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Existence Results for Nonlinear Problems with φ- Laplacian Operators and Nonlocal Boundary Conditions

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ABSTRACT: Using Leray-Schauder degree theory, we study the existence of at least one solution for the boundary value problem of the type

$$\begin{cases} (\varphi(u'))' = f(t,u,u') \\ u'(0) = u(0), \ u'(T) = bu'(0), \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\varphi(0) = 0$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, T a positive real number, and b some non zero real number.

Key Words: Boundary value problem, Leray-Schauder degree, Brouwer degree, nonlocal boundary conditions, Fixed point theorem.

Contents

1. Introduction

The purpose of this article is to obtain some existence results for the nonlinear boundary value problem of the form

$$\begin{cases} (\varphi(u'))' = f(t,u,u') \\ u'(0) = u(0), \ u'(T) = bu'(0), \end{cases} \tag{1.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\varphi(0) = 0$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, T a positive real number, and b some non zero real number. By a *solution* of (1.1) we mean a function $u : [0,T] \to \mathbb{R}$ of class C^1 with $\varphi(u')$ continuously differentiable, which satisfies the boundary conditions and $(\varphi(u'(t)))' = f(t,u(t),u'(t))$ for all $t \in [0,T]$.

In particular, regular periodic problems with $\varphi-$ or p-Laplacian on the left hand side were considered by several authors, see e.g. del Pino, Manásevich and Murúa [5] or Yan [9].

Recently, V. Bouchez and J. Mawhin in [2] have studied the following boundary value problem:

$$\begin{cases} (\varphi(u))' = f(t,u) \\ u(T) = bu(0), \end{cases}$$

where $\varphi : \mathbb{R} \to (-a,a)$ is a homeomorphism such that $\varphi(0) = 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a and T are positive real numbers and b some non zero real number. The authors obtained the existence of solutions using topological methods based upon Leray-Schauder degree [7].

The main aim of this paper is to study the existence of at least one solution for the boundary value problem (1.1) using Schauder fixed point theorem or Leray-Schauder degree. For this, we reduce the nonlinear boundary value problem to some fixed points problem. The first consequence of this reduction

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is that this operator is defined in C^1. Second, it is completely continuous. Next, adapts a technique introduced by Ward [8] for the search of a priori bounds for the possible fixed points required by a Leray-Schauder approach. The main contribution of this paper is the extension of some results found in [3] to a more general type of boundary conditions. Such a problem does not seem to have been studied in the literature.

The paper is organized as follows. In Section 2, we establish the notation and terminology used throughout the work. In Section 3, we formulate the fixed point operator equivalent to the problem (1.1). In Section 4, we give the main results of this paper. For these results, we adapt the ideas of [1,2] to the present situation. Finally, in section 5, we give some examples to illustrate the results obtained.

2. Notation and terminology

We first introduce some notation. For fixed T, we denote the usual norm in $L^1 = L^1([0,T],\mathbb{R})$ by $\|\cdot\|_{L^1}$. By $C = C([0,T],\mathbb{R})$ we indicate the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|\cdot\|_\infty$ and by $C^1 = C^1([0,T],\mathbb{R})$ we designate the Banach space of continuously differentiable functions from $[0,T]$ into $\mathbb{R}$ endowed with the usual norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

We introduce the following operators:

the *Nemytskii operator* $N_f : C^1 \to C$

$$N_f(u)(t) = f(t,u(t),u'(t)),$$

the *integration operator* $H : C \to C^1$

$$H(u)(t) = \int_0^t u(s)ds,$$

the following continuous linear operators:

$$Q : C \to C, \quad Q(u)(t) = \tfrac{1}{T}\int_0^T u(s)ds,$$

$$P : C \to C, \quad P(u)(t) = u(0),$$

and finally, we introduced the continuous function

$$B_{\varphi,b} : \mathbb{R} \to \mathbb{R}, \quad B_{\varphi,b}(x) = \varphi(bx) - \varphi(x).$$

For $u \in C$, we write

$$u_m = \min_{[0,T]} u, \;\; u_M = \max_{[0,T]} u, \;\; u^+ = \max\{u,0\}, \;\; u^- = \max\{-u,0\}.$$

