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Renormalized Solutions for Some Nonlinear Nonhomogeneous Elliptic Problems with Neumann Boundary Conditions and Right Hand Side Measure

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ABSTRACT: Our aim in this paper is to study the existence of renormalized solution for a class of nonlinear $p(x)$-Laplace problems with Neumann nonhomogeneous boundary conditions and diffuse Radon measure data which does not charge the sets of zero $p(.)$-capacity.

Key Words: Renormalized solution, Nonlinear, Nonhomogeneous, Elliptic, Neumann condition, Measure.

Contents

1. Introduction

In this paper, we consider the following nonhomogeneous and nonlinear Neumann boundary value problem:

$$(P_\mu)\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u + \alpha(u)|\nabla u|^{p(x)} = \mu & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \eta} + \gamma(u) = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq I\!R^N (N \geq 3)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$, α, γ are real functions defined on $I\!R$ or $I\!R^N$, $g \in L^1(\partial\Omega)$ and μ is a diffuse measure such that $\mu = \mu\lfloor\Omega$. We note that in [1] the authors treated the problem (P_μ) where the right hand side $\mu = f \in L^1(\Omega)$.

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The operator $-\Delta_{p(x)}u$ is called $p(x)$-Laplacian which become p-Laplacian when $p(x) \equiv p$ is a constant. It possesses more complicated nonlinearities than the p-Laplacian. As the exponent which appear in (P_μ) depends on the variable x, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(.)}$ and $W^{1,p(.)}$. The study of PDEs with variable exponent has experienced a revival of interest over the past few years (see for example [10,13,25] and references therein). The interest of study problem involving variable exponent is due to the fact that they can model various phenomena which arise in the study of elastic mechanics, electrorheological fluids or image restauration (for more details see [10,13]).

In the present paper we use the framework of renormalized solutions. This notion was introduced by DiPerna and Lions in [12] for the first order equations and has been developed for elliptic problems with Dirichlet boundary conditions and with $L^1(\Omega)$ data in [17]. In [11] the authors gave a definition of a renormalized solution for elliptic problems with general measure data and proved the existence of such a solution. Observe that for elliptic equations with boundary Dirichlet conditions and L^1-data, this notion is equivalent to the notion of entropy solutions (see [1,2,3,18]) and to the notion of solutions obtained as limit of approximations. As far as elliptic equations with L^1-data and Dirichlet boundary conditions are concerned we refer to [1,2,3,7,18] among a wide literature.

The concept of renormalized solutions in the context of variable exponent was studied for the first time by wittbold and Zimmerman in [23] where they considered an homogeneous Dirichlet boundary condition. In our paper, we consider an inhomogeneous Neumann boundary condition and diffuse Radon measure which bring some difficulty to treat. In fact, the Neumann boundary condition that appears in (P_μ) is quitely different from the one used in [19]. In order to get our main result, we define the space $\mathcal{T}^{1,p(x)}_{tr}(\Omega)$ which will help us to take into account the boundary condition. This space in the context of variable exponent was for the first time introduced by Ouaro and Tchousso (see [20]).

We define $\mathcal{M}_b(X)$ as the space of bounded Radon measure in X, equipped with its standard norm $||.||_{\mathcal{M}_b(X)}$.

We mention that Sobolev capacities are needed to understand point-wise behavior of Sobolev functions. They also play an important role in studies of solutions of partial differential equations (see [13]). In the context of variable exponent, the $p(.)$-capacity of any subset $B \subset X$ is defined by

$$Cap_{p(.)}(B,X) = \inf_{u\in\ S_{p(.)}(B)} \{\int_X (|u|^{p(x)} + |\nabla u|^{p(x)})\, dx\}$$

with

$$S_{p(.)}(B) = \{u \in W^{1,p(x)}_0(X) : u \geq 1 \text{ in an open set containing B and } u \geq 0 \text{ in } X\}.$$

If

$$S_{p(.)}(B) = \emptyset, \text{ we set } Cap_{p(.)}(B,X) = +\infty$$

$$E : W^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(U_\Omega),$$

where U_Ω is the open bounded subset of $I\!R^N$ which extend Ω via the operator E such that:
(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(x)}(\Omega)$,
(ii) $||E(u)||_{W_0^{1,p(x)}(U_\Omega)} \leq c||u||_{W^{1,p(x)}(\Omega)}$, where c is a constant depending only on Ω.
We introduce the set

$$\mathfrak{M}_b^{p(.)}(\Omega) = \{\mu \in \mathcal{M}_b^{p(.)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set U_Ω.
Note that for $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$, we have

$$< \mu, E(u) >= \int_\Omega u \, d\mu.$$

On the other and as μ is diffuse, there exist $f \in L^1(U_\Omega)$, and $F \in (L^{p'(.)}(U_\Omega))^N$ such that

$$\mu = f - \operatorname{div}(F) \text{ in } \mathcal{D}'(U_\Omega).$$

Therefore, we can also write

$$< \mu, E(u) >= \int_{U_\Omega} f E(u) dx + \int_{U_\Omega} F.\nabla E(u) dx.$$

2. Preliminaries

As the exponent $p(x)$ appearing in (P_μ) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents, under the following assumptions on the data:

$$\begin{cases} p(.) : \overline{\Omega} \to I\!R \text{ is a continuous function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{2.1}$$

where $p^- := ess \inf_{x \in \Omega} p(x)$ and $p^+ := ess \sup_{x \in \Omega} p(x)$.
We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u : \Omega \to I\!R$ for which the convex modular

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p^+ < +\infty$, then the expression

$$||u||_{p(x)} := \inf\{\lambda > 0 : \rho_{p(x)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space

$$(L^{p(x)}(\Omega), \|.\|_{p(x)})$$

is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(x)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{p(x)}\|v\|_{p'(x)} \tag{2.2}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.
Let

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}(\Omega), ||.||_{1,p(x)})$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(x)}$ of the space $L^{p(x)}(\Omega)$. We have the following results :

Proposition 2.1. *(see [15,25]) If $u_n, u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$, then the following assertion hold:*
(i)

$$\|u\|_{p(x)} < 1 \text{ (resp. } = 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 \text{ (resp. } = 1; > 1);$$

(ii)

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|^{p^-}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p^+}_{p(x)};$$
$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|^{p^+}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p^-}_{p(x)};$$

