f^2)^{\frac{k+1}{2}} + \left(K_1 + \frac{K_1}{2a} + \frac{K_2}{a}\right)\|f\| \\
&\leq \frac{K_1}{2}C_*^k C_2^{k+1}((1+x), f^2)^{\frac{k+1}{2}} + K_3((1+x), f^2)^{\frac{1}{2}},
\end{aligned}$$

where $K_3 = \left(K_1 + \frac{K_1}{2a} + \frac{K_2}{a}\right)$. Returning to (5.11) and using (4.15), we find

$$\begin{aligned}
a\|w\|^2 - k(2^{\frac{k-3}{2}} + 2^{\frac{3(k-1)}{2}})\frac{K_1}{2}C_*^k C_2^{2k}((1+x), f^2)^k\|w\|^2 \\
-k(2^{\frac{k-3}{2}} + 2^{\frac{3(k-1)}{2}})C_2^{k-1}K_3((1+x), f^2)^{\frac{k}{2}}\|w\|^2 \leq 0.
\end{aligned}$$

Assuming $((1+x), f^2)^{\frac{1}{2}} \leq 1$, then $((1+x), f^2)^k \leq ((1+x), f^2)^{\frac{k}{2}}$. Therefore

$$\left(a - k(2^{\frac{k-3}{2}} + 2^{\frac{3(k-1)}{2}})\left(\frac{K_1}{2}C_*^k C_2^{2k} + K_3 C_2^{k-1}\right)((1+x), f^2)^{\frac{k}{2}}\right)\|w\|^2 \leq 0.$$

For fixed k and a assume that

$$((1+x), f^2)^{\frac{1}{2}} < \min\left\{\left(\frac{1}{2aC_3}\right)^{\frac{4-k}{4k}}, \left(\frac{a}{K_4}\right)^{\frac{1}{k}}\right\}, \tag{5.13}$$

where $K_4 = k(2^{\frac{k-3}{2}} + 2^{\frac{3(k-1)}{2}})(\frac{K_1}{2}C_*^k(a\beta)^{-k} + K_3(a\beta)^{-\frac{k-1}{2}})$. Then

$$C_2^{2k} < \left(\frac{1}{a\beta}\right)^k, \ C_2^{k-1} < \left(\frac{1}{a\beta}\right)^{\frac{k-1}{2}}$$

and

$$\left(a - k(2^{\frac{k-3}{2}} + 2^{\frac{3(k-1)}{2}})\left(\frac{K_1}{2}C_*^k C_2^{2k} + K_3 C_2^{k-1}\right)((1+x), f^2)^{\frac{k}{2}}\right) > 0.$$

This implies $\|w\| = 0$ and uniqueness is proved for $l = 1$ and $1 \leq k < 4$.

Critical case $\mathbf{k = 4}$.

In this case, (5.11) becomes

$$a\|w\|^2 - 2^{\frac{5}{2}}\|u_1\|^3_{H_0^1} \sup_{x\in(0,L)} |Du_1(x)|\|w\|^2$$
$$-2^{\frac{13}{2}}(\|u_1\|^3_{H_0^1} + \|u_2\|^3_{H_0^1}) \sup_{x\in(0,L)} |Du_2(x)|\|w\|^2 \leq 0. \tag{5.14}$$

By (4.11),(4.23),

$$\|D^3 u_i\| \leq 2\|f\| + C_*^4 \left(\frac{1}{2a\gamma_1}\right)^{\frac{5}{2}} ((1+x), f^2)^{\frac{5}{2}}, \ i = 1, 2, \tag{5.15}$$

where $\gamma_1 = \min\{\frac{a}{2}, \frac{3}{2} - \frac{C_*^4}{6a^4}\|f\|^4\}$. Then (3.3),(4.11),(5.15) implies

$$\sup_{x\in(0,L)} |Du_i(x)| \leq \frac{K_1}{2} C_*^4 \left(\frac{1}{2a\gamma_1}\right)^{\frac{5}{2}} ((1+x), f^2)^{\frac{5}{2}} + K_3((1+x), f^2)^{\frac{1}{2}}.$$

