On the Derivative of a Polynomial

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Abstract: Let \( P(z) = c_n z^n + \sum_{\nu=\mu}^{n-\mu} c_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu < n \), be a polynomial of degree at most \( n \) having no zeros in \( |z| < k, \ k \leq 1 \), and \( Q(z) = z^n P(1/z) \), it is proved by Dewan et al. [5] that if \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}.
\]

In this paper, we generalize the above inequality for the polynomials of type \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, \quad 1 \leq \mu \leq n \).

Key Words: Polynomial, Inequality, Maximum modulus, Restricted zeros.

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1. Introduction and statement of results

Let \( P(z) \) be a polynomial of degree \( n \), then according to the well known Bernstein's inequality on the derivative of a polynomial, we have
\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}
\]
The result is best possible and equality holds for the polynomials having all its zeros at the origin.

For polynomials having no zeros in \( |z| < 1 \), Erdös conjectured and later Lax [8] proved that if \( P(z) \neq 0 \) in \( |z| < 1 \), then (1.1) can be replaced by
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}
\]

With equality for those polynomials, which have all their zeros on \( |z| = 1 \).

In the literature, there already exists various refinements and generalizations of
(1.2), for example (see Aziz [1], Bidkham et.al [2,3,4], Khojastehnezhad and Bidkham [7], Zireh [14] etc). As an extension of (1.2) Malik [12] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)|. \quad (1.3)$$

Further Govil [9] proved that for the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ which has no zeros in $|z| < k$, $k \leq 1$, if $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |P(z)|. \quad (1.4)$$

Whereas the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ having all its zeros on $|z| = k$, $k \leq 1$, Govil [10] proved

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)|. \quad (1.5)$$

Recently Dewan and Hans [5] obtained a generalization of (1.4) and proved for $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{\nu} z^{\nu}$, $1 \leq \mu < n$ that having no zeros in $|z| < k$, $k \leq 1$, if $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}. \quad (1.6)$$

For $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{\nu} z^{\nu}$, $1 \leq \mu < n$ that having all its zeros on $|z| = k$, $k \leq 1$, Dewan [5] also proved

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^n} \max_{|z|=1} |P(z)|. \quad (1.7)$$

In this paper, first we obtain the following result

**Theorem 1.1.** Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}$, $1 \leq \mu \leq n$ is a polynomial of degree $n$, having no zeros in $|z| < k$, $k \leq 1$ and $Q(z) = z^n P(1/z)$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^n + \mu - 1} \max_{|z|=1} |P(z)|. \quad (1.8)$$

**Remark 1.2.** If we take $\mu = 1$ in Theorem 1.1, then inequality (1.8) reduces to inequality (1.4) due to Govil.

Next we prove the following interesting result which is a refinement of inequality (1.8).

**Theorem 1.3.** Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}$, $1 \leq \mu \leq n$ is a polynomial of degree $n$, having no zeros in $|z| < k$, $k \leq 1$ and $Q(z) = z^n P(1/z)$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}. \quad (1.9)$$
Remark 1.4. If we take $\mu = 1$ in Theorem 1.3, then inequality (1.9) reduces to the following result which proved by Aziz and Ahmad [1].

Corollary 1.5. Let $P(z)$ is a polynomial of degree $n$, having no zeros in $|z| < k$, $k \leq 1$ and $Q(z) = z^n P(1/z)$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}. \quad (1.10)$$

Finally we prove the following result.