(iii)

$$\|u_n\|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n) \to 0;$$
$$\|u_n\|_{p(x)} \to +\infty \Leftrightarrow \rho_{p(x)}(u_n) \to +\infty;$$

(iv)

$$\rho_{p(x)}(u/\|u\|_{p(x)}) = 1.$$

For a measurable function $u : \Omega \to I\!R$, we introduce the following notation:

$$\rho_{1,p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Proposition 2.2. *(see [22,24]) If $u \in W^{1,p(x)}(\Omega)$, then the following assertion hold:*
(i)

$$||u||_{1,p(x)} < 1 \text{ (resp. } = 1; > 1) \Leftrightarrow \rho_{1,p(x)}(u) < 1 \text{ (resp. } = 1; > 1).$$

(ii)

$$||u||_{1,p(x)} > 1 \Rightarrow ||u||^{p^-}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^+}_{1,p(x)};$$
$$||u||_{1,p(x)} < 1 \Rightarrow ||u||^{p^+}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq ||u||^{p^-}_{1,p(x)};$$

Put

$$p^{\partial}(x) := (p(x))^{\partial} := \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x) < N \\ \infty, \text{ if } p(x) \geq N. \end{cases}$$

Proposition 2.3. *(see [24]) Let $p \in \mathcal{C}(\bar{\Omega})$ and $p^- > 1$. If $q \in \mathcal{C}(\partial\Omega)$ satisfies the condition*

$$1 \leq q(x) < p^{\partial}(x), \ \forall\, x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.
In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.
Let us introduce the following notation: Given two bounded measurable functions $p(x), q(x) : \Omega \to I\!R$, we write

$$q(x) \ll p(x) \text{ if } ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

Lemma 2.4. *Let $\xi, \eta \in I\!R^N$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta).$$

Lemma 2.5. *(Lebesgue generalized convergence theorem)*
Let $(f_n)_{n\in I\!N}$ be a sequence of measurable functions and f a measurable function such that

$$f_n \to f \text{ a.e. in } \Omega.$$

Let $(g_n)_{n\in I\!N} \subset L^1(\Omega)$ such that for all $n \in I\!N, |f_n| \leq g_n$ a.e. in Ω and $g_n \to g$ in $L^1(\Omega)$. Then:

$$\int_\Omega f_n dx \to \int_\Omega f dx.$$

In the sequel, we need the following two technical Lemmas (see [14,26]).

Lemma 2.6. *Let $(v_n)_{n\in I\!N}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(\cdot)}(\Omega)$ for some $1 << p(.) \in L^\infty(\Omega)$, then v_n converges strongly to v in $L^1(\Omega)$*

Lemma 2.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \longrightarrow [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \varepsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

We end this section by recalling the result of decomposition of measure.

Theorem 2.8. *(see [19]) Let $p(.) : \bar{X}_1 \subset X \to [1, \infty]$ with $1 < p_- < p_+ < +\infty$ be a continuous function and $\mu \in \mathcal{M}_b(X)$. Then*
$\mu \in \mathcal{M}_b^{p(.)}(X)$ if and only if $\mu \in L^1(X) + W^{-1,p'(.)}(X)$.

3. Basic Assumptions on the data and definition of a renormalized solution

For the solution of the problem (P_μ), the following conditions are assumed:
(H1) f and g are positive functions such that $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.
(H2) α and γ are increasing continuous functions defined on $I\!R$ such that $\alpha(0) = \gamma(0) = 0$.
(H3) $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$.

Now, we recall some notations and results.
For any $k > 0$, we define the truncation function T_k by

$$T_k(s) := \max\{-k, \min\{k, s\}\}.$$

For all $u \in W^{1,p(x)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. In the sequel, we will identify at the boundary u and $\tau(u)$.
Set

$$\mathcal{T}^{1,p(x)}(\Omega) = \{u : \Omega \to I\!R, \text{ measurable such that } T_k(u) \in W^{1,p(x)}(\Omega), \forall k > 0\}.$$

Proposition 3.1. *(see [6]) Let $u \in \mathcal{T}^{1,p(x)}(\Omega)$. Then, there exists a unique measurable function $v : \Omega \to I\!R^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u.*
Moreover, if $u \in W^{1,p(x)}(\Omega)$ then $v \in (L^{p(x)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.

We denoted by $\mathcal{T}_{tr}^{1,p(x)}(\Omega)$ [4,5,20,21] the set of functions $u \in \mathcal{T}^{1,p(x)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in I\!N} \subset W^{1,p(x)}(\Omega)$ satisfying the following conditions:

(A_1) $u_n \to u$ a.e. in Ω.
(A_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.
(A_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4,5]. In the sequel the trace of $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(x)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.
We can now introduce the notion of renormalized solution of (P_μ).

Definition 3.2. *A measurable function $u : \Omega \to I\!R$ is a renormalized solution of problem (P_μ) if:*
(i)

$$u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega), \;\; \textit{and} \;\; \lim_{h\to+\infty} \frac{1}{h}\int_{\{h<|u|<2h\}} |\nabla u|^{p(x)} = 0, \tag{3.1}$$

(ii)

$$\begin{aligned}&\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(S(u)\varphi)\,dx + \int_\Omega |u|^{p(x)-2}uS(u)\varphi\,dx\\ &+\int_\Omega \alpha(u)|\nabla u|^{p(x)}S(u)\varphi\,dx + \int_{\partial\Omega}\gamma(u)S(u)\varphi\,d\sigma\\ &= \int_\Omega S(u)\varphi\,d\mu + \int_{\partial\Omega} gS(u)\varphi\,d\sigma,\end{aligned} \tag{3.2}$$

for every $\varphi \in W^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$ and for any smooth function with compact support S in $I\!R$.

4. Existence result

Now we announce the main result of this section.

