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Smarandache Curves of Bertrand Curves Pair According to Frenet Frame

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abstract: In this paper, the curvature and the torsion of Smarandache curves obtained by the vectors of the Bertrand partner curve are calculated. These values are expressed depending upon the α curve. Besides, we illustrate example with our main results.

Key Words: Bertrand curves pair, Smarandache Curves, Frenet invariants.

Contents

1. Introduction

It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of these: Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curves is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrand curves Partner. The following properties of Bertrand curves are well known: If two curves have the same principal normals, (i) corresponding points are a fixed distance apart; (ii) the tangents at corresponding points are at a fixed angle. These well known properties of Bertrand curves in Euclidean 3-space was extended by L. R. Pears in [8]. The Bertrand curves are the Inclined curve pairs. On the other hand, it gave the notion of Bertrand Representation and found that the Bertrand Representation is spherical, [6].

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [13]. Special Smarandache curves have been studied by some authors. Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache

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TB_2 curves in the space $\mathbb{E}_1^4$ [13]. Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case, [1]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves, [3]. Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in $\mathbb{E}^3$, [2]. Nurten Bayrak, Özcan Bektaş and Salim Yüce studied some special Smarandache curves in $\mathbb{E}_1^3$. Kemal Taşköprü and Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 [12]. Şenyurt and Sivas defined NC-Smarandache curve, then they calculated the curvature and torsion of NB and TNB- Smarandache curves together with NC-Smarandache curve, [11]. It studied that the special Smarandache curve in terms of Sabban frame of Fixed Pole curve and they gave some characterization of Smarandache curves, [4]. When the unit Darboux vector of the partner curve of Mannheim curve were taken as the position vectors, the curvature and the torsion of Smarandache curve were calculated. These values were expressed depending upon the Mannheim curve, [5]. In [10], they calculated curvature and torsion of spatial quaternionic involute curve according to n_1^* normal vector and w^* unit Darboux vector of Smarandache curve.

In this paper, special Smarandache curves belonging to α^* curve such as T^*N^*, N^*B^*, T^*B^* and $T^*N^*B^*$ drawn by Frenet frame are defined and some related results are given.

2. Preliminaries

In $\mathbb{E}^3$, inner product is given by

$$\langle , \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in \mathbb{E}^3$. Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [7], [9]

$$T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = -\tau N. \tag{2.1}$$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector N of the curve α is linearly dependent on the principal vector N^* of the curve α^*, then the pair (α, α^*) is said to be Bertrand curves pair. The distance between corresponding points of the Bertrand curves pair in $\mathbb{E}^3$ is constant, [7].
The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are given by [9].

$$\begin{cases} T^* = \cos\theta T + \sin\theta B \\ N^* = N \\ B^* = -\sin\theta T + \cos\theta B. \end{cases} \tag{2.2}$$

Where $\angle(T, T^*) = \theta$

Theorem 2.1. *Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have, [9]*

$$\kappa^* = \frac{\lambda\kappa - \sin^2\theta}{\lambda(1-\lambda\kappa)}, \ \tau^* = \frac{\sin^2\theta}{\lambda^2\tau}, \ \lambda = constant. \tag{2.3}$$

3. Smarandache Curves of Bertrand curves Pair According to Frenet Frame

Here, we give some characterizations of Smarandache curves in Euclidean 3-space. We find parametrization of Smarandache curves for tangent, principal normal and binormal of α^* curve. Furthermore we investigate the curvature and the torsion of Smarandache curves of Bertrand pair and these values are expressed depending upon the α curve.

Definition 3.1. *Let (α, α^*) be a Bertrand curves pair in E^3 and $\{T^*N^*B^*\}$ be the Frenet frame of the Bertrand curves α^* at $\alpha^*(s)$.*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(T^* + N^*). \tag{3.1}$$

regular curve drawn by the vector is called T^*N^* - Smarandache curve.