Theorem 1.6. Let $P(z) = a_0 + \sum_{\nu=1}^n a_{\nu}z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree $n$, having all its zeros on $|z| = k$, $k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n+\mu-1} + k^{n+\mu-2}} \max_{|z|=1} |P(z)|. \quad (1.11)$$

Remark 1.7. If we take $\mu = 1$ in Theorem 1.6, then inequality (1.11) reduces to inequality (1.5) due to Govil.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. [13] Let $P(z)$ be a polynomial of degree $n$, then for $R \geq 1$

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \quad (2.1)$$

Lemma 2.2. Let $P(z) = c_n z^n + \sum_{\nu=1}^n c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ be a polynomial of degree $n$, having all its zeros in $|z| \leq k$, $k \geq 1$, then for $|z| = 1$

$$k^{n-\mu+1} |Q'(z)| \leq |P'(k^2 z)|, \quad (2.2)$$

where $Q(z) = z^n P(1/z)$.

Proof: Let $F(z) = P(kz)$, then $F(z)$ has all its zeros in $|z| \leq 1$. If $G(z) = z^n F(1/z) = z^n P(k/z) = k^n Q(z/k)$, then all the zeros of $G(z)$ lie in $|z| \geq 1$. Since $|F(z)| = |G(z)|$ on $|z| = 1$, we can say that an application of maximum modulus principle to the function $G(z)$ will yield $|G(z)| \leq |F(z)|$, $|z| \geq 1$. Therefore the polynomial $G(z) - \lambda F(z)$, will not vanish in $|z| > 1$ for every $\lambda$ with $|\lambda| > 1$. Gauss-Lucas theorem will then imply that polynomial $G'(z) - \lambda F'(z)$ will not vanish in $|z| > 1$ for every $\lambda$ with $|\lambda| > 1$ and therefore $|G'(z)| \leq |F'(z)|$, $|z| \geq 1$. Substituting for $F'(z)$ and $G'(z)$, we get

$$k^{n-1} |Q'(z/k)| \leq k |P'(kz)|, \quad (2.3)$$
where $|z| \geq 1$. 

Since $Q(z) = \sum_{\nu=\mu} c_n z^n$, then

$$k^{n-\mu} |\sum_{\nu=\mu}^{n} \nu \sigma_{n-\nu} \left(\frac{z}{k}\right)^{\nu-\mu}| \leq k |P'(kz)|,$$

i.e.,

$$k^{n-\mu} |\sum_{\nu=\mu}^{n} \nu \sigma_{n-\nu} \left(\frac{z}{k}\right)^{\nu-\mu}| \leq k |P'(k^2z)|,$$

(2.4)

where $|z| \geq 1$.

If we take $kz$ instead of $z$ in inequality (2.4), then we have

$$k^{n-\mu} |\sum_{\nu=\mu}^{n} \nu \sigma_{n-\nu} z^{\nu-\mu}| \leq k |P'(k^2z)|,$$

(2.5)

where $|z| \geq 1/k$.

Since $1/k \leq 1$, we have in particular,

$$k^{n-\mu} |\sum_{\nu=\mu}^{n} \nu \sigma_{n-\nu} z^{\nu-\mu}| \leq k |P'(k^2z)|,$$

(2.6)

where $|z| \geq 1$.

This implies

$$k^{n-\mu} |\sum_{\nu=\mu}^{n} \nu \sigma_{n-\nu} z^{\nu-1}| \leq k |P'(k^2z)|,$$

(2.7)

where $|z| = 1$.

This completes the proof of Lemma 2.2. 

\[\Box\]

**Lemma 2.3.** Let $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ be a polynomial of degree $n$, having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |Q'(z)| \leq k^{n+\mu-1} \max_{|z|=1} |P'(z)|,$$

(2.8)

where $Q(z) = z^n P(1/z)$.

**Proof:** On applying Lemma 2.2 we have

$$k^{n-\mu} |Q'(z)| \leq |P'(k^2z)|.$$  

(2.9)

Now using Lemma 2.1 for the polynomial $P'(k^2z)$, of degree $n - 1$. We have

$$\max_{|z|=k^2} |P'(z)| \leq k^{2n-2} \max_{|z|=1} |P'(z)|.$$  

(2.10)

Combining (2.9) and (2.10), we have desired result. 

\[\Box\]
Lemma 2.4. Let \( P(z) = a_0 + \sum_{\mu=\mu}^{n} a_\mu z^\mu \), \( 1 \leq \mu \leq n \) be a polynomial of degree \( n \), has no zeros in \( |z| < k \), \( k \leq 1 \), then

\[
k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|,
\]

where \( Q(z) = z^n P(1/z) \).