Theorem 4.1. *Let assumptions (H1)–(H3) hold true. Then there exists at least one renormalized solution u of the elliptic problem (P_μ).*

Proof. The proof of Theorem (4.1) is divided into several steps:

Step1. The approximate problem Since $\mu \in \mathcal{M}_b^{p(.)}(U_\Omega)$, recall that

$$\mu = f - \operatorname{div}(F) \text{ in } \mathcal{D}'(U_\Omega)$$

with $f \in L^1(U_\Omega)$ and $F \in (L^{p'(.)}(U_\Omega))^N$, again U_Ω is the open bounded subset of $I\!R^N$ which extend Ω via the operator E.
We regularize μ as follows: $\forall\varepsilon > 0, \forall x \in u_\Omega$, we define

$$f_\varepsilon(x) = T_{\frac{1}{\varepsilon}}(f(x))\chi_\Omega(x).$$

We consider $F_R = \chi_\Omega F$ and $\mu_\varepsilon = f_\varepsilon - \operatorname{div}(F_R)$.
For any $\varepsilon > 0$, one has $\mu_\varepsilon \in \mathfrak{M}_b^{p(.)}(\Omega)\cap L^\infty(\Omega)$ and $\mu_\varepsilon \rightharpoonup \mu$ in $\mathcal{M}_b^{p(.)}(U_\Omega)$.
Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(x)}(\Omega)$,

$$|\int_\Omega T_k(\xi)d\mu_\varepsilon| \le kc(\mu,\Omega).$$

Now, we consider the approximated problem:

$$(P_{\mu_\varepsilon})\begin{cases} -\Delta_{p(x)}u_\varepsilon + |u_\varepsilon|^{p(x)-2}u_\varepsilon + T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}) = \mu_\varepsilon & \text{in } \Omega\\ |\nabla u_\varepsilon|^{p(x)-2}\dfrac{\partial u_\varepsilon}{\partial\eta} + T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon)) = T_{\frac{1}{\varepsilon}}(g) & \text{on } \partial\Omega.\end{cases}$$

Lemma 4.2. *There exists at least one weak solution u_ε for the problem (P_{μ_ε}) in the sense that $u_\varepsilon \in W^{1,p(.)}(\Omega)$ and for all $v \in W^{1,p(.)}(\Omega)$,*

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla v\, dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon v\, dx + \int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})v\, dx$$
$$+\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))v\, d\sigma = \int_\Omega v\, d\mu_\varepsilon + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)v\, d\sigma. \tag{4.1}$$

Proof of Lemma 4.2 We define the following reflexive space

$$E = W^{1,p(x)}(\Omega) \times L^{p(x)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u,v) \in E : v = \tau(u)\}$$

In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p(.)}(\Omega)$.
We define the operator A_ε by,

$$< A_\varepsilon u, v >=< Au, v > + \int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})v\, dx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))v\, d\sigma,$$

where

$$< Au, v >= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx + \int_\Omega |u|^{p(x)-2}uv\, dx\ \forall u, v \in X_0.$$

According to [1], the operator A_ε is bounded, coercive, hemi-continuous and it is of type (M) from X_0 into $X_0^{'}$.
Thus, for $F_\varepsilon \in E^{'} \subset X_0^{'}$ defined by

$$< F_\varepsilon, v >= \int_\Omega v\, d\mu_\varepsilon + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)v\, d\sigma,$$

we deduce the existence of a function $u_\varepsilon \in X_0$ such that :

$$< A_\varepsilon u_\varepsilon, v >=< F_\varepsilon, v >,\ \forall\ v \in X_0,$$

i.e.

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla v\, dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon v\, dx + \int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})v\, dx$$
$$+\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))v\, d\sigma = \int_\Omega v\, d\mu_\varepsilon + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)v\, d\sigma.$$

Step2. A priori Estimates
Assertion 1. $(\nabla T_k(u_\varepsilon))_{\varepsilon>0}$ is bounded in $(L^{p(x)}(\Omega))^N$.
Proof: We choose $T_k(u_\varepsilon)$ as a test function in (4.1) we obtain,

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_k(u_\varepsilon)\,dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)\,dx \\
&+ \int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})T_k(u_\varepsilon)\,dx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))T_k(u_\varepsilon)\,d\sigma \\
&= \int_\Omega T_k(u_\varepsilon)\,d\mu_\varepsilon + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)T_k(u_\varepsilon)\,d\sigma.
\end{aligned} \tag{4.2}$$

The third and fourth terms in the left-hand side of the above equality are non negative, then:

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_k(u_\varepsilon)\,dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)\,dx \\
&\leq k||g||_{L^1(\partial\Omega)} + \int_\Omega T_k(u_\varepsilon)\,d\mu_\varepsilon.
\end{aligned} \tag{4.3}$$

One has

$$\begin{aligned}
\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)\,dx &= \int_{\{|u_\varepsilon|\leq k\}} |T_k(u_\varepsilon)|^{p(x)}\,dx \\
&\quad + \int_{\{|u_\varepsilon|>k\}} |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)\,dx \\
&\geq \int_{\{|u_\varepsilon|\leq k\}} |T_k(u_\varepsilon)|^{p(x)}\,dx + \int_{\{|u_\varepsilon|>k\}} k^{p(x)}\,dx \\
&\geq \int_{\{|u_\varepsilon|\leq k\}} |T_k(u_\varepsilon)|^{p(x)}\,dx + \int_{\{|u_\varepsilon|>k\}} |T_k(u_\varepsilon)|^{p(x)}\,dx
\end{aligned}$$

then

$$\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)\,dx \geq \int_\Omega |T_k(u_\varepsilon)|^{p(x)}\,dx, \tag{4.4}$$

we have

$$\begin{aligned}
\int_\Omega T_k(u_\varepsilon)\,d\mu_\varepsilon &= \int_\Omega E(T_k(u_\varepsilon))\,d\mu_\varepsilon \qquad (4.5)\\
&= < \mu_\varepsilon, E(T_k(u_\varepsilon)) > \\
&= \int_{U_\Omega} f_\varepsilon E(T_k(u_\varepsilon))\,dx + \int_{U_\Omega} F_k.\nabla E(T_k(u_\varepsilon))\,dx \\
&= \int_\Omega T_{\frac{1}{\varepsilon}}(f)E(T_k(u_\varepsilon))\,dx + \int_\Omega F_k.\nabla E(\chi_\Omega T_k(u_\varepsilon))\,dx. \qquad (4.6)
\end{aligned}$$

Firstly, we have

$$|\int_\Omega T_{\frac{1}{\varepsilon}}(f)E(T_k(u_\varepsilon))\,dx| \quad \leq k||f||_{L^1(\Omega)}. \tag{4.7}$$