Theorem 3.2. *The Frenet invariants of the T^*N^* - Smarandache curve are given as following;*

$$\begin{cases} T_{\beta_1}(s) = \dfrac{-\kappa T + (\kappa\cos\theta - \tau\sin\theta)N + \tau B}{\sqrt{\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2}}, \ N_{\beta_1} = \dfrac{\gamma_1 T + \upsilon_1 N + \chi_1 B}{\sqrt{{\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2}} \\[2ex] B_{\beta_1} = \dfrac{\big(\chi_1(\kappa\cos\theta - \tau\sin\theta) - \upsilon_1\tau\big)T + (\kappa\chi_1 + \tau\gamma_1)N}{\sqrt{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}} \\[2ex] \qquad - \dfrac{\big(\upsilon_1\kappa + \gamma_1(\kappa\cos\theta - \tau\sin\theta)\big)B}{\sqrt{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}}, \\[2ex] \kappa_{\beta_1} = \dfrac{\sqrt{2({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}}{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)^2}, \ \tau_{\beta_1} = \dfrac{\sqrt{2}(\vartheta_1\eta_1 + \sigma_1\lambda_1 + \mu_1\rho_3)}{{\vartheta_1}^2 + {\sigma_1}^2 + {\mu_1}^2} \end{cases} \tag{3.2}$$

Herein, the coefficients are

$$\begin{cases} \gamma_1 = & (-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta)\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & +\kappa\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big), \\ \upsilon_1 = & (-\|W\|^2 + \kappa'\cos\theta - \tau'\sin\theta)\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & -(\kappa\cos\theta - \tau\sin\theta)\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big), \\ \chi_1 = & (\kappa\tau\cos\theta - \tau^2\sin\theta + \tau')\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & -\tau\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big), \end{cases}$$

$$\begin{cases} \eta_1 = & (-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta)' - \kappa\big(- \|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big), \\ \lambda_1 = & \kappa(-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta) + \big(- \|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big)' \\ & -\tau(\kappa\tau\cos\theta - \tau^2\sin\theta + \tau'), \\ \rho_1 = & \tau\big(- \|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big) + (\kappa\tau\cos\theta - \tau^2\sin\theta + \tau')', \\ \vartheta_1 = & (\kappa\cos\theta - \tau\sin\theta)(\kappa\tau\cos\theta - \tau^2\sin\theta + \tau') - \tau\big(- \|W\|^2 \\ & +(\kappa\cos\theta - \tau\sin\theta)'\big), \\ \sigma_1 = & \kappa(\kappa\tau\cos\theta - \tau^2\sin\theta + \tau') + \tau(-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta), \\ \mu_1 = & -\kappa\big(- \|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big) + (\kappa\cos\theta - \tau\sin\theta) \\ & \cdot(-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta). \end{cases}$$

Proof If we use the equation (2.2) in the equation (3.1), we obtain

$$\beta_1(s) = \frac{\cos\theta T + N + \sin\theta B}{\sqrt{2}}\cdot \tag{3.3}$$

The derivative of this equation with respect to s is as follows:

$$\beta_1' = T_{\beta_1}\frac{ds_{\beta_1}}{ds} = \frac{-\kappa T + (\kappa\cos\theta - \tau\sin\theta)N + \tau B}{\sqrt{2}} \tag{3.4}$$

and by substitution, we get

$$T_{\beta_1}(s) = \frac{-\kappa T + (\kappa\cos\theta - \tau\sin\theta)N + \tau B}{\sqrt{\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2}} \tag{3.5}$$

where

$$\frac{ds_{\beta_1}}{ds} = \sqrt{\frac{\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2}{2}}\cdot \tag{3.6}$$