3. Fixed point formulations

Let us consider the operator

$$M_1 : C^1 \to C^1,$$
$$u \mapsto Q(N_f(u)) - \tfrac{B_{\varphi,b}(Pu)}{T} + H\left(\varphi^{-1}\left[\varphi(Pu) + H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(Pu)}{T}\right]\right) + P(u)$$

where I denotes the function which maps t on t and φ^{-1} is understood as the operator $\varphi^{-1} : C \to C$ defined by $\varphi^{-1}(v)(t) = \varphi^{-1}(v(t))$. It is clear that φ^{-1} is continuous and maps bounded sets into bounded sets.

Using the theorem of Arzelà-Ascoli we show that the operator M_1 is completely continuous.

Lemma 3.1. *The operator $M_1 : C^1 \to C^1$ is completely continuous.*

Proof. Let $\Lambda \subset C^1$ be a bounded set. Then, if $u \in \Lambda$, there exists a constant $\rho > 0$ such that

$$\|u\|_1 \le \rho. \tag{3.1}$$

Next, we show that $\overline{M_1(\Lambda)} \subset C^1$ is a compact set. Let $(v_n)_n$ be a sequence in $M_1(\Lambda)$, and let $(u_n)_n$ be a sequence in Λ such that $v_n = M_1(u_n)$. Using (3.1), we have that there exists a constant $L_1 > 0$ such that, for all $n \in \mathbb{N}$,

$$\|N_f(u_n)\|_\infty \le L_1,$$

which implies that

$$\|H(N_f(u_n) - Q(N_f(u_n)))\|_\infty \le 2L_1T.$$

Hence the sequence $(H(N_f(u_n) - Q(N_f(u_n))))_n$ is bounded in C. Moreover, for $t, t_1 \in [0,T]$ and for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|H(N_f(u_n) - Q(N_f(u_n)))(t) - H(N_f(u_n) - Q(N_f(u_n)))(t_1)| \\
&\le \left|\int_{t_1}^{t} N_f(u_n)(s)ds\right| + \left|\int_{t_1}^{t} Q(N_f(u_n))(s)ds\right| \\
&\le L_1 |t-t_1| + |t-t_1| \, \|Q(N_f(u_n))\|_\infty \\
&\le 2L_1 |t-t_1|,
\end{aligned}$$

which implies that $(H(N_f(u_n) - Q(N_f(u_n))))_n$ is equicontinuous. Thus, by the Arzelà-Ascoli theorem there is a subsequence of $(H(N_f(u_n) - Q(N_f(u_n))))_n$, which we call $\left(H(N_f(u_{n_j}) - Q(N_f(u_{n_j})))\right)_j$, which is convergent in C. Then, passing to a subsequence if necessary, we obtain that the sequence

$$\left(H(N_f(u_{n_j}) - Q(N_f(u_{n_j}))) + \frac{IB_{\varphi,b}(Pu_{n_j})}{T} + \varphi(P(u_{n_j}))\right)_j$$

is convergent in C. Using the fact that $\varphi^{-1} : C \to C$ is continuous, it follows from

$$M_1(u_{n_j})' = \varphi^{-1}\left[\left(H(N_f(u_{n_j}) - Q(N_f(u_{n_j}))) + \tfrac{IB_{\varphi,b}(Pu_{n_j})}{T} + \varphi(P(u_{n_j}))\right)\right]$$

that the sequence $(M_1(u_{n_j})')_j$ is convergent in C. Therefore, passing if necessary to a subsequence, we have that $(v_{n_j})_j = (M_1(u_{n_j}))_j$ is convergent in C^1. Finally, let $(v_n)_n$ be a sequence in $\overline{M_1(\Lambda)}$. Let $(z_n)_n \subseteq M_1(\Lambda)$ be such that

$$\lim_{n\to\infty} \|z_n - v_n\|_1 = 0.$$

Let $(z_{n_j})_j$ be a subsequence of $(z_n)_n$ such that converge to z. It follows that $z \in \overline{M_1(\Lambda)}$ and $(v_{n_j})_j$ converge to z. This concludes the proof. □