Secondly, we have

$$
\begin{aligned}
|\int_{\Omega} F.\nabla E(\chi_{\Omega} T_k(u_\varepsilon))\, dx| &= |\int_{\{|u_\varepsilon|<k\}} F.\nabla T_k(u_\varepsilon)\, dx| \\
&\leq \int_{\Omega} |F||\nabla T_k(u_\varepsilon)|\, dx. \qquad (4.8)
\end{aligned}
$$

Using Young's inequality, we get:

$$
\begin{aligned}
\int_{\Omega} |F||\nabla T_k(u_\varepsilon)|\, dx &= \int_{\Omega} \frac{|\nabla T_k(u_\varepsilon)|}{(p_+ \times 2^{p_+})^{\frac{1}{p(x)}}}.(p_+ \times 2^{p_+})^{\frac{1}{p(x)}} |F|\, dx \\
&\leq \int_{\Omega} \frac{1}{p(x)}.\frac{1}{p_+ \times 2^{p_+}} |\nabla T_k(u_\varepsilon)|^{p(x)} dx \\
&\quad + \int_{\Omega} \frac{1}{p'(x)} (p_+ \times 2^{p_+})^{\frac{p'(x)}{p(x)}} |F|^{p'(x)}\, dx \\
&\leq \frac{1}{p_-}.\frac{1}{p_+ \times 2^{p_+}} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^{p(x)}\, dx \\
&\quad + \frac{1}{p'_-} (p_+ \times 2^{p_+})^{\frac{p'_+}{p_-}} \int_{\Omega} |F|^{p'(x)}\, dx. \qquad (4.9)
\end{aligned}
$$

Combining (4.7) and (4.9), the equality (4.5) becomes

$$
\begin{aligned}
\int_{\Omega} T_k(u_\varepsilon)\, d\mu_\varepsilon &\leq k||f||_{L^1(\Omega)} + \frac{1}{p_-}.\frac{1}{p_+ \times 2^{p_+}} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^{p(x)}\, dx \\
&\quad + \frac{1}{p'_-}(p_+ \times 2^{p_+}) \int_{\Omega} |F|^{p'(x)}\, dx. \qquad (4.10)
\end{aligned}
$$

Then, according to (4.3), we get

$$
\begin{aligned}
&\int_{\Omega} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla T_k(u_\varepsilon)\, dx + \int_{\Omega} |u_\varepsilon|^{p(x)-2} u_\varepsilon T_k(u_\varepsilon)\, dx \\
&\leq k(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}) + \tfrac{1}{p_-}.\tfrac{1}{(p_+ \times 2^{p_+})} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^{p(x)}\, dx \qquad (4.11) \\
&+ \tfrac{1}{p'_-}(p_+ \times 2^{p_+})^{\frac{p'_+}{p_-}} \int_{\Omega} |F|^{p'(x)}\, dx.
\end{aligned}
$$

So, we set from (4.11);

$$
\begin{aligned}
\int_{\Omega} |\nabla T_k(u_\varepsilon)|^{p(x)}\, dx + \int_{\Omega} |T_k(u_\varepsilon)|^{p(x)}\, dx &\leq k(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}) \\
&\quad + \frac{1}{p_-}.\frac{1}{(p_+ \times 2^{p_+})} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^{p(x)}\, dx \\
&\quad + \frac{1}{p'_-}(p_+ \times 2^{p_+})^{\frac{p'_+}{p_-}} \int_{\Omega} |F|^{p'(x)}\, dx.
\end{aligned}
$$

Then

$$[1-\frac{1}{p_-(p_+\times 2^{p_+})}]\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}\,dx+\int_\Omega |T_k(u_\varepsilon)|^{p(x)}\,dx$$

$$\leq k(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)})+\frac{1}{p'_-}(p_+\times 2^{p_+})^{\frac{p'_+}{p_-}}\int_\Omega |F|^{p'(x)}\,dx.$$

Therefore

$$[1-\frac{1}{p_-(p_+\times 2^{p_+})}]\varrho_{1,p(.)}(T_k(u_\varepsilon)) \leq k(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}) +\frac{1}{p'_-}(p_+\times 2^{p_+})^{\frac{p'_+}{p_-}}\int_\Omega |F|^{p'(x)}\,dx. \quad (4.12)$$

Consequently,

$$\varrho_{1,p(.)}(T_k(u_\varepsilon)) \leq kc_1+c_2 \quad (4.13)$$

Where $c_1=const(f,g,p_-,p_+)$ and $c_2=const(F,p_-,p_+)$. Thus

$$||T_k(u_\varepsilon)||_{1,p(.)}\leq 1+(kc_1+c_2)^{\frac{1}{p_-}} \quad (4.14)$$

We deduce that for any $k>0$, the sequence $(T_k(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $W^{1,p(.)}(\Omega)$. Then, up to a subsequence, we can assume that for any $k>0$, $T_k(u_\varepsilon)\rightharpoonup v_k$ in $W^{1,p(.)}(\Omega)$. Furthermore, by compact embedding, we have $T_k(u_\varepsilon)\to v_k$ in $L^{p(.)}(\Omega)$ and a.e. in Ω.

Assertion 2. $(u_\varepsilon)_{\varepsilon>0}$ converges in measure to some function u.
To prove this, we show that $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in measure.
Thanks to (4.12), we conclude that

$$\int_{\{|u_\varepsilon|>k\}} k^{p_-}\,dx\leq \int_{\{|u_\varepsilon|>k\}} k^{p(x)}\,dx\leq k(c_1+c_2).$$

It follows that

$$\text{meas}\{|u_\varepsilon|>k\}\leq k^{1-p_-}(c_1+c_2).$$

Therefore

$$\text{meas}\{|u_\varepsilon|>k\}\to 0 \text{ as } k\to+\infty \text{ since } 1-p_-<0. \quad (4.15)$$

Moreover, for every fixed $t>0$ and every positive $k>0$, it is clear that

$$\{|u_{\varepsilon_1}-u_{\varepsilon_2}|>t\}\subset\{|u_{\varepsilon_1}|>k\}\cup\{|u_{\varepsilon_2}|>k\}\cup\{|T_k(u_{\varepsilon_1})-T_k(u_{\varepsilon_2})|>t\},$$

hence

$$\text{meas}\{|u_{\varepsilon_1}-u_{\varepsilon_2}|>t\} \leq \text{meas}\{|u_{\varepsilon_1}|>k\}+\text{meas}\{|u_{\varepsilon_2}|>k\} +\text{meas}(\{|T_k(u_{\varepsilon_1})-T_k(u_{\varepsilon_2})|>t\}). \quad (4.16)$$