In order to determine the first curvature and the principal normal of the curve $\beta_1(s)$, we formalize

$$T_{\beta_1}'(s) = \frac{\sqrt{2}(\gamma_1 T + \upsilon_1 N + \chi_1 B)}{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)^2} \tag{3.7}$$

where

$$\begin{cases}\gamma_1 = & (-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta)\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & +\kappa\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big), \\ \upsilon_1 = & (-\|W\|^2 + \kappa'\cos\theta - \tau'\sin\theta)\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & -(\kappa\cos\theta - \tau\sin\theta)\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big), \\ \chi_1 = & (\kappa\tau\cos\theta - \tau^2\sin\theta + \tau')\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big) \\ & -\tau\big(\|W\|\|W\|' + (\kappa\cos\theta - \tau\sin\theta)(\kappa'\cos\theta - \tau'\sin\theta)\big)\cdot\end{cases}$$

The first curvature is

$$\begin{aligned}\kappa_{\beta_1} &= \|T'_{\beta_1}\|, \\ \kappa_{\beta_1} &= \frac{\sqrt{2({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}}{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)^2}.\end{aligned}$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_{\beta_1} = \frac{\gamma_1 T + \upsilon_1 N + \chi_1 B}{\sqrt{{\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2}}\ , \tag{3.8}$$

$$\begin{aligned}B_{\beta_1} &= \frac{\big(\chi_1(\kappa\cos\theta - \tau\sin\theta) - \upsilon_1\tau\big)T + (\kappa\chi_1 + \tau\gamma_1)N}{\sqrt{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}} \\ &- \frac{\big(\upsilon_1\kappa + \gamma_1(\kappa\cos\theta - \tau\sin\theta)\big)B}{\sqrt{\big(\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)^2\big)({\gamma_1}^2 + {\upsilon_1}^2 + {\chi_1}^2)}}\ .\end{aligned} \tag{3.9}$$

The torsion is then given by

$$\tau_{\beta_1} = \frac{det(\beta_1', \beta_1'', \beta_1''')}{\|\beta_1' \wedge \beta_1''\|^2},$$

$$\tau_{\beta_1} = \frac{\sqrt{2}(\vartheta_1\eta_1 + \vartheta_2\eta_2 + \vartheta_3\eta_3)}{{\vartheta_1}^2 + {\vartheta_2}^2 + {\vartheta_3}^2}$$

where

$$\begin{cases}\eta_1 = & (-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta)' - \kappa\big(-\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big), \\ \lambda_1 = & \kappa(-\kappa' - \kappa^2\cos\theta + \kappa\tau\sin\theta) + \big(-\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big)' \\ & -\tau(\kappa\tau\cos\theta - \tau^2\sin\theta + \tau'), \\ \rho_1 = & \tau\big(-\|W\|^2 + (\kappa\cos\theta - \tau\sin\theta)'\big) + (\kappa\tau\cos\theta - \tau^2\sin\theta + \tau')',\end{cases}$$

$$\begin{cases} \vartheta_1 = & (\kappa\cos\theta-\tau\sin\theta)(\kappa\tau\cos\theta-\tau^2\sin\theta+\tau')-\tau\big(-\|W\|^2 \\ & +(\kappa\cos\theta-\tau\sin\theta)'\big), \\ \sigma_1 = & \kappa(\kappa\tau\cos\theta-\tau^2\sin\theta+\tau')+\tau(-\kappa'-\kappa^2\cos\theta+\kappa\tau\sin\theta) \\ \mu_1 = & -\kappa\big(-\|W\|^2+(\kappa\cos\theta-\tau\sin\theta)'\big)+(\kappa\cos\theta-\tau\sin\theta) \\ & \cdot(-\kappa'-\kappa^2\cos\theta+\kappa\tau\sin\theta). \end{cases}$$

Definition 3.3. *Let (α,α^*) be a Bertrand curves pair in E^3 and $\{T^*N^*B^*\}$ be the Frenet frame of the Bertrand curves α^* at $\alpha^*(s)$*

$$\beta_2(s)=\frac{1}{\sqrt{2}}(N^*+B^*) \tag{3.10}$$

regular curve drawn by the vector is called N^*B^* - Smarandache curve.