Lemma 3.2. *$u \in C^1$ is a solution of (1.1) if and only if u is a fixed point of the operator M_1.*

Proof. Let $u \in C^1$, we have the following equivalences:

$$(\varphi(u'))' = N_f(u), \; u'(T) = bu'(0), \; u'(0) = u(0)$$

$$\Leftrightarrow (\varphi(u'))' = N_f(u) - \left(Q(N_f(u)) - \tfrac{B_{\varphi,b}(u'(0))}{T}\right), \quad Q(N_f(u)) - \tfrac{B_{\varphi,b}(u'(0))}{T} = 0, \; u'(0) = u(0)$$

$$\Leftrightarrow \varphi(u') = H\left(N_f(u) - Q(N_f(u))\right) + \tfrac{IB_{\varphi,b}(u'(0))}{T} + \varphi(u'(0)), \quad Q(N_f(u)) - \tfrac{B_{\varphi,b}(u'(0))}{T} = 0, \; u'(0) = u(0)$$

$$\Leftrightarrow u' = \varphi^{-1}\left[H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(u'(0))}{T} + \varphi(u'(0))\right], \quad Q(N_f(u)) - \tfrac{B_{\varphi,b}(u'(0))}{T} = 0, \; u'(0) = u(0)$$

$$\Leftrightarrow u = H\left(\varphi^{-1}\left[H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(u'(0))}{T} + \varphi(u'(0))\right]\right) + u(0), \quad Q(N_f(u)) - \tfrac{B_{\varphi,b}(u'(0))}{T} = 0, \; u'(0) = u(0)$$

$$\Leftrightarrow u = H\left(\varphi^{-1}\left[H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(u(0))}{T} + \varphi(u(0))\right]\right) + u(0),$$
$$Q(N_f(u)) - \tfrac{B_{\varphi,b}(u(0))}{T} = 0$$

$$\Leftrightarrow u = Q(N_f(u))$$
$$-\tfrac{B_{\varphi,b}(Pu)}{T} + H\left(\varphi^{-1}\left[H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(Pu)}{T} + \varphi(Pu)\right]\right) + Pu.$$ □

Remark 3.3. *Note that if u is a fixed point of M_1, we have the following equivalence: $u'(T) = bu'(0) \Leftrightarrow Q(N_f(u)) = \frac{B_{\varphi,b}(u'(0))}{T}$.*

In order to apply Leray-Schauder degree to the operator M_1, we introduce a family of problems depending on a parameter λ. We remember that for each continuous function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we associate its Nemytskii operator $N_f : C^1 \to C$ defined by

$$N_f(u)(t) = f(t, u(t), u'(t)).$$

For $\lambda \in [0,1]$, we consider the family of boundary value problems

$$\begin{cases} (\varphi(u'))' = \lambda N_f(u) + (1-\lambda)Q(N_f(u)) \\ u'(0) = u(0), \ u'(T) = bu'(0). \end{cases} \tag{3.2}$$

Notice that (3.2) coincides with (1.1) for $\lambda = 1$. So, for each $\lambda \in [0,1]$, the operator associated to (3.2) by Lemma 3.2 is the operator $M(\lambda, \cdot)$, where M is defined on $[0,1] \times C^1$ by

$$M(\lambda, u) = Q(N_f(u)) - \tfrac{B_{\varphi,b}(Pu)}{T} + H\left(\varphi^{-1}\left[\varphi(Pu) + \lambda H(N_f(u) - Q(N_f(u))) + \tfrac{IB_{\varphi,b}(Pu)}{T}\right]\right) + P(u).$$

Using the same arguments as in the proof of Lemma 3.1, we show that the operator M is completely continuous. Moreover, using the same ideas as above, the system (3.2) (see Lemma 3.2) is equivalent to the problem

$$u = M(\lambda, u). \tag{3.3}$$

In order to prove the existence of at least one solution of (1.1), we introduce the family of problems

$$\begin{cases} (\varphi(u'))' = \lambda Q(N_f(u)) \\ \int_0^T f(t, u(t), u'(t))dt = \varphi(bu(0)) - \varphi(u(0)), \ u'(0) = u(0). \end{cases} \tag{3.4}$$

We also introduce the homotopy $Z : [0,1] \times C^1 \to C^1$ defined by

$$Z(\lambda, u) = P(u) + Q(N_f(u)) - \frac{B_{\varphi,b}(Pu)}{T} + H\left(\varphi^{-1}\left[\lambda \frac{IB_{\varphi,b}(Pu)}{T} + \varphi(Pu)\right]\right),$$

where $Z(1, \cdot) = M(0, \cdot)$. By the same argument as above, the operator Z (see Lemma 3.1) is completely continuous.

