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Existence Results Involving Fractional Liouville Derivative

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abstract: In this paper we investigate the question of existence of nonnegative solution to some fractional liouville equation. Our main tools based on the well known Krasnoselskii's fixed point theorem.

Key Words: Caputo fractional derivative, Green function, Guo-Krasnoselskii fixed point theorem.

Contents

1. Introduction

The purpose of this work is to study the existence of nonnegative solution to the following nonlinear boundary value problems of fractional differential equations:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u'(0) = 0, \quad \lambda u(0) - \mu u(1) = \gamma u(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, D^{α}_{0+} denotes the Caputo fractional derivative, $f \in C([0, 1]\times\mathbb{R}, \mathbb{R})$, λ, μ and γ are positive real numbers such that

$$\lambda > \gamma + \mu. \tag{1.2}$$

The theory of fractional calculus may be used to the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann- Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [2,3,8,9] and references therein. Bai and Lü [1] investigated the following nonlinear fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

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where $1 < \alpha \leq 2$. By means of Guo-Krasnoselskii fixed point theorem and Leggett-Williams fixed point theoremm, the authors prove the existence and multiplicity of positive solutions to problem (1.3).
Zhao et al. [10] studied the following nonlinear fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $1 < \alpha \leq 2$. By the properties of the Green function, the lower and upper solution method and fixed point theorem, the authors prove the existence of multiple positive solutions to problem (1.4).
In this paper, analogy with boundary-value problem for differential equations of integer order, we firstly derive the corresponding Green function. Consequently, problem (1.1) is reduced to a equivalent integral equation. Finally, using some fixed-point theorem, the existence of nonnegative solutions are obtained.
We define

$$f^{\theta} = \lim_{x \to \theta} \sup_{0 \leq t \leq 1} \frac{f(t,x)}{x} \quad \text{and} \quad f_{\theta} = \lim_{x \to \theta} \inf_{0 \leq t \leq 1} \frac{f(t,x)}{x}, \tag{1.5}$$

where θ denotes either 0 or ∞. The main result of this paper is the following:

Theorem 1.1. *Assume that $f \in C([0,1] \times [0,\infty), [0,\infty))$, then, there exist tow positive constants m and M such that the fractional boundary value problem* (1.1) *has at least one nonnegative solution in one of the following tow cases:*
(i) $0 \leq f^0 \leq M$ *and* $m \leq f_{\infty} \leq \infty$. *or*
(ii) $0 \leq f^{\infty} \leq M$ *and* $m \leq f_0 \leq \infty$.

This paper is organized as follows. In Section 2, , we present some preliminaries that will be used to prove our results. The proof of Theorem 1.1 is given in Section 3.

2. Preliminaries

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [2,3,9].

Definition 2.1. *The Caputo fractional derivatives of order $\alpha > 0$ of a continuous function $u : (0,\infty) \to R$ is given by*

$$D_{0^+}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.1}$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of the real number α and provided that the right side integral is pointwise defined on $[0,\infty)$.

Definition 2.2. *The Riemann-Liouville standard fractional integral of order $\alpha > 0$ of a continuous function $u : (0,\infty) \to R$ is given by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \tag{2.2}$$

provided that the right side integral is pointwise defined on $[0,\infty)$.

Lemma 2.3. *let $n-1<\alpha\leq n$ $(n\in\mathbb{N})$. Then,*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{2.3}$$

for some $c_i\in\mathbb{R}$, $i=0,1,\ldots,n-1$.

Lemma 2.4. *let $\alpha>\beta>0$. If $u(t)\in C(0,1)\cap L(0,1)$, then,*

$$D_{0+}^{\beta}I_{0+}^{\alpha}u(t)=I_{0+}^{\alpha-\beta}u(t).$$

Lemma 2.5. *let $n-1<\alpha\leq n$ $(n\in\mathbb{N})$. The fractional differential equation $D_{0+}^{\alpha}x(t)=0$ has a solution*

$$x(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{2.4}$$

for some $c_i\in\mathbb{R}$, $i=0,1,\ldots,n-1$.