Let $\delta>0$, using (4.15), we choose $k=k(\delta)$ such that

$$\text{meas}\{|u_{\varepsilon_1}|>k\}\leq\tfrac{\delta}{3} \text{ and } \text{meas}\{|u_{\varepsilon_2}|>k\}\leq\tfrac{\delta}{3}. \quad (4.17)$$

Since $(T_k(u_\varepsilon))_{\varepsilon>0}$ converges strongly in $L^{p(x)}(\Omega)$, then it is a Cauchy sequence in $L^{p(x)}(\Omega)$.
Thus, for all $\varepsilon_1, \varepsilon_2 \geq no(t,\delta)$ we have

$$\mathrm{meas}(\{|T_k(u_{\varepsilon_1}) - T_k(u_{\varepsilon_2})| > t\}) \leq \tfrac{\delta}{3}. \tag{4.18}$$

Combining (4.16), (4.17) and (4.18) we obtain

$$\mathrm{meas}\{|u_{\varepsilon_1} - u_{\varepsilon_2}| > t\} \leq \delta \text{ for all } \varepsilon_1, \varepsilon_2 \geq no(t,\delta). \tag{4.19}$$

which prove that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in measure, and then converges almost everywhere to some measurable function u.
Therefore

$$\begin{array}{c} T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ in } W^{1,p(x)}(\Omega) \\ T_k(u_\varepsilon) \to T_k(u) \text{ in } L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{array} \tag{4.20}$$

Assertion 3. $(\nabla u_\varepsilon)_{\varepsilon>0}$ converges in measure to the weak gradient of u
Proof: Indeed, let δ, t, k, ν be positive real numbers (it is assumed that $\nu < 1$) and let $\varepsilon > 0$. We have

$$\begin{aligned} \{|\nabla u_\varepsilon - \nabla u| > t\} \quad \subset \quad & \{|u_\varepsilon| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_\varepsilon)| > k\} \\ & \cup \{|\nabla T_k(u)| > k\} \cup \{|u_\varepsilon - u| > \nu\} \cup G \end{aligned}$$

where

$$G = \{|\nabla u_\varepsilon - \nabla u| > t, |u| \leq k, |u_\varepsilon| \leq k, |\nabla T_k(u_\varepsilon)| \leq k, |\nabla T_k(u)| \leq k, |u_\varepsilon - u| \leq \nu\}.$$

The same strategy used in the proof of Assertion 2 allows us to obtain for k sufficiently large,

$$\mathrm{meas}(\{|u_\varepsilon| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_\varepsilon)| > k\} \cup \{|\nabla T_k(u)| > k\}) \leq \tfrac{\delta}{3}. \tag{4.21}$$

On the other hand, the mapping

$$H : I\!R^N \times I\!R^N \to I\!R$$

$$(\zeta_1, \zeta_2) \to (\Phi(\zeta_1) - \Phi(\zeta_2)).(\zeta_1 - \zeta_2),$$

where $\Phi(\zeta) = |\zeta|^{p(x)-2}\zeta$, is continuous.
The set

$$\mathcal{A} = \{(\zeta_1, \zeta_2) \in I\!R^N \times I\!R^N / |\zeta_1| \leq k, |\zeta_2| \leq k, |\zeta_1 - \zeta_2| > t\}$$

is compact and

$$(\Phi(\zeta_1) - \Phi(\zeta_2)).(\zeta_1 - \zeta_2) > 0 \ \forall \zeta_1 \neq \zeta_2.$$

Then, the mapping H attains its minimum on $\mathcal{A}$, we denote it by β.

Therefore, we have $\beta > 0$ and

$$\begin{aligned}\int_G \beta\, dx &\leq \int_G [\Phi(\nabla u_\varepsilon) - \Phi(\nabla u)][\nabla u_\varepsilon - \nabla u]\, dx \\ &\leq \int_G [\Phi(\nabla u_\varepsilon) - \Phi(\nabla T_k(u))].\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx \\ &\leq \int_G \Phi(\nabla u_\varepsilon).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx \\ &\quad - \int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx.\end{aligned}$$

We take $\vartheta = T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))$ in (4.1) to obtain

$$\begin{aligned}&\int_\Omega \Phi(\nabla u_\varepsilon).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx + \int_\Omega |u_\varepsilon|^{p(x)-2} u_\varepsilon T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx \\ &\leq \nu(||T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})||_{L^1(\Omega)} + ||T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))||_{L^1(\partial\Omega)} + ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}) \\ &\qquad + \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))\, dx.\end{aligned}$$

Then

$$\begin{aligned}&\int_\Omega \Phi(\nabla u_\varepsilon).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx \\ \leq \nu(||T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})||_{L^1(\Omega)} \\ &+||T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))||_{L^1(\partial\Omega)} + ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}) \\ &+ \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))\, dx \\ &- \int_\Omega |u_\varepsilon|^{p(x)-2} u_\varepsilon T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx\end{aligned}$$

Taking $\vartheta = \frac{1}{k} T_k(u_\varepsilon)$ in (4.1), we get:

$$\begin{aligned}&\int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}) \frac{1}{k} T_k(u_\varepsilon)\, dx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon)) \frac{1}{k} T_k(u_\varepsilon)\, d\sigma \\ &\leq \quad ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)} + \int_{U_\Omega} F.\nabla E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon))\, dx.\end{aligned} \tag{4.22}$$

Since

$$\lim_{k\to 0} \frac{1}{k} T_k(u_\varepsilon) = sign(u_\varepsilon),$$

then, using the Lebesgue dominated convergence theorem as $k \to 0$, we deduce that

$$\int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}) \frac{1}{k} T_k(u_\varepsilon)\, dx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon)) \frac{1}{k} T_k(u_\varepsilon)\, d\sigma \to$$

$$||T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})||_{L^1(\Omega)} + ||T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))||_{L^1(\partial\Omega)} \tag{4.23}$$

The sequence $(E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon)))_{\varepsilon>0}$ is bounded in $W_0^{1,p(.)}(U_\Omega)$. Indeed, $(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$ and we use the inequality

$$||E(u)||_{W_0^{1,p(x)}(U_\Omega)} \leq c||u||_{W^{1,p(x)}(\Omega)}, \text{ for all } u \in W^{1,p(.)}(\Omega).$$