Lemma 3.4. *If $(\lambda, u) \in [0,1] \times C^1$ is such that $u = Z(\lambda, u)$, then u is a solution of (3.4).*

Proof. Let $(\lambda, u) \in [0,1] \times C^1$ be such that $u = Z(\lambda, u)$. It follows that

$$\int_0^T f(t, u(t), u'(t))dt = \varphi(bu(0)) - \varphi(u(0))$$

and

$$u'(t) = \varphi^{-1}\left[t\lambda \frac{\varphi(bu(0)) - \varphi(u(0))}{T} + \varphi(u(0))\right] \tag{3.5}$$

for all $t \in [0,T]$. Applying φ to both members and differentiating, we deduce that

$$(\varphi(u'(t)))' = \lambda \tfrac{\varphi(bu(0)) - \varphi(u(0))}{T} = \lambda Q(N_f(u))$$

for all $t \in [0,T]$.

On the other hand, using (3.5) for $t = 0$, we obtain $u'(0) = u(0)$. This completes the proof. □

4. Main results

In this section, we present and prove our main results. These results are inspired on works by Bereanu and Mawhin [1] and Manásevich and Mawhin [6]. We denote by deg_B the Brouwer degree and for deg_{LS} the Leray-Schauder degree, and define the mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto \left(\frac{B_{\varphi,b}(x)}{T} - \frac{1}{T} \int_0^T f(t, x + yt, y)dt, -x + y \right). \tag{4.1}$$

Theorem 4.1. *Assume that Ω is an open bounded set in C^1 such that the following conditions hold.*

1. *If $(\lambda, u) \in [0, 1] \times C^1$ is such that $u = Z(\lambda, u)$, then $u \notin \partial\Omega$.*
2. *The Brouwer degree*

$$deg_B(G, \Omega \cap \mathbb{R}^2, 0) \neq 0,$$

where we consider the natural identification $(x, y) \approx x + yt$ of $\mathbb{R}^2$ with related functions in C^1.

3. *For each $\lambda \in (0, 1]$ the problem (3.2) has no solution on $\partial\Omega$.*

Then (1.1) has a solution.

Proof. Using hypothesis 1 and that Z is completely continuous, we deduce that for each $\lambda \in [0, 1]$, the Leray-Schauder degree $deg_{LS}(\mathbf{I} - Z(\lambda, \cdot), \Omega, 0)$ is well-defined. The homotopy invariance implies that

$$deg_{LS}(\mathbf{I} - Z(1, \cdot), \Omega, 0) = deg_{LS}(\mathbf{I} - Z(0, \cdot), \Omega, 0).$$

On the other hand, we have

$$deg_{LS}(\mathbf{I} - Z(0, \cdot), \Omega, 0) = deg_{LS}(\mathbf{I} - (P + QN_f - \tfrac{B_{\varphi,b}P}{T} + HP), \Omega, 0).$$

But the range of the mapping

$$u \longrightarrow P(u) + Q(N_f(u)) - \tfrac{B_{\varphi,b}(P(u))}{T} + H(P(u))$$

is contained in the subspace of related functions, isomorphic to $\mathbb{R}^2$. Thus, using a reduction property of Leray-Schauder degree [4,7]

$$\begin{aligned} &deg_{LS}(\mathbf{I} - (P + QN_f - \frac{B_{\varphi,b}P}{T} + HP), \Omega, 0) \\ &= deg_B \left(\mathbf{I} - (P + QN_f - \frac{B_{\varphi,b}P}{T} + HP) \left|_{\overline{\Omega \cap \mathbb{R}^2}} \right., \Omega \cap \mathbb{R}^2, 0 \right) \\ &= deg_B(G, \Omega \cap \mathbb{R}^2, 0) \neq 0. \end{aligned}$$