Theorem 2.6. *Let $h\in C(0,1)$, $0<\alpha\leq 1$ and $\lambda,\mu,\gamma\in\mathbb{R}$ such that $\lambda\neq\gamma+\mu$. Then, the solution of the boundary problem*

$$\begin{cases} D^{\alpha}u(t)=h(t), & 0<t<1,\\ \lambda u(0)-\mu u(1)=\gamma u(\eta), & 0<\eta<1, \end{cases} \tag{2.5}$$

can be represented by

$$u(t)=\int_0^1 G(t,s)h(s)ds,$$

where $G(t,s)$ is defined in $[0,1]\times[0,1]$ as:

$$\Gamma(\alpha)G(t,s)=\begin{cases} (t-s)^{\alpha-1}+\frac{\mu(1-s)^{\alpha-1}}{\lambda-\mu-\gamma}+\frac{\gamma(\eta-s)^{\alpha-1}}{\lambda-\mu-\gamma} & \text{if } 0\leq s\leq min(t,\eta),\\ \frac{\mu(1-s)^{\alpha-1}}{\lambda-\mu-\gamma}+\frac{\gamma(\eta-s)^{\alpha-1}}{\lambda-\mu-\gamma} & \text{if } 0\leq t\leq s\leq\eta\leq 1,\\ (t-s)^{\alpha-1}+\frac{\mu(1-s)^{\alpha-1}}{\lambda-\mu-\gamma} & \text{if } 0\leq s\leq t\leq\eta\leq 1,\\ \frac{\mu(1-s)^{\alpha-1}}{\lambda-\mu-\gamma} & \text{if } max(t,\eta)\leq s\leq 1 \end{cases} \tag{2.6}$$

Proof. Assume that u is a solution of equation (2.5), then, from Lemma 2.3 we get:

$$u(t)=I^{\alpha}h(t)+c_0+c_1t.$$

So, from the boundary condition $u'(0)=0$, we get

$$u(t)=I^{\alpha}h(t)+c_0.$$

On the other hand, from the boundary condition $\lambda u(0)-\mu u(1)=\gamma u(\eta)$, one has

$$c_0=\frac{\mu}{\lambda-\mu-\gamma}I^{\alpha}g(1)+\frac{\gamma}{\lambda-\mu-\gamma}I^{\alpha}g(\eta).$$

Consequently

$$\begin{aligned} u(t) &= I^{\alpha}h(t)+\frac{\mu}{\lambda-\mu-\gamma}I^{\alpha}g(1)+\frac{\gamma}{\lambda-\mu-\gamma}I^{\alpha}g(\eta)\\ &= \int_0^1 G(t,s)h(s)ds, \end{aligned}$$

where $G(t,s)$ is given by (2.6) □

Lemma 2.7. *Let $h \in C([0,1])$ be a given function, then, the function $G(t,s)$ defined by (2.6) has the following properties:*
(i) *$G(t,s) \in C([0,1]\times[0,1])$, and $G(t,s) > 0$ for all $t, s \in (0,1)$.*
(ii) *There exists a positive real number $\sigma \in (0,1)$ such that for all $(t,s) \in (0,1)\times(0,1)$ we have:*

$$\sigma k(s) \leq G(t,s) \leq k(s),$$

where

$$k(s) = \frac{(\lambda-\gamma)(1-s)^{\alpha-1} + \gamma(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)}.$$

Proof. It is not difficult to see that for all $(t,s) \in (0,1)\times(0,1)$, we have $G(t,s) > 0$. Now, let us prove assertion **(ii)**.
First, a simple calculation chows that for all $(T,s) \in (0,1)\times(0,1)$, we have

$$\frac{\mu(1-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)} \leq G(t,s) \leq k(s). \tag{2.7}$$

Put

$$\sigma = \frac{\mu}{\lambda} \in (0,1).$$

Then, we get

$$\begin{aligned} \frac{\mu(1-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)} - \sigma k(s) &= \frac{\mu(1-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)} \\ &\quad -\frac{\mu}{\lambda}\left(\frac{(\lambda-\gamma)(1-s)^{\alpha-1} + \gamma(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)}\right) \\ &= \frac{\mu\gamma}{\lambda\Gamma(\alpha)(\lambda-\mu-\gamma)}\left((1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right) \\ &\geq 0. \end{aligned} \tag{2.8}$$

That is

$$\frac{\mu(1-s)^{\alpha-1}}{\Gamma(\alpha)(\lambda-\mu-\gamma)} \geq \sigma k(s). \tag{2.9}$$

Combining (2.7) and (2.9), we obtain

$$\sigma k(s) \leq G(t,s) \leq k(s), \quad \forall\, (t,s) \in (0,1)\times(0,1).$$

□

The proof of our main results is based upon an application of the following fixed point theorems (See [5], [6]).