We also have

$$E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon)) = \chi_\Omega \frac{1}{k} T_k(u_\varepsilon) \text{ a.e in } U_\Omega$$

and

$$\chi_\Omega \frac{1}{k} T_k(u_\varepsilon) \to \chi_\Omega sign(u_\varepsilon) \text{ a.e in } U_\Omega \text{ as } k \to 0.$$

Hence

$$E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon)) \to E(\chi_\Omega sign(u_\varepsilon)) \text{ a.e in } U_\Omega \text{ as } k \to 0.$$

Then,

$$\nabla E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon)) \rightharpoonup 0 \text{ in } (L^{p(.)}(U_\Omega))^N$$

Finally, we get

$$\lim_{k\to 0} \int_{U_\Omega} F.\nabla E(\chi_\Omega \frac{1}{k} T_k(u_\varepsilon))\, dx = 0. \tag{4.24}$$

Therefore by passing to the limit as $k \to 0$ in (4.22) and using (4.23) and (4.24), we get

$$||T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}||_{L^1(\Omega)} + ||T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))||_{L^1(\partial\Omega)} \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

It follows that

$$\begin{aligned} &\int_\Omega \Phi(\nabla u_\varepsilon).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\, dx \leq \nu c_3 \\ &+ \int_\Omega |u_\varepsilon|^{p(x)-1} |T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))|\, dx \\ &+ \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))\, dx \end{aligned} \tag{4.25}$$

Now, let us show that

$$\lim_{\nu\to 0} \lim_{\varepsilon\to 0} \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))\, dx = 0. \tag{4.26}$$

The sequence $(E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))))_{\varepsilon>0}$ is bounded in $W_0^{1,p(.)}(U_\Omega)$. We have

$$E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))) = \chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)) \text{ a.e in } U_\Omega$$

and

$$\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)) \to \chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u)) \text{ a.e in } U_\Omega \text{ as } \varepsilon \to 0.$$

Hence

$$E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))) \to E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u))) \text{ a.e in } U_\Omega \text{ as } \varepsilon \to 0.$$

Then

$$\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))) \rightharpoonup \nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u))) \text{ in } (L^{p(\cdot)}(U_\Omega))^N.$$

Since $F \in (L^{p'(\cdot)}(U_\Omega))^N$, we deduce that

$$\lim_{\varepsilon\to 0} \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))\, dx = \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u)))\, dx. \tag{4.27}$$

we have

$$\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u)))\, dx \to 0 \text{ a.e in } U_\Omega \text{ as } \nu \to 0,$$

and as $\nu < 1$ we have

$$F.E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u))) \le |F|.|E(\chi_\Omega T_1(T_{k+1}(u) - T_k(u)))|.$$

Using Hölder inequality, we get

$$|F|.|E(\chi_\Omega T_1(T_{k+1}(u) - T_k(u)))| \in L^1(U_\Omega).$$

Thanks to the Lebesgue dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0} \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u) - T_k(u)))dx = 0,$$

consequently, letting $\nu \to 0$ in (4.27) yields

$$\lim_{\nu\to 0} \lim_{\varepsilon\to 0} \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))dx = 0.$$

Since

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{p(x)-1} |T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))|\, dx &\le \nu \int_\Omega |u_\varepsilon|^{p(x)-1}\, dx \\ &\le \nu(\varrho_{p'(.)}(|u_\varepsilon|^{p(x)-1}) + \varrho_{p(.)}(1)) \le \nu(\text{meas}(\Omega) + \varrho_{p(.)}(u_\varepsilon)). \end{aligned} \tag{4.28}$$

So, letting $\nu \to 0$ in (4.28) and using the fact that Ω is bounded and $\varrho_{p(.)}(u_\varepsilon)$ is finite, we deduce that

$$\lim_{\nu\to 0} \int_\Omega |u_\varepsilon|^{p(x)-1} |T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))|\, dx = 0. \tag{4.29}$$

According to the Assertion 1, the sequence $(T_{k+\nu}(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $W^{1,p(.)}(\Omega)$.
Then

$$T_{k+\nu}(u_\varepsilon) \to T_{k+\nu}(u) \text{ in } W^{1,p(.)}(\Omega)$$

and

$$\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)) \rightharpoonup \nabla T_\nu(T_{k+\nu}(u) - T_k(u)) \text{ in } (L^{p(.)}(\Omega))^N$$

consequently,

$$\lim_{\varepsilon\to 0}\int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\,dx = \int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u) - T_k(u))\,dx \tag{4.30}$$

Since

$$\lim_{\nu\to 0}\nabla T_\nu(T_{k+\nu}(u) - T_k(u)) = 0$$

and as $\nu < 1$, we have:

$$|\Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u) - T_k(u))| \le |\nabla T_k(u)|^{p(x)-1}|\nabla T_1(T_{k+1}(u) - T_k(u))| \in L^1(\Omega)$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{\nu\to 0}\int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u) - T_k(u))\,dx = 0$$

Let $\varrho > 0$ and $\nu < \frac{\varrho}{4c_3}$ be fixed such that

$$|\int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u) - T_k(u))\,dx| \le \frac{\varrho}{4}, \tag{4.31}$$

then, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$,

$$\Big|\int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\,dx - \int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u) - T_k(u))\,dx\Big| \le \frac{\varrho}{4} \tag{4.32}$$

Combining (4.31) and (4.32), we obtain

$$|\int_\Omega \Phi(\nabla T_k(u)).\nabla T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))\,dx| \le \frac{\varrho}{2}, \forall \varepsilon < \varepsilon_1. \tag{4.33}$$

Also, there exists $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$,

$$\nu c_3 + \int_\Omega |u_\varepsilon|^{p(x)-1}|T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u))|\,dx + \int_{U_\Omega} F.\nabla E(\chi_\Omega T_\nu(T_{k+\nu}(u_\varepsilon) - T_k(u)))dx \le \frac{\varrho}{2}. \tag{4.34}$$

Then, using (4.33) and (4.34), we get:

$$\int_G \beta \, dx \leq \varrho.$$

Thus, by applying the Lemma 2.7, we obtain

$$\text{meas}(G) \leq \tfrac{\delta}{3}. \tag{4.35}$$

Moreover, by using the Assertion 2, we deduce the existence of $\varepsilon_3 > 0$, such that

$$\text{meas}(\{|u_\varepsilon - u| > \nu\}) \leq \tfrac{\delta}{3}, \ \forall \varepsilon \leq \varepsilon_3. \tag{4.36}$$