Proof: Since \( P(z) \) has no zeros in \( |z| < k \), then \( Q(z) = z^n P(1/z) \) has all its zeros in \( |z| \leq 1/k \), \( 1/k \geq 1 \). On applying Lemma 2.3 to the polynomial \( Q(z) \), we have

\[
k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|.
\]

The following lemma is due to Malik [6].

Lemma 2.5. Let \( P(z) \) be a polynomial of degree \( n \), has no zero in \( |z| < k \), \( k \geq 1 \), then for \( |z| = 1 \)

\[
k |P'(z)| \leq |Q'(z)|
\]

where \( Q(z) = z^n P(1/z) \).

Lemma 2.6. Let \( P(z) \) be a polynomial of degree \( n \), having all its zeros on \( |z| = k \), \( k \leq 1 \), then for \( |z| = 1 \)

\[
|Q'(z)| \leq k |P'(z)|
\]

where \( Q(z) = z^n P(1/z) \).

Proof: Since \( P(z) \) has all its zeros on \( |z| = k \), then \( Q(z) = z^n P(1/z) \) has all its zeros in \( |z| = 1/k \), \( 1/k \geq 1 \). On applying Lemma 2.5 to the polynomial \( Q(z) \), we have

\[
1/k |Q'(z)| \leq |P'(z)|.
\]

The following lemma is a special case of a result due to Govil and Rahman [11].

Lemma 2.7. Let \( P(z) \) be a polynomial of degree \( n \), then for \( |z| = 1 \)

\[
|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|
\]

where \( Q(z) = z^n P(1/z) \).
3. Proofs of the theorems

Proof of Theorem 1.1. Since $|P'(z)|$ and $|Q'(z)|$ attained maximum at the same point on $|z| = 1$. This implies there exist a point $z_0$ such that $|P'(z_0)| = \max_{|z|=1} |P'(z)| = \max_{|z|=1} |Q'(z)| = |Q'(z_0)|$. On the other hand by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$

On applying Lemma 2.4, we have

$$|P'(z_0)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

This implies

$$\max_{|z|=1} |P'(z)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 1.1. \qed

Proof of Theorem 1.3. Let $m = \min_{|z|=k} |P(z)|$. For $\alpha$ with $|\alpha| < 1$, we have $|\alpha m| < m \leq |P(z)|$, where $|z| = k$.

Therefore by implying Rouche’s theorem, the polynomial $G(z) = P(z) - \alpha m$ has no zeros in $|z| < k$. On applying Theorem 1.1 to the polynomial $G(z)$, we have

$$\max_{|z|=1} |G'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |G(z)|,$$

i.e.,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |P(z) - \alpha m|.$$

If we choose a point $z_0$ on $|z| = 1$ such that $\max_{|z|=1} |P(z)| = |P(z_0)|$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} |P(z_0) - \alpha m|.$$

Now by suitable choice of argument of $\alpha$, we get

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} (|P(z_0)| - |\alpha m|).$$

By making $|\alpha| \to 1$, the result follows. \qed

Proof of Theorem 1.6. If $z_0$ is a point on $|z| = 1$ such that $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$. Then by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$

On applying Lemma 2.6, we have

$$\frac{1}{k} |Q'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$
i.e,
\[
\left(\frac{1}{k} + 1\right) \max_{|z|=1} |Q'(z)| \leq n \max_{|z|=1} |P(z)|.
\]
Now applying Lemma 2.4, we have
\[
\left(\frac{1}{k} + 1\right)^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.
\]
This completes the proof of Theorem 1.6. \(\square\)

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