Theorem 2.8. *(Guo-Krasnoselskii) Let $(E, ||.||)$ be a Banach space, and $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$T:\, P \cap (\overline{\Omega_2}\backslash\Omega_1) \longrightarrow P$$

be a completely continuous operator such that either
(i) $||Tu|| \leq ||u||$ *for* $u \in P \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||$ *for* $u \in P \cap \partial\Omega_2$*; or*
(ii) $||Tu|| \geq ||u||$ *for* $u \in P \cap \partial\Omega_1$ *and* $||Tu|| \leq ||u||$ *for* $u \in P \cap \partial\Omega_2$*.*
Then T *has a fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$*.*

3. Proof of our main result

First, Let $E = C([0,1])$ be endowed with the maximum norm,

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|.$$

Define the cone $K \subset E$ by

$$K = \left\{ x \in E : u \geq 0 \text{ and } \min_{0<t<1} u(t) \geq \sigma \|u\| \right\}, \tag{3.1}$$

and define the operator $T : K \to E$ by:

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) ds, \tag{3.2}$$

where G is given by (2.6).

Lemma 3.1. *Assume that* $f \in C([0,1] \times [0,\infty), [0,\infty))$*, then the operator* $T : K \to K$ *is complettely continuous.*

Proof. Firstly, we prove that $T : K \to K$. Indeed, in view of nonnegativeness of $G(t,s)$ and $f(s,u(s))$, for any $u \in K$ we have $Tu(t) \geq 0$. On the other hand, in view of Lemma 2.7, for all $t \in (0,\sigma)$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) f(s,u(s)) ds \\ &\geq \sigma \int_0^1 k(s) f(s,u(s)) ds \\ &\geq \sigma \int_0^1 G(\tau,s) f(s,u(s)) ds, \end{aligned} \tag{3.3}$$

where τ is such that

$$\|Tu\| = Tu(\tau). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$Tu(t) \geq \sigma \|Tu\|, \;\; \forall \, t \in (0,\eta).$$

So,

$$\min_{0<t<1} Tu(t) \geq \sigma \|Tu\|.$$

Which implies that $T : K \to K$.
Let $P \subset K$ be bounded, then, there exists $L > 0$ such that

$$\|u\| \leq L, \ \forall \, u \in P.$$

Put $M = \max\limits_{0<t<1;0<s<L} f(t,s) + 1$, then, for $u \in P$ and from Lemma 2.7, we get

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))ds \leq M \int_0^1 k(s)ds.$$

Therefor, $T(P)$ is bounded. That is T is uniformly bounded.
Let us prove that T is equicontinuous. For all $u \in P$ and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have
case 1: $0 < t_1 < t_2 < \eta$. In this case we have:

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 (G(t_2,s) - G(t_1,s)) f(s,u(s))ds \right| \\
&\leq \int_0^{t_1} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{t_1}^{t_2} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{t_2}^{\eta} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{\eta}^{1} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\leq M \left(\int_0^{t_1} |G(t_2,s) - G(t_1,s)| \, ds + \int_{t_1}^{t_2} |G(t_2,s) - G(t_1,s)| \, ds \right) \\
&\quad + M \left(\int_{t_2}^{\eta} |G(t_2,s) - G(t_1,s)| \, ds + \int_{\eta}^{1} |G(t_2,s) - G(t_1,s)| \, ds \right) \\
&\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right) \\
&\leq \frac{M}{\Gamma(\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha} - 2(t_2 - t_1)^{\alpha} \right).
\end{aligned}$$

case 2: $0 < t_1 < \eta < t_2$. In this case we have:

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 (G(t_2,s) - G(t_1,s)) f(s,u(s))ds \right| \\
&\leq \int_0^{t_1} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{t_1}^{\eta} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{\eta}^{t_2} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds \\
&\quad + \int_{t_2}^{1} |G(t_2,s) - G(t_1,s)| \, f(s,u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\le M\left(\int_0^{t_1} |G(t_2,s)-G(t_1,s)|\,ds+\int_{t_1}^{\eta} |G(t_2,s)-G(t_1,s)|\,ds\right)\\
&\quad + M\left(\int_{\eta}^{t_2} |G(t_2,s)-G(t_1,s)|\,ds+\int_{t_2}^{1} |G(t_2,s)-G(t_1,s)|\,ds\right)\\
&\le\frac{M}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds\right)\\
&\quad+\frac{M}{\Gamma(\alpha)}\left(\int_{t_1}^{\eta}(t_2-s)^{\alpha-1}ds+\int_{\eta}^{t_2}(t_2-s)^{\alpha-1}ds\right)\\
&\le\frac{M}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}-2(t_2-t_1)^{\alpha}\right).
\end{aligned}
$$

