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Existence Results on Nonlinear Fractional Differential Inclusions

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abstract: In this paper, we study the existence of solution for a boundary value problem of nonlinear fractional differential inclusion of order $\alpha \in (0,1)$ with initial boundary value problems (BVP for short) and the standard Riemann-Liouville fractional derivative.

Our approach is based on the topological transversality method in fixed point theory. we use a powerful method due to Granas to prove the existence of solution to BVP. Granas method is commonly as topological transversality and relies on the idea of an essential map. The method has been highly useful proving existence of solutions for initial and boundary value problem for integer order differential equations.

Key Words: Riemann-Liouville derivative, Fixed point, Fractional differential inclusion.

Contents

1. Introduction

In this paper, we study the existence of solution for a boundary value problem of nonlinear fractional differential inclusion of order $\alpha \in (0,1)$ with initial boundary value problems (BVP for short) given by

$$D^{\alpha}u(t) + D^{\beta}u(t) \in F(t,u(t)), t \in J = [0,1] \qquad (1.1)$$

$$u(0) = 0 \qquad (1.2)$$

where $0 < \beta < \alpha < 1$ and D^{α}, D^{β} is the standard Riemann-Liouville fractional derivative, $F : J \times R^n \to P(R^n)$ is a Caratheodory multifunction, Here $P(R^n)$ denotes the family of all nonempty subsets of R^n.
Fractional Differential inclusions have gained considerable importance due to their application in various science, such as physics, mechanics, chemistry, engineering, control, etc.(see [6]- [10]and [17]).
Recently, there has been a significant development in the study of ordinary and partial differential equations and inclusions involving fractional derivatives, see the

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monographs of Kilbas [15], Lakshmikantham [11], Miller and Ross [12], Podlubny [13], Samko [14], Rezapour [1] , Ouahab [16], .El-Sayed and Ibrahim [15] initiated the study of fractional multivalued differential inclusions and existence results for fractional boundary value problem and relaxation theorem, where studied by Ouahab [16].
In this paper we use a powerful method due to Granas [2] to prove the existence of solution to BVP (1.1)-(1.2). Granas method is commonly as topological transversality and relies on the idea of an essential map. The method has been highly useful proving existence of solutions for initial and boundary value problem for integer order differential equations. see for example [3], [4], [18].
This paper is organized as follows: in Section 2 we introduce some backgrounds on fractional calculus and the topological transversality theorem. In Section 3 we present our main results.

2. The Preliminary

For the convenience of the reader, we present the necessary definitions from fractional calculus theory. These definitions can be found in the recent literaturs and books(see [5]).

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

provided the right side is pointwise defind on $(0, \infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds.$$

where $n = [\alpha] + 1$, provided the right side is pointwise defind on $(0, \infty)$.

We denote by $\|y\|$ the norm of any element $y \in R^n$ and $C(J, R^n)$ is the Banach space of all continuous functions from J into R^n with the usual norm

$$\|y\|_{\infty} = \sup\{\|y(t)\| : 0 \leq t \leq 1\}.$$

$AC(J, R^n)$ is the space of absolutely continuous functions $y : J \to R^n$ and $L^1(J, R^n)$ denote the Banach space of functions $y : J \to R^n$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^1 \|y(t)\| dt.$$

Let X, Y be banach spaces. A set-valued map $F : X \to P(Y)$ is said to be compact if $F(X) = \sqcup\{F(y) : y \in X\}$ is compact. F has convex (closed, compact) values if $F(y)$ is convex (closed, compact) for every $y \in X$. F is bounded on bounded subsets of X if $F(B)$ is bounded in Y for every bounded subset B of X. A set-valued map F is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set V containing Fz_0, there exists a neighborhood U of z_0 such that $F(U) \subset V$. F is usc on X if it is usc at every point of X if F is nonempty and compact-valued then F is usc if and only if F has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $bcc(X)$. A set-valued map $F : J \to P(X)$ is measurable if for each $y \in X$, the function $t \mapsto dist(y, F(t))$ is measurable on J. If $X \subset Y$,F has a fixed point if there exists $y \in X$ such that $y \in Fy$. Also, $\|F(y)\|_p = \sup\{|x| : x \in F(y)\}$.

Definition 2.3. *A multivalued map $F : [0,1] \times R^n \to P(R^n)$ is said to be L^1-Caratheodory if*
(i) $t \mapsto F(t,x)$ is measurable for each $x \in R^n$,
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.
(iii) for each $\rho > 0$, there exists $\phi_\rho \in L^1([0,1], R^+)$ such that

$$\|F(t,x)\|_p = \sup\{\|v\| : v \in F(t,x) \leq \phi_\rho(t), \|x\| \leq \rho, a.e.t \in [0,1]\}.$$