Theorem 3.4. *The Frenet invariants of the N^*B^* - Smarandache curve are given as following;*

$$\begin{cases} T_{\beta_2} = & \dfrac{-\kappa T-(\kappa\sin\theta+\tau\cos\theta)N+\tau B}{\sqrt{\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)^2}},\ N_{\beta_2}=\dfrac{\gamma_2 T+\upsilon_2 N+\chi_2 B}{\sqrt{{\gamma_2}^2+{\upsilon_2}^2+{\chi_2}^2}}, \\ B_{\beta_2} = & \dfrac{-(\chi_2(\kappa\sin\theta+\tau\cos\theta)+\upsilon_2\tau)T+(\chi_2\kappa+\gamma_2\tau)N}{\sqrt{\big(\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)^2\big)({\gamma_2}^2+{\upsilon_2}^2+{\chi_2}^2)}} \\ & +\dfrac{(-\upsilon_2\kappa+\gamma_2(\kappa\sin\theta+\tau\cos\theta))B}{\sqrt{\big(\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)^2\big)({\gamma_2}^2+{\upsilon_2}^2+{\chi_2}^2)}}, \\ \kappa_{\beta_2} = & \dfrac{\sqrt{2\gamma_2^2+2\upsilon_2^2+2\chi_2^2}}{\Big(\|W\|^2+\big(\kappa\sin\theta+\tau\cos\theta\big)^2\Big)^2},\ \tau_{\beta_2}=\dfrac{\sqrt{2}(\vartheta_2\eta_2+\sigma_2\lambda_2+\mu_2\rho_2)}{{\vartheta_2}^2+{\sigma_2}^2+{\mu_2}^2}. \end{cases}$$

Herein, the coefficients are

$$\begin{cases} \gamma_2 = & \big(-\kappa'+\kappa(\kappa\sin\theta+\tau\cos\theta)\big)\big(\|W\|^2+(\kappa\sin\theta \\ & +\tau\cos\theta)^2\big)+\kappa\big(\|W\|\|W\|'+(\kappa\sin\theta+\tau\cos\theta)\big) \\ & \cdot(\kappa'\sin\theta+\tau'\cos\theta)(\kappa'\sin\theta+\tau'\cos\theta), \\ \upsilon_2 = & (-\kappa^2-(\kappa'\sin\theta+\tau'\cos\theta)-\tau^2)\big(\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)^2\big) \\ & +(\kappa\sin\theta+\tau\cos\theta)\big(\|W\|\|W\|'+(\kappa\sin\theta+\tau\cos\theta)\big) \\ & \cdot(\kappa'\sin\theta+\tau'\cos\theta), \\ \chi_2 = & (-\tau(\kappa\sin\theta+\tau\cos\theta)+\tau')\big(\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)^2\big) \\ & -\tau\big(\|W\|\|W\|'+(\kappa\sin\theta+\tau\cos\theta)\big)(\kappa'\sin\theta+\tau'\cos\theta), \end{cases}$$

$$\begin{cases} \eta_2 = & (-\kappa' - \kappa(\kappa\sin\theta + \tau\cos\theta))' - \kappa\big(- \|W\|^2 + (\kappa'\sin\theta + \tau'\cos\theta)\big), \\ \lambda_2 = & \kappa(-\kappa' + \kappa(\kappa\sin\theta + \tau\cos\theta)) + \big(- \|W\|^2 - (\kappa'\sin\theta + \tau'\cos\theta)\big)' \\ & -\tau(-\tau(\kappa\sin\theta + \tau\cos\theta) + \tau'), \\ \rho_2 = & \tau\big(- \|W\|^2 + (\kappa'\sin\theta + \tau'\cos\theta)\big) + (-\tau(\kappa\sin\theta + \tau\cos\theta) + \tau')', \\ \vartheta_2 = & (-\kappa\sin\theta - \tau\cos\theta)\big(- \tau(\kappa\sin\theta + \tau\cos\theta) + \tau'\big) \\ & +\tau\big(\|W\|^2 + (\kappa'\sin\theta + \tau'\cos\theta)\big), \\ \sigma_2 = & \kappa(-\tau(\kappa\sin\theta + \tau\cos\theta) + \tau') + \tau(-\kappa' + \kappa(\kappa\sin\theta + \tau\cos\theta)), \\ \mu_2 = & (\kappa\sin\theta + \tau\cos\theta)(-\kappa' + \kappa(\kappa\sin\theta + \tau\cos\theta)) \\ & +\kappa\big(\|W\|^2 + (\kappa'\sin\theta + \tau'\cos\theta)\big). \end{cases}$$

Proof The proof is similar to proof of Theorem 3.2.