Making use of (4.23), we rewrite (5.14) as

$$a\|w\|^2 - (2^{\frac{5}{2}} + 2^{\frac{15}{2}})\frac{K_1}{2} C_*^4 \left(\frac{1}{2a\gamma_l}\right)^4 ((1+x), f^2)^4 \|w\|^2$$
$$-(2^{\frac{5}{2}} + 2^{\frac{15}{2}})K_3 \left(\frac{1}{2a\gamma_l}\right)^{\frac{3}{2}} ((1+x), f^2)^2 \|w\|^2 \leq 0.$$

Assuming $((1+x), f^2)^{\frac{1}{2}} \leq 1$, then $((1+x), f^2)^4 \leq ((1+x), f^2)^2$. This implies

$$a\|w\|^2 - (2^{\frac{5}{2}} + 2^{\frac{15}{2}})\frac{K_1}{2} C_*^4 \left(\frac{1}{2a\gamma_l}\right)^4 ((1+x), f^2)^2 \|w\|^2$$
$$-(2^{\frac{5}{2}} + 2^{\frac{15}{2}})K_3 \left(\frac{1}{2a\gamma_l}\right)^{\frac{3}{2}} ((1+x), f^2)^2 \|w\|^2 \leq 0.$$

For a fixed a, suppose that

$$((1+x), f^2)^{\frac{1}{2}} < \min\left\{\frac{\sqrt{3}a}{C_*}, \left(\frac{a}{K_5}\right)^{\frac{1}{4}}\right\}, \tag{5.16}$$

where $K_5 = (2^{\frac{5}{2}} + 2^{\frac{15}{2}})(\frac{K_1}{2}C_*^4(2a\gamma_l)^{-4} + K_3(2a\gamma_l)^{-\frac{3}{2}})$. Then (2.4) holds and

$$\left(a - (2^{\frac{5}{2}} + 2^{\frac{15}{2}})\left(\frac{K_1}{2} C_*^4 \left(\frac{1}{2a\gamma_l}\right)^4 + K_3 \left(\frac{1}{2a\gamma_l}\right)^{\frac{3}{2}}\right) ((1+x), f^2)^2\right) > 0.$$

It follows that $\|w\| = 0$ and uniqueness is proved for $l = 1$ and $k = 4$.

This completes the proof of the uniqueness part of Theorem 2.2.

To show continuous dependence of solutions, consider the case when $l \geq 2$ and $1 \leq k < 4l$. Let $f_1, f_2 \in L^2(0, L)$ satisfy (5.5) and u_1, u_2 be solutions of (2.1)-(2.2) with the right-hand sides f_1 and f_2 respectively. Then, similarly to (5.4), $u_1 - u_2$ satisfies the following inequality:

$$\left(a - (2^{\frac{k-2}{2}} + 2^{\frac{3k}{2}})k\tilde{C_2}^k M\right) \|u_1 - u_2\| \leq \|f_1 - f_2\|,$$

where

$$M = \max\{((1+x), f_1^2)^{\frac{1}{2}}, ((1+x), f_2^2)^{\frac{1}{2}}\}$$

and

$$\tilde{C_2} = \frac{1}{\sqrt{\beta}} \left[C_3 M^{\frac{4lk}{4l-k}} + \frac{1}{2a}\right]^{\frac{1}{2}}.$$

Making use of (5.5), we obtain

$$\|u_1 - u_2\| \leq C_6 \|f_1 - f_2\|$$

with $C_6 = \left(a - (2^{\frac{k-2}{2}} + 2^{\frac{3k}{2}})k\tilde{C_2}^k M\right)^{-1} > 0$. This proves the continuous dependence for $l \geq 2$ and $1 \leq k < 4l$. The other cases can be proved in a similar way taking $((1+x), {f_i}^2)^{\frac{1}{2}}$, $i = 1, 2$ satisfying (5.6), (5.13) and (5.16). Therefore the proof of the Theorem 2.2 is complete. □

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