On the other hand, using the fact that M is completely continuous, that $Z(1, \cdot)$ coincides with the operator $M(0, \cdot)$ and by hypothesis 3, we deduce that for each $\lambda \in [0, 1]$, $deg_{LS}(\mathbf{I} - M(\lambda, \cdot), \Omega, 0)$ is well-defined and by the homotopy invariance, we have

$$deg_{LS}(\mathbf{I} - M(1, \cdot), \Omega, 0) = deg_{LS}(\mathbf{I} - M(0, \cdot), \Omega, 0).$$

Hence, $deg_{LS}(\mathbf{I} - M(1, \cdot), \Omega, 0) \neq 0$. This, in turn, implies that there exists $u \in \Omega$ such that $M_1(u) = u$, which is a solution for (1.1). □

The problem (1.1) can be studied by requiring some special conditions on $f(t, x, y)$.

Theorem 4.2. *Assume that the following conditions hold.*

1. *There exists a function $h \in C$ such that*

$$|f(t,x,y)| \le h(t) \;\; for \;\; all \;\; (t,x,y) \in [0,T] \times \mathbb{R}^2.$$

2. *There exists* $M_1 < M_2$ *such that for all* $u \in C^1$,

$$\int_0^T f(t,u(t),u'(t))dt - B_{\varphi,b}(u'(0)) \neq 0 \;\; if \;\; u'_m \ge M_2,$$

$$\int_0^T f(t,u(t),u'(t))dt - B_{\varphi,b}(u'(0)) \neq 0 \;\; if \;\; u'_M \le M_1.$$

3. *The Brouwer degree*

$$deg_B(G, B_\rho(0) \cap \mathbb{R}^2, 0) \neq 0,$$

where $\rho \ge max\{R_1, R_2\}$ *with* $R_1 = r_1(2+T)$ *and* $R_2 = r_2(2+T)$ *where*

$$r_1 = max\{\left|\varphi^{-1}(L + 2\|h\|_{L^1})\right|, \;\; \left|\varphi^{-1}(-L - 2\|h\|_{L^1})\right|\}$$

and

$$r_2 = max\{\left|\varphi^{-1}(L + \|h\|_{L^1})\right|, \;\; \left|\varphi^{-1}(-L - \|h\|_{L^1})\right|\},$$

for $L = max\{|\varphi(M_2)|, \;\; |\varphi(M_1)|\}$.

Then problem (1.1) has at least one solution.

Proof. Let $(\lambda, u) \in [0,1] \times C^1$ be such that u is a solution of (3.2). Using (3.3), we have that

$$u = M(\lambda,u) = Q(N_f(u)) - \frac{B_{\varphi,b}(Pu)}{T} + H\left(\varphi^{-1}\left[\varphi(Pu) + \lambda H(N_f(u) - Q(N_f(u))) + \frac{IB_{\varphi,b}(Pu)}{T}\right]\right) + P(u).$$

By evaluation of u at 0, we obtain

$$\int_0^T f(t,u(t),u'(t))dt - B_{\varphi,b}(u(0)) = 0.$$

Differentiating u and using the fact that $u'(0) = u(0)$, we deduce that

$$\int_0^T f(t,u(t),u'(t))dt - B_{\varphi,b}(u'(0)) = 0.$$

Now by hypothesis 2 it follows that

$$u'_m < M_2 \;\; \text{and} \;\; u'_M > M_1.$$

Then, there exists $\omega \in [0,T]$ such that $M_1 < u'(\omega) < M_2$. Moreover,

$$\int_\omega^t (\varphi(u'(s)))'ds = \lambda \int_\omega^t N_f(u)(s)ds + (1-\lambda)\int_\omega^t Q(N_f(u))(s)ds$$

for all $t \in [0,T]$. By hypothesis 1, it follows that

$$|\varphi(u'(t))| \le |\varphi(u'(\omega))| + 2\|h\|_{L^1} < L + 2\|h\|_{L^1},$$

where $L = \max\{|\varphi(M_2)|, |\varphi(M_1)|\}$. Hence,

$$\|u'\|_\infty < r_1,$$

where $r_1 = \max\{\left|\varphi^{-1}(L + 2\|h\|_{L^1})\right|, \left|\varphi^{-1}(-L - 2\|h\|_{L^1})\right|\}$. Using the fact that $u'(0) = u(0)$, we obtain

$$|u(t)| \le |u(0)| + \int_0^T |u'(s)|\, dt < r_1 + r_1 T \;\; (t \in [0,T]),$$

and hence

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty < r_1 + r_1 T + r_1 = r_1(2+T) = R_1.$$