Definition 3.5. *Let (α, α^*) be a Bertrand curves pair in E^3 and $\{T^* N^* B^*\}$ be the Frenet frame of the Bertrand curves α^* at $\alpha^*(s)$.*

$$\beta_3(s) = \frac{1}{\sqrt{2}}(T^* + B^*). \tag{3.11}$$

regular curve drawn by the vector is called T^*B^* - Smarandache curve.

Theorem 3.6. *The Frenet invariants of the N^*B^* - Smarandache curve are given as following;*

$$\begin{cases} T_{\beta_3} = \dfrac{\big(\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big)N}{\sqrt{\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta}}, N_{\beta_3} = \dfrac{\gamma_3 T + \upsilon_3 N + \chi_3 B}{\sqrt{{\gamma_3}^2 + {\upsilon_3}^2 + {\chi_3}^2}}, \\ \\ B_{\beta_3} = \dfrac{(\chi_3 T - \gamma_3 B)[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)]}{\sqrt{(\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta)({\gamma_3}^2 + {\upsilon_3}^2 + {\chi_3}^2)}} \\ \\ \kappa_{\beta_3} = \dfrac{\sqrt{2({\gamma_3}^2 + {\upsilon_3}^2 + {\chi_3}^2)}}{(\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta)^2}, \tau_{\beta_3} = \dfrac{-\tau\kappa' + \kappa\tau'}{\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]\|W\|^2}. \end{cases}$$

Herein, the coefficients are

$$\begin{cases} \gamma_3 = & -\kappa\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big](\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta), \\ \upsilon_3 = & \big[\kappa'(\cos\theta - \sin\theta) - \tau'(\sin\theta + \cos\theta)\big](\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta) \\ & -\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)(\kappa\kappa' - \tau\tau' - \|W\|\|W\|'\sin 2\theta)\big], \\ \chi_3 = & \tau\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big](\kappa^2 - \tau^2 - \|W\|^2\sin 2\theta). \end{cases}$$

Proof The proof is similar to proof of Theorem 3.2.

Definition 3.7. *Let (α, α^*) be a Bertrand curves pair in E^3 and $\{T^* N^* B^*\}$ be the Frenet frame of the Bertrand curves α^* at $\alpha^*(s)$.*

$$\beta_4(s) = \frac{1}{\sqrt{3}}(T^* + N^* + B^*). \tag{3.12}$$

regular curve drawn by the vector is called $T^*N^*B^*$ - Smarandache curve.

Theorem 3.8. *The Frenet invariants of the $T^*N^*B^*$ - Smarandache curve are given as following;*

$$
\begin{aligned}
T_{\beta_4} &= \frac{-\kappa T + \big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]N+\tau B}{\sqrt{\|W\|^2+\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2}}\ ,\\
N_{\beta_4} &= \frac{\gamma_4 T+\upsilon_4 N+\chi_4 B}{\sqrt{{\gamma_4}^2+{\upsilon_4}^2+{\chi_4}^2}}\ ,\\
B_{\beta_4} &= \frac{\begin{array}{l}\Big[\Big(\chi_4\big(\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big)-\upsilon_4\tau\Big)T\\ +(\chi_4\kappa+\gamma_4\tau)N-\Big(\upsilon_4\kappa+\gamma_4\big(\kappa(\cos\theta-\sin\theta)\\ -\tau(\sin\theta+\cos\theta)\big)\Big)B\Big]({\gamma_4}^2+{\upsilon_4}^2+{\chi_4}^2)^{\frac{-1}{2}}\end{array}}{\sqrt{\|W\|^2+\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2}}\ ,\\
\kappa_{\beta_4} &= \frac{\sqrt{3({\gamma_4}^2+{\upsilon_4}^2+{\chi_4}^2)}}{\Big(\|W\|^2+\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2\Big)^2}\ ,\\
\tau_{\beta_4} &= \frac{\sqrt{3}(\vartheta_4\eta_4+\sigma_4\lambda_4+\mu_4\rho_4)}{{\vartheta_4}^2+{\sigma_4}^2+{\mu_4}^2}.
\end{aligned}
$$