For each $x \in C(J,R^n)$, define the set of selections of F by

$$S^1_{F,x} = \{v \in L^1(J,R^n) : v \in F(t,x(t))a.e.t \in J = [0,1]\}.$$

Note that for an L^1-Caratheodory multifunction $F : J \times R^n \to P(R^n)$ the set $S^1_{F,x}$ is not empty (see [19]).
Let C be a convex subset of X and U an open subset of C. $K_{\partial U}(\overline{U}, P(C))$ denotes the set of all set-valued maps $G : \overline{U} \to P(C)$ which are compact, usc with closed convex values and have no fixed points on ∂U (i.e., $\underline{u} \in Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0,1] \times \overline{U} \to P(C)$ which is compact, usc with closed convex values.
If $u \in H(\lambda, u)$ for every $\lambda \in [0,1]$, $\underline{u} \in \partial U$, H is said to be fixed point free on ∂U. Two set valued maps $F, G \in K_{\partial U}(\overline{U}, P(C))$ are called homotopic in $K_{\partial U}(\overline{U}, P(C))$ if there exists a compact homotopy $H : [0,1] \times \overline{U} \to P(C)$ which is fixed point free on ∂U and such that $H(0,.) = F$ and $\underline{H}(1,.) = G$. The function $G \in K_{\partial U}(\overline{U}, P(C))$ is called essential if every $F \in K_{\partial U}(\overline{U}, P(C))$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential.

Theorem 2.4. *[2]Let $G : \overline{U} \to P(C)$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential.*

Theorem 2.5. *(Topological transversality theorem) [2]. Let F, G be two homotopic maps in $K_{\partial U}(\overline{U}, P(C))$. Then F is essential if and only if G is essential.*

For further details of the Topological Transversality Theory we refer the reader to [18].

3. Main results

We are concerned with the existence of solutions for the problem (1.1),(1.2). Consider the following spaces

$$AC_B(J,R^n) = \{u \in AC(J,R^n) : u(0) = 0\}$$

$$AC^\alpha(J,R^n) = \{u \in AC_B(J,R^n) : \int_0^t \mid D^\alpha u(t) \mid ds < \infty\}$$

$$AC^{\alpha,\beta}(J,R^n) = \{u \in AC_B(J,R^n) : \int_0^t \mid D^\alpha u(t) + D^\beta u(t) \mid ds < \infty\}$$

$AC^{\alpha,\beta}(J,R^n)$ is a Banach space with the norm

$$\parallel u \parallel_{AC^{\alpha,\beta}} = \max\{\parallel u \parallel_\infty, \parallel D^\alpha u + D^\beta u \parallel_{L^1}\}$$

For the existence of solutions for the problem (1.1),(1.2) we have the following result which is useful in what follows.

Definition 3.1. *A function $u \in AC^{\alpha,\beta}(J,R^n)$ is said a solution to BVP (1.1),(1.2) if there exists a function $v \in L^1(J,R)$ with $v(t) \in F(t,u(t))$ for a.e $t \in J$, $0 < \beta < \alpha < 1$, and the function u satisfies condition (1.2).*

Let $h : J \to R^n$ be continuous, and consider the linear fractional order differential equation

$$D^\alpha u(t) + D^\beta u(t) = h(t), t \in J, 0 < \beta < \alpha < 1 \tag{3.1}$$

For the existence of solutions for the problem (1.1)-(1.2), we have the following result which is useful in what follows.

Lemma 3.2. *[21]Let $0 < \beta < \alpha < 1$ and let $h : J \to R^n$ be continuous. A function u is a solution of (3.1), if and only if*

$$u(t) = \int_0^t G(t-s)h(s)ds, \qquad t \in J = [0,1]$$

where

$$G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}). \tag{3.2}$$

Theorem 3.3. *Assume the following hypothesis hold:*
(A_1) The function $F : J \times R^n \to bcc(R^n)$ is a L^1-Caratheodory multivalued map,
(A_2) There exists a function $p \in L^1(J,R^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, such that for each $(t,u) \in J \times R^n$

$$\parallel F(t,u) \parallel_p \leq p(t)\psi(\parallel u \parallel)$$

(A_3) $\limsup_{r \mapsto \infty} \frac{r}{\psi(r)} = \infty$
Then, the fractional BVP(1.1)-(1.2) has a Least one solution on J.

This proof will be given in several steps;
Step 1.Consider the set valued operator $\gamma : C(J, R^n) \to P(L^1(J, R^n))$ defined by $(\gamma u)(t) = F(t, u(t))$, γ is well defined upper semicontinuous, with convex values and sends bounded subsets of $C(J, R^n)$ into bounded subsets of $L^1(J, R^n)$. In fact, we have

$$\gamma u := \{v : J \to R^n, measurable, v(t) \in F(t, u(t)), a.e, t \in J\}$$

Let $z \in C(J, R^n)$, and $v \in \gamma z$. Then

$$\|v(t)\| \leq p(t)\psi(\|z(t)\|) \leq p(t)\psi(\|z\|_\infty)$$

Hence, $\|v\|_{L^1} \leq K_0 := \|p\|_{L^1}\psi(\|z\|_\infty)$. This shows that γ is well defined. It is clear that γ is convex valued. Now, let B be a bounded subset of $C(J, R^n)$. Then, there exists $k > 0$ such that $\|u\|_\infty \leq k$ for $u \in B$.