Let $(\lambda, u) \in [0,1] \times C^1$ be such that $u = Z(\lambda, u)$. Using Lemma 3.4, u is a solution of (3.4), which implies that

$$\int_0^T f(t,u(t),u'(t))dt - B_{\varphi,b}(u'(0)) = 0.$$

Using hypothesis 2, it follows that there exists $\tau \in [0,T]$ such that $M_1 < u'(\tau) < M_2$. Moreover,

$$|\varphi(u'(t))| \leq |\varphi(u'(\tau))| + \left|\lambda \int_\tau^t Q(N_f(u))(s)ds\right|$$

for all $t \in [0,T]$. Now by hypothesis 1, it follows that

$$|\varphi(u'(t))| < L + \|h\|_{L^1}.$$

Hence,

$$\|u'\|_\infty < r_2,$$

where $r_2 = \max\{|\varphi^{-1}(L+\|h\|_{L^1})|, |\varphi^{-1}(-L-\|h\|_{L^1})|\}$. Now for $t \in [0,T]$

$$|u(t)| \leq |u(0)| + \int_0^T |u'(s)|\, dt < r_2 + r_2 T,$$

and hence

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty < r_2 + r_2T + r_2 = r_2(2+T) = R_2.$$

Defining $\Omega = B\rho(0)$ in Theorem 4.1, where $B\rho(0)$ is the open ball in C^1 center 0 and radius $\rho \geq \max\{R_1, R_2\}$, we can guarantee the existence of at least a solution of (1.1). □

In the next lemma, we adapt the ideas of Ward [8] to obtain the required a priori bounds.

Lemma 4.3. *Assume that f satisfies the following conditions.*

1. *There exists $c \in C$ such that*

$$f(t,x,y) \geq c(t)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

2. *There exists $M_1 < M_2$ such that for all $u \in C^1$,*

$$\int_0^T f(t,u(t),u'(t))dt \neq 0 \quad \text{if} \quad u'_m \geq M_2,$$

$$\int_0^T f(t,u(t),u'(t))dt \neq 0 \quad \text{if} \quad u'_M \leq M_1.$$

If $b = 1$ and $(\lambda, u) \in [0,1] \times C^1$ is such that $u = M(\lambda, u)$, then

$$\|u'\|_\infty < r,$$

where

$$r = \max\left\{\left|\varphi^{-1}\left(L + 2\left\|c^-\right\|_{L^1}\right)\right|, \left|\varphi^{-1}\left(-L - 2\left\|c^-\right\|_{L^1}\right)\right|\right\},$$
$$L = \max\left\{|\varphi(M_2)|,\ |\varphi(M_1)|\right\}.$$

Proof. Use the same arguments as in the proof of Theorem 4.2 and the following inequality

$$|f(t,u(t),u'(t))| \leq f(t,u(t),u'(t)) + 2c^-(t) \text{ for all } t \in [0,T].$$

□

Now we can prove an existence theorem for (1.1).

Theorem 4.4. *Let f be continuous and satisfy conditions (1) and (2) of Lemma 4.3. Assume that the following conditions hold for some $\rho \geq r(2+T)$.*

1. *The equation*

$$G(x,y) = (0,0),$$

has no solution on $\partial B_\rho(0) \cap \mathbb{R}^2$*, where we consider the natural identification* $(x,y) \approx x + yt$ *of* $\mathbb{R}^2$ *with related functions in* C^1.