Herein, the coefficients are

$$
\begin{cases}
\gamma_4 = & \Big[-\kappa'-\kappa\big(\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big)\Big]\Big(\|W\|^2+\big[\kappa(\cos\theta\\
& -\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2\Big)+\kappa\Big(\|W\|\|W\|'+\big[\kappa(\cos\theta-\sin\theta)\\
& -\tau(\sin\theta+\cos\theta)\big]\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]'\Big),\\
\upsilon_4 = & \Big[-\|W\|^2+\big(\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big)'\Big]\Big(\|W\|^2+\big[\kappa(\cos\theta\\
& -\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2\Big)-\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]\\
& \cdot\Big(\|W\|\|W\|'+\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]\big[\kappa(\cos\theta\\
& -\sin\theta)-\tau(\sin\theta+\cos\theta)\big]'\Big),\\
\chi_4 = & \Big[\tau\big(\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big)+\tau'\Big]\Big(\|W\|^2+\big[\kappa(\cos\theta\\
& -\sin\theta)-\tau(\sin\theta+\cos\theta)\big]^2\Big)-\tau\Big(\|W\|\|W\|'+\big[\kappa(\cos\theta-\sin\theta)\\
& -\tau(\sin\theta+\cos\theta)\big]\big[\kappa(\cos\theta-\sin\theta)-\tau(\sin\theta+\cos\theta)\big]'\Big),
\end{cases}
$$

$$\begin{cases} \eta_4 = & \Big(-\kappa' - \kappa\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]\Big)' \\ & -\kappa\Big(-\|W\|^2 + \big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]'\Big), \\ \lambda_4 = & \kappa\Big(-\kappa' - \kappa\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]\Big) \\ & +\Big(-\|W\|^2 + \big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]'\Big)' \\ & -\tau\Big(\tau\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big] + \tau'\Big), \\ \rho_4 = & \tau\Big(-\|W\|^2 + \big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]'\Big) \\ & +\Big(\tau\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big] + \tau'\Big)', \end{cases}$$

$$\begin{cases} \vartheta_4 = & \big(\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big)\Big(\tau\big[\kappa(\cos\theta \\ & - \sin\theta) - \tau(\sin\theta + \cos\theta)\big] + \tau'\Big) \\ & -\tau\Big(-\|W\|^2 + \big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]'\Big), \\ \sigma_4 = & \kappa\Big(\tau\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big] + \tau'\Big) \\ & +\tau\Big(-\kappa' - \kappa\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]\Big), \\ \mu_4 = & \kappa\Big(-\|W\|^2 + \big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]'\Big) \\ & +\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big] \\ & \Big(-\kappa' - \kappa\big[\kappa(\cos\theta - \sin\theta) - \tau(\sin\theta + \cos\theta)\big]\Big). \end{cases}$$

Proof The proof is similar to proof of Theorem 3.2.