Therefore, for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, it follows that

$$\text{meas}(\{|\nabla u_\varepsilon - \nabla u| > t\}) \leq \delta, \ \forall \varepsilon \leq \varepsilon_0 \tag{4.37}$$

So, ∇u_ε converges in measure to ∇u.
Assertion 4. $(u_\varepsilon)_{\varepsilon>0}$ converges a.e on $\partial\Omega$ to some function ϑ.
Proof: we know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant $c_4 > 0$ such that

$$||T_k(u_\varepsilon) - T_k(u)||_{L^1(\partial\Omega)} \leq c_4 ||T_k(u_\varepsilon) - T_k(u)||_{W^{1,1}(\Omega)}.$$

Therefore

$$T_k(u_\varepsilon) \to T_k(u) \text{ in } L^1(\partial\Omega) \text{ and a.e in } \partial\Omega,$$

we deduce that, there exists $A \subset \partial\Omega$ such that $T_k(u_\varepsilon)$ converges to $T_k(u)$ on $\partial\Omega \backslash A$ with $\sigma(A) = 0$, where σ is the area measure on $\partial\Omega$.
For every $k > 0$, let $A_k = \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $B = \partial\Omega \backslash \bigcup_{k>0} A_k$.
We have

$$\sigma(B) = \tfrac{1}{k} \int_B |T_k(u)| d\sigma \leq \frac{c_4}{k} ||T_k(u)||_{W^{1,1}(\Omega)} \leq \frac{c_5}{k} ||T_k(u)||_{1,p(x)}, \tag{4.38}$$

we know that for all $k > 1$, $\varrho_{1,p(.)}(T_k(u_\varepsilon)) \leq kM$ where M is a positive constant that does not depends on ε.
Then

$$\int_\Omega |T_k(u_\varepsilon)|^{p(x)} dx + \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)} dx \leq kM \tag{4.39}$$

we now use the Fatou's lemma in (4.39) to get

$$\int_\Omega |T_k(u)|^{p(x)} dx + \int_\Omega |\nabla T_k(u)|^{p(x)} dx \leq kM$$

which is equivalent to

$$\varrho_{1,p(x)}(T_k(u)) \leq kM. \tag{4.40}$$

According to(4.40), we deduce that

$$||T_k(u)||_{W^{1,p(x)}(\Omega)} \leq c_6(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}).$$

Therefore, we get by letting $k \to +\infty$ in (4.38) that $\sigma(B) = 0$.
Let us now define in $\partial\Omega$ the function ϑ by

$$\vartheta(x) = T_k(u(x)) \text{ if } x \in A_k$$

we take $x \in \partial\Omega \backslash (A \cup B)$, then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_\varepsilon(x) - \vartheta(x) = (u_\varepsilon(x) - T_k(u_\varepsilon(x))) + (T_k(u_\varepsilon(x)) - T_k(u(x))),$$

since $x \in A_k$, we have $|T_k(u_\varepsilon(x))| < k$ from which we deduce that $|u_\varepsilon(x)| < k$. Therefore,

$$u_\varepsilon(x) - \vartheta(x) = T_k(u_\varepsilon(x)) - T_k(u(x)) \to 0, \text{ as } \varepsilon \to 0.$$

This means that u_ε converges to ϑ a.e on $\partial\Omega$.

Assertion 5. u is a renormalized solution of the problem (P_μ)
Proof: Since the sequence $(\nabla T_k(u_\varepsilon))_{\varepsilon>0}$ converges in measure to $\nabla T_k(u)$, then by (4.14) and
Lemma 2.3 we get

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(u) \text{ in } (L^1(\Omega))^N, \forall k > 0 \tag{4.41}$$

Consequently, Assertion 2, 4 and (4.41) give $u \in \mathcal{T}^{1,p(.)}_{tr}(\Omega)$.
Let $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and let S be a smooth function with compact support in $I\!R$, we take $\vartheta = S(u_\varepsilon)\varphi$ as a test function in (4.1) to get

$$\begin{aligned} &\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla(S(u_\varepsilon)\varphi)\,dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon S(u_\varepsilon)\varphi\,dx \\ &+ \textstyle\int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})S(u_\varepsilon)\varphi\,dx + \displaystyle\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))S(u_\varepsilon)\varphi\,d\sigma \\ &= \textstyle\int_\Omega S(u_\varepsilon)\varphi\,d\mu_\varepsilon + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)S(u_\varepsilon)\varphi\,d\sigma, \end{aligned} \tag{4.42}$$

The function S has compact support, then there exists a positive real number k such that
$supp(S) \subset [-k, k]$ which leads to

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla(S(u_\varepsilon)\varphi)\,dx &= \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)-2}\nabla T_k(u_\varepsilon)S(u_\varepsilon)\nabla\varphi\,dx \\ &+ \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}S^{'}(u_\varepsilon)\varphi\,dx. \end{aligned} \tag{4.43}$$

Since

$$|\nabla T_k(u_\varepsilon)|^{p(x)-2}\nabla T_k(u_\varepsilon) \rightharpoonup |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \text{ weakly in } (L^{p^{'}(x)}(\Omega))^N$$

and $S(u_\varepsilon)\nabla\varphi \to S(u)\nabla\varphi$ strongly in $L^{p(x)}(\Omega)$.
Hence

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla T_k(u_\varepsilon)S(u_\varepsilon)\nabla\varphi\,dx \to \int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)S(u)\nabla\varphi\,dx$$

and as

$$|\nabla T_k(u_\varepsilon)|^{p(x)} \to |\nabla T_k(u)|^{p(x)} \text{ in } L^1(\Omega)$$

it follows that

$$\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}S'(u_\varepsilon)\varphi\,dx \to \int_\Omega |\nabla T_k(u)|^{p(x)}S'(u)\varphi\,dx.$$

Then

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\nabla(S(u_\varepsilon)\varphi)\,dx \to \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(S(u)\varphi)\,dx. \tag{4.44}$$

In the same way, it is easy to see that

$$\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon S(u_\varepsilon)\varphi\,dx \to \int_\Omega |u|^{p(x)-2}uS(u)\varphi\,dx \tag{4.45}$$

and

$$\int_\Omega T_{\frac{1}{\varepsilon}}(\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})S(u_\varepsilon)\varphi\,dx \to \int_\Omega \alpha(u)|\nabla u|^{p(x)}S(u)\varphi\,dx. \tag{4.46}$$

Moreover, we have u_ε converges to u on $\partial\Omega$.