2. *The Brouwer degree*

$$deg_B(G, B_\rho(0) \cap \mathbb{R}^2, 0) \neq 0.$$

Then problem (1.1) with $b = 1$ *has a solution.*

Proof. If $b = 1$ and $(\lambda, u) \in [0,1] \times C^1$ is such that $u = Z(\lambda, u)$, by evaluation of u at 0, we have that

$$\int_0^T f(t, u(t), u'(t))dt = 0. \tag{4.2}$$

Moreover, u is a function of the form $u(t) = x + yt,\ y = x$. Thus, by (4.2)

$$\textstyle\int_0^T f(t, x + yt, y)dt = 0,$$

which, together with hypothesis 1, implies that $u = x + tx \notin \partial B_\rho(0)$.

Let $b = 1$ and $(\lambda, u) \in [0,1] \times C^1$ be such that $u = M(\lambda, u)$. Using Lemma 4.3, we have that $\|u\|_1 < r(2+T)$. Thus we have proved that (3.2) has no solution in $\partial B_\rho(0)$ for $b = 1$ and $(\lambda, u) \in [0,1] \times C^1$, hence the conditions of Theorem 4.1 are satisfied, the proof is complete. □

Our next theorem is a generalization of Theorem 4.2 with $b < 0$. We need first define the following operators. The *differential operator*

$$D : \mathrm{dom}(D) \to C, \quad u \mapsto u',$$

where

$$\mathrm{dom}(D) = \left\{u \in C_b^1 : \varphi(u') \in C^1\right\},$$

and

$$C_b^1 = \left\{u \in C^1 : u'(T) = bu'(0), \quad u'(0) = u(0)\right\}.$$

The operator

$$D_\varphi : \mathrm{dom}(D_\varphi) \to C, \quad u \mapsto (\varphi(u))',$$

where $\mathrm{dom}(D_\varphi) = \left\{u \in C : \varphi(u) \in C^1\right\}$.

The operator

$$\widetilde{D}_\varphi = D_\varphi D : \mathrm{dom}(D) \to C, \quad u \mapsto (\varphi(u'))'.$$

When $b < 0$, $-\varphi(\cdot)$ and $\varphi(b\ \cdot)$ are simultaneously increasing or decreasing. In this case, $B_{\varphi,b}(\cdot) = \varphi(b\ \cdot) - \varphi(\cdot)$ is injective. Thus, the operator $\widetilde{D}_\varphi$ has an inverse given by

$$u \mapsto H\left(\varphi^{-1}\left[\varphi\left(B_{\varphi,b}^{-1}\left(\textstyle\int_0^T u(s)ds\right)\right) + \int_0^t u(s)ds\right]\right) + B_{\varphi,b}^{-1}\left(\textstyle\int_0^T u(s)ds\right).$$

Hence,

$$\begin{aligned} &(\varphi(u'))' = N_f(u),\ u'(T) = bu'(0),\ u'(T) = bu'(0) \\ &\Leftrightarrow (D_\varphi D)(u) = N_f(u),\ u \in \mathrm{dom}(D) \\ &\Leftrightarrow u = (D_\varphi D)^{-1} N_f(u),\ u \in C^1. \end{aligned}$$

Hence our problem is finding a fixed point of the operator

$$\Gamma := (D_\varphi D)^{-1} N_f : C^1 \to \mathrm{dom}(D).$$
$$u \mapsto H\left(\varphi^{-1}\left[\varphi\left(B_{\varphi,b}^{-1}\left(\int_0^T N_f(u)(s)ds\right)\right) + \int_0^t N_f(u)(s)ds\right]\right) + B_{\varphi,b}^{-1}\left(\int_0^T N_f(u)(s)ds\right).$$

In the next theorem, we adapt the ideas of Bouches and Mawhin [2] to obtain the existence of at least one solution of (1.1).

Theorem 4.5. *If there exists a function $h \in C$ such that*

$$|f(t,x,y)| \le h(t)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R}^2$, then problem (1.1) with $b < 0$ has a solution.