3.1. Example

Let (α, α^*) be Bertrand curves pair, these curves are as follows

$$\alpha(s) = \frac{1}{\sqrt{2}}\big(-\cos s, -\sin s, s\big), \alpha^*(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)\cdot$$

The Frenet invariants of the Bertrand partner curve $\alpha^*(s)$ are given as following:

$$\begin{cases} T^*(s) = & \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1),\ N^*(s) = \big(-\cos s, -\sin s, 0\big) \\ B^*(s) = & \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1),\ \kappa^*(s) = \frac{1}{\sqrt{2}},\ \tau^*(s) = \frac{1}{\sqrt{2}}\cdot \end{cases}$$

In terms of definitions, we obtain special Smarandache curves as following:

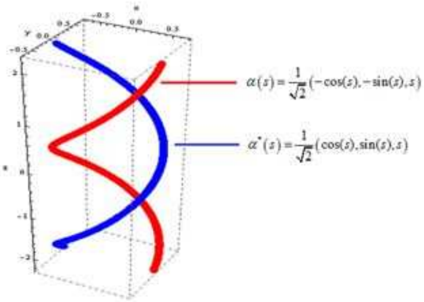
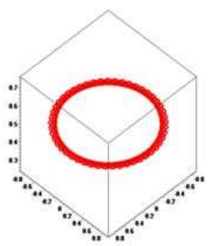
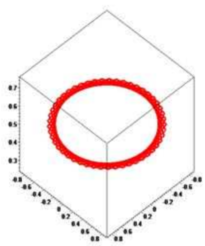


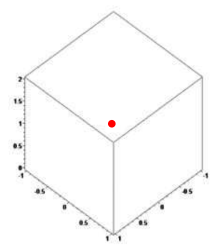
Figure 1: Bertrand Curve Pair



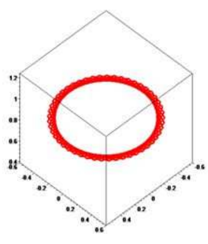
$T^*N^* - Smarandache\ Curve$



$N^*B^* - Smarandache\ Curve$



$T^*B^* - Smarandache\ Curve$



$T^*N^*B^* - Smarandache\ Curve$

References

1. Ali, A. T., *Special Smarandache Curves in the Euclidean Space*. Intenational Journal of Mathematical Combinatorics 2, 30–36, (2010).

2. Bektaş, Ö. and Yüce, S., *Special Smarandache Curves According to Dardoux Frame in Euclidean 3-Space*, Romanian Journal of Mathematics and Computer science 3(1), 48–59, (2013).

3. Çalışkan, A. and Şenyurt, S., *Smarandache Curves In terms of Sabban Frame of Spherical Indicatrix Curves*, Gen. Math. Notes 31(2), 1–15, (2015).

4. Çalışkan, A. and Şenyurt, S., *Smarandache Curves In Terms of Sabban Frame of Fixed Pole Curve*, Boletim da Sociedade parananse de Mathemtica 34(2), 53–62, (2016).

5. Çalışkan, A. and Şenyurt, S., *N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame*, International J.Math. Combin. 1, 1–13, (2015).

6. Görgülü, A. and Özdamar, E., *A Generalizations of the Bertrand curves as general inclined curve in E^n*, Commun. Fac. Sci. Uni. Ankara Series A1, 35, 53–60, (1986).

7. Hacısalihoğlu, H.H., *Differential Geometry*, İnönü University, Malatya, Mat. no. 7, (1983).

8. Pears, L. R., *Bertrand curves in Riemannian space*, J. London Math. Soc. 1-10(2), 180–183, (1935).

9. Sabuncuoğlu, A., *Differential Geometry*, Nobel Publications, Ankara, (2006).

10. Şenyurt, S., Cevahir, C. and Altun, Y., *On Spatial Quaternionic Involute Curve a New View*, Adv. Appl. Clifford Algebras, 27, 1815–1824, (2017).

11. Şenyurt, S. and Sivas, S., *An Application of Smarandache Curve*, Ordu Univ. J. Sci. Tech. 3(1), 46–60, (2013).

12. Taşköprü, K. and Tosun, M., *Smarandache Curves According to Sabban Frame on S^2*, Boletim da Sociedade parananse de Mathemtica 3 srie., 32(1), 51–59, (2014).

13. Turgut, M. and Yılmaz, S., *Smarandache Curves in Minkowski space-time*, International Journal of Mathematical Combinatorics 3, 51–55, (2008).

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