So, by continuity of γ, it follows that

$$\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(\gamma(u_\varepsilon))S(u_\varepsilon)\varphi\,d\sigma \to \int_{\partial\Omega}\gamma(u)S(u)\varphi\,d\sigma. \tag{4.47}$$

Let us prove that

$$\int_\Omega S(u_\varepsilon)\varphi\,d\mu_\varepsilon \to \int_\Omega S(u)\varphi\,d\mu. \tag{4.48}$$

We have

$$\begin{aligned}\int_\Omega S(u_\varepsilon)\varphi d\mu_\varepsilon &= <\mu_\varepsilon, E(S(u_\varepsilon)\varphi)> \\ &= \int_{U_\Omega} f_\varepsilon E(S(u_\varepsilon)\varphi)dx + \int_{U_\Omega} F_R\nabla E(S(u_\varepsilon)\varphi)dx \\ &= \int_{U_\Omega} T_{\frac{1}{\varepsilon}}(f)\chi_\Omega E(S(u_\varepsilon)\varphi)dx + \int_{U_\Omega}(F\chi_\Omega)\nabla E(S(u_\varepsilon)\varphi)dx.\end{aligned}$$

$$= \int_\Omega T_{\frac{1}{\varepsilon}}(f)(S(u_\varepsilon)\varphi)dx + \int_{U_\Omega} F.\nabla E(\chi_\Omega S(u_\varepsilon)\varphi)dx. \tag{4.49}$$

Thanks to the Lebesgue dominated convergence theorem, we have

$$\int_\Omega T_{\frac{1}{\varepsilon}}(f)S(u_\varepsilon)\varphi dx \to \int_\Omega fS(u)\varphi dx. \tag{4.50}$$

Since the sequence $(\chi_\Omega S(u_\varepsilon)\varphi)_{\varepsilon>0}$ is bounded in $W^{1,p(\cdot)}(\Omega)$, by using the property (ii) of the operator E, we deduce that $(E(\chi_\Omega S(u_\varepsilon)\varphi))_{\varepsilon>0}$ is bounded in $W_0^{1,p(\cdot)}(U_\Omega)$. Moreover, we have

$$E(\chi_\Omega S(u_\varepsilon)\varphi) = \chi_\Omega S(u_\varepsilon)\varphi \text{ a.e in } U_\Omega$$

and

$$\chi_\Omega S(u_\varepsilon)\varphi \to \chi_\Omega S(u)\varphi \text{ in } U_\Omega \text{ as } \varepsilon \to 0,$$

which implies that

$$E(\chi_\Omega S(u_\varepsilon)\varphi) \to E(S(u)\varphi) \text{ in } U_\Omega \text{ as } \varepsilon \to 0.$$

Consequently,

$$\nabla E(\chi_\Omega S(u_\varepsilon)\varphi) \rightharpoonup \nabla E(\chi_\Omega S(u)\varphi) \text{ in } (L^{p(\cdot)}(U_\Omega))^N.$$

Thus, by using the fact that $F \in (L^{p'(\cdot)}(U_\Omega))^N$, we deduce that

$$\lim_{\varepsilon\to 0}\int_{U_\Omega} F.\nabla E(\chi_\Omega S(u_\varepsilon)\varphi)dx = \int_{U_\Omega} F.\nabla E(\chi_\Omega S(u)\varphi)dx. \tag{4.51}$$

Hence, by passing to the limit in (4.49) and using (4.50) and (4.51), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega S(u_\varepsilon)\, d\mu_\varepsilon &= \int_\Omega fS(u)\varphi dx + \int_{U_\Omega} F\nabla E(\chi_\Omega S(u)\varphi)\, dx \\
&= \int_{U_\Omega} fE(\chi_\Omega S(u)\varphi)dx + \int_{U_\Omega} F.\nabla E(\chi_\Omega S(u)\varphi)dx \\
&= < \mu, E(S(u)\varphi) > \\
&= \int_\Omega S(u)\varphi dx.
\end{aligned}$$

By using again the Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)S(u_\varepsilon)\varphi dx = \int_{\partial\Omega} gS(u)\varphi d\sigma. \tag{4.52}$$

Using (4.42), (4.44), (4.45), (4.46), (4.47) and (4.52), we get

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(S(u)\varphi)\, dx + \int_\Omega |u|^{p(x)-2}uS(u)\varphi\, dx \\
&+ \int_\Omega \alpha(u)|\nabla u|^{p(x)}S(u)\varphi\, dx + \int_{\partial\Omega}\gamma(u)S(u)\varphi\, d\sigma \\
&= \int_\Omega S(u)\varphi d\mu + \int_{\partial\Omega} gS(u)\varphi d\sigma.
\end{aligned}$$

Now, we claim that

$$\lim_{h\to+\infty}\frac{1}{h}\int_{\{h<|u|<2h\}}|\nabla u|^{p(x)}dx=0. \tag{4.53}$$

Indeed, by taking $\vartheta = S_h(u_\varepsilon) = T_h(u_\varepsilon - T_h(u_\varepsilon))$ in (4.1), where

$$S_h(u_\varepsilon)=\begin{cases} S-h.sign(S) & \text{if } h<|S|<2h \\ h.sign(S) & \text{if } |S|\geq 2h \\ 0 & \text{if } |S|\leq h, \end{cases} \tag{4.54}$$

we get

$$\int_\Omega |\nabla S_h(u_\varepsilon)|^{p(x)} \leq \int_\Omega S_h(u_\varepsilon)d\mu_\varepsilon + \int_{\partial\Omega}|g|S_h(u_\varepsilon)d\sigma$$

passing to the limit as $\varepsilon\to 0$, we obtain

$$\int_\Omega |\nabla S_h(u)|^{p(x)} \leq \int_\Omega S_h(u)d\mu + \int_{\partial\Omega}|g|S_h(u)d\sigma$$

Then, it follows that

$$\lim_{h\to\infty}\frac{1}{h}\int_\Omega|\nabla S_h(u)|^{p(x)}\leq 0,$$

which completes the proof of Theorem 4.1.

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