Proof. Let us consider $v = \Gamma(u) := (D_\varphi D)^{-1} N_f(u)$. Then,

$$v'(T) = bv'(0),\ \ v'(0) = v(0)$$

and

$$N_f(u) = (D_\varphi D)(v) = (\varphi(v'))'.$$

Because $v \in C^1$ is such that $v'(T) = bv'(0)$, there exists $\tau \in [0,T]$ such that $v'(\tau) = 0$, which implies $\varphi(v'(\tau)) = 0$ and

$$\begin{aligned}
&|\varphi(v'(t))| \le \left|\int_\tau^t (\varphi(v'(s)))' ds\right| \\
&\le \int_\tau^t |N_f(u)(s)|\, ds \le \int_0^T |f(s,u(s),u'(s))|\, ds \\
&\le \|h\|_{L^1} \quad (t \in [0,T]),
\end{aligned}$$

which implies

$$\|v'\|_\infty \le \beta,$$

where $\beta = \max\left\{\left|\varphi^{-1}\left(\|h\|_{L^1}\right)\right|,\ \left|\varphi^{-1}\left(-\|h\|_{L^1}\right)\right|\right\}$. Using the fact that $v'(0) = v(0)$, we deduce that

$$|v(t)| \le |v(0)| + \int_0^t |v'(s)|\, ds \le |v(0)| + \int_0^T |v'(s)|\, ds \le \beta + \beta T$$

for all $t \in [0,T]$, and hence

$$\|v\|_1 = \|v\|_\infty + \|v'\|_\infty \le \beta + \beta T + \beta = \beta(2+T).$$

Because the Γ is completely continuous and bounded, we can use Schauder's Fixed Point Theorem to deduce the existence of at least one fixed point in $\overline{B_{\beta(2+T)}(0)}$. The proof is complete. □

5. Examples

In order to illustrate the above results, we consider some examples.

Example 5.1. *Let us consider the problem*

$$\begin{cases} \left((u')^3\right)' = \frac{e^{u'}}{2} - 1 \\ u(0) = u'(0) = u'(T). \end{cases} \tag{5.1}$$

Let $M_1 = -1$ and $M_2 = 1$. If we suppose that $u'_m \ge M_2$ or $u'_M \le M_1$, then

$$\int_0^T \left(\frac{e^{u'(t)}}{2} - 1\right)dt \ge \left(\frac{e^{M_2}}{2} - 1\right)T > 0, \qquad \int_0^T \left(\frac{e^{u'(t)}}{2} - 1\right)dt \le \left(\frac{e^{M_1}}{2} - 1\right)T < 0.$$

On the other hand, if we choose $\rho \geq (1+2T)^{1/3}(2+T)$ *and* $c(t) = -1$ *for all* $t \in [0, T]$*, we have that the equation*

$$\begin{aligned} G(a,b) &= \left(aT + bT^2 - bT - \frac{1}{T}\int_0^T f(t, a+bt, b)dt, b - a - bT\right) = (0,0) \\ &= \left(aT + bT^2 - bT - \frac{1}{T}\int_0^T (\frac{e^b}{2} - 1)dt, b - a - bT\right) = (0,0) \\ &= \left(aT + bT^2 - bT - \frac{e^b}{2} + 1, b - a - bT\right) = (0,0) \end{aligned}$$

has no solution on $\partial B_\rho(0) \cap \mathbb{R}^2$*. Then we have that the Brouwer degree*

$$deg_B(G, B_\rho(0) \cap \mathbb{R}^2, (0,0))$$

is well defined and, by the properties of that degree, we have

$$deg_B(G, B_\rho(0) \cap \mathbb{R}^2, (0,0)) = \sum_{x \in G^{-1}(0,0)} sgn J_G(x) \neq 0,$$

where $(0,0)$ *is a regular value of* G *and* $J_G(x) = det G'(x)$ *is the Jacobian of* G *at* x*. So, using Theorem 4.4, we obtain that the boundary value problem (5.1) has at least one solution.*

Example 5.2. *We consider the following boundary value problem*

$$\begin{cases} \left(|u'|^{p-2}\, u'\right)' = \frac{e^{-u^2}}{2} + t^2 \cos u' + 2 \\ u(0) = u'(0), \;\; u'(T) = bu'(0), \end{cases} \tag{5.2}$$

where $p \in (1, \infty)$*,* T *is a positive real number, and* $b < 0$*. Then, by Theorem 4.5, we obtain that (5.2) has at least one solution.*

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