So, for $w \in \gamma u$ we have $\|w\|_{L^1} \leq k_1$, where $k_1 = \|p\|_{L^1}\psi(k)$. Also, we can argue as in [20] to show that γ is usc.
Step 2. Priori bounds on solutions. We shall show that if u be a possible solution of (1.1)-(1.2), then there exists a positive constant R^*, independent of u, such that

$$\|u\|_{AC^{\alpha,\beta}} \leq R^*$$

Let u be a possible solution of (1.1)-(1.2), by lemma [3.2] there exists $v \in S^1_{F,u}$ such that, for each $t \in J$

$$u(t) = \int_0^t G(t-s)v(s)ds$$

where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $|G(t)| \leq |\frac{-t^\alpha}{1+(-t)^{\alpha-\beta}}| \leq t^\alpha$ where $|t| \leq 1$ Then, let $G_0 = \sup\{\|G(t-s)\| : t, s \in J \times J\}$ and $P_0 = \sup\{p(t) : t \in J\}$. Hence for $t \in J$,

$$\|u(t)\| \leq \int_0^t |G(t-s)v(s)|ds \leq \int_0^t \|G(t-s)\|\|v(s)\|ds$$

$$\leq G_0 \int_0^t \|v(s)\|ds \leq G_0 \int_0^t p(s)\psi(\|u(s)\|)ds$$

Since ψ is nondecreasing, we have

$$\|u\|_\infty \leq G_0P_0\psi(\|u\|_\infty)t \leq G_0P_0\psi(\|u\|_\infty)$$

Thus

$$\frac{\|u\|_\infty}{\psi(\|u\|_\infty)} \leq G_0P_0 = \widetilde{R}$$

So

$$\frac{\|u\|_\infty}{\psi(\|u\|_\infty)} \leq \widetilde{R} \qquad (3.5)$$

Now, the condition ψ in (A_3) shows that there exists $R_1^* > 0$ such that for all $R > R_1^*$

$$\frac{R}{\psi(R)} > \widetilde{R} \qquad (3.6)$$

Comparing these last two inequalities (3.5) and (3.6) we see that $R_0 \leq R_1^*$. Consequently, we obtain $\|u(t)\| \leq R_1^*$ for all $t \in J$.
Now from (1.1) and (A_2) we have

$$\int_0^t \|D^\alpha u(s) + D^\beta u(s)\| ds \leq \psi(R_1^*) \int_0^t p(s) ds := R_2^*$$

Hence

$$\|u\|_{AC^{\alpha,\beta}} \leq \max\{R_1^*, R_2^*\} := R^*$$

.

Step 3. Existence of solutions
For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$$D^\alpha u(t) + D^\beta u(t) \in \lambda F(t, u(t)), t \in J = [0,1], 0 < \beta < \alpha < 1 \qquad (1_\lambda)$$

$$u(0) = 0 \qquad (2_\lambda)$$

which reduces to (1.1)-(1.2) for $\lambda = 1$. For $0 \leq \lambda \leq 1$, we define the operator $\gamma_\lambda : C(J, R^n) \to P(L^1(J, R^n))$ by $(\gamma_\lambda u)(t) = \lambda F(t, u(t))$.
Step 1. Shows that γ_λ is usc, has convex values and sends bounded subsets of $C(J, R^n)$ into bounded subsets of $L^1(J, R^n)$ and if u is a solution of $(1_\lambda) - (2_\lambda)$ for some $\lambda \in [0,1]$, then $\|u\|_{AC^{\alpha,\beta}} \leq R^*$, where R^* does not depend on λ.
For $\lambda \in [0,1]$, we define the operators $\Phi : AC^{\alpha,\beta}(J, R^n) \to C(J, R^n)$ and $\Theta : AC^{\alpha,\beta}(J, R^n) \to L^1(J, R^n)$ by

$$(\Phi u)(t) = u(t), (\Theta u)(t) = D^\alpha u(t) + D^\beta u(t)$$

It is clear that Φ is continuous and completely continuous and Θ is linear, continuous and has a bounded inverse denoted by Θ^{-1}, let

$$V := \{u \in AC^{\alpha,\beta}(J, R^n); \|u\|_{AC^{\alpha,\beta}} < R^* + 1\}$$

Define a map $H : [0,1] \times \overline{V} \to AC^{\alpha,\beta}(J, R^n)$ by

$$H(\lambda, u) = (\Theta^{-1} o \gamma_\lambda o \Phi)(u)$$

We can show that the fixed points of $H(\lambda, .)$ are solutions of $(1_\lambda) - (2_\lambda)$. Moreover, H is a compact homotopy between $H(0, .) \equiv 0$ and $H(1, .)$. In fact, H is compact since Φ is completely continuous. γ_λ is continuous and Θ^{-1} is continuous. Since solutions of (1_λ) satisfy

$$\|u\|_{AC^{\alpha,\beta}} \leq R^*$$

we see that $H(\lambda, .)$ has no fixed points on ∂V. Now $H(0, .)$ is essential by Theorem [2.1] . Hence by Theorem [2.2] , $H(1, .)$ is essential. This implies that $\Theta^{-1} o \gamma_\lambda o \Phi$ has a fixed point which is a solution to problem (1.1)-(1..2).

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