case 3: $\eta < t_1 < t_2$. In this case we have:

$$
\begin{aligned}
|Tu(t_2)-Tu(t_1)| &= \left|\int_0^1 (G(t_2,s)-G(t_1,s))f(s,u(s))ds\right|\\
&\le \int_0^{\eta} |G(t_2,s)-G(t_1,s)|\, f(s,u(s))ds\\
&\quad+\int_{\eta}^{t_1} |G(t_2,s)-G(t_1,s)|\, f(s,u(s))ds\\
&\quad+\int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)|\, f(s,u(s))ds\\
&\quad+\int_{t_2}^{1} |G(t_2,s)-G(t_1,s)|\, f(s,u(s))ds\\
&\le M\left(\int_0^{\eta} |G(t_2,s)-G(t_1,s)|\,ds+\int_{\eta}^{t_1} |G(t_2,s)-G(t_1,s)|\,ds\right)\\
&\quad + M\left(\int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)|\,ds+\int_{t_2}^{1} |G(t_2,s)-G(t_1,s)|\,ds\right)\\
&\le\frac{M}{\Gamma(\alpha)}\left(\int_0^{\eta}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds\right)\\
&\quad+\frac{M}{\Gamma(\alpha)}\left(\int_{\eta}^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds\right)\\
&+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&\le\frac{M}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}-2(t_2-t_1)^{\alpha}\right).
\end{aligned}
$$

Since t^{α} is uniformly continuous when $t \in [0,1]$ and $1 < \alpha \le 2$, it's easy to prove that $T(P)$ is equicontinuous. The Arzela-Ascoli Theorem implies that $\overline{T(P)}$ is compact. That is, $T : K \longrightarrow K$ is completely continuous. □

We will prove that T has a fixed point in K in the case:

$$0 \le f^0 \le M \ \text{ and } \ m \le f_\infty \le \infty,$$

where m and M are given by

$$m = \left(\sigma^2 \int_0^1 k(s)ds\right)^{-1} \text{ and } M = \left(\int_0^1 k(s)ds\right)^{-1}.$$

Since $0 \leq f^0 \leq M$, we may choose $R_1 > 0$ such that for each $0 \leq x \leq R_1$ and $t \in [0,1]$, we have:

$$f(t,x) \leq Mx. \tag{3.5}$$

Put

$$\Omega_1 = \{u \in E : ||u|| < R_1\}.$$

Then, it follows from (3.5) and Lemma 2.6 that for all $(t,u) \in I \times (K \cap \partial\Omega_1)$

$$\begin{aligned} Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds &\leq \int_0^1 k(s)Mu(s)ds \\ &\leq M||u|| \int_0^1 k(s)ds = ||u||. \end{aligned}$$

Hence

$$||Tu|| \leq ||u|| \ \ \forall\, u \in K \cap \partial\Omega_1. \tag{3.6}$$

On the other hand, since $m \leq f_\infty \leq \infty$, we may choose $R > 0$ such that

$$f(t,x) \geq mx, \ \ \forall x \geq R. \tag{3.7}$$

Let $R_2 = max(2R, \frac{R}{\sigma})$ and

$$\Omega_2 = \{u \in E : ||u|| < R_2\}.$$

It follows that for all $u \in K \cap \partial\Omega_2$ and $t \in [0,1]$, we have

$$u(t) \geq \sigma||u|| = \sigma R_2 \geq R.$$

So, we deduce by (3.7) and Lemma 2.6, that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \int_0^1 \sigma k(s)mu(s)ds \\ &\geq m\sigma^2\|u\| \int_0^1 k(s)ds = ||u||. \end{aligned}$$

Consequently,

$$||Tu|| \geq ||u|| \ \ \ \forall\, u \in K \cap \partial\Omega_2. \tag{3.8}$$

Combining (3.6) and (3.8), it follows from the first part of Theorem 2.8 that T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Now, we consider the case: $0 \leq f^{\infty} \leq M$ and $m \leq f_0 \leq \infty$.
Since $m \leq f_0 \leq \infty$, we may choose $R_3 > 0$ such that for all $t \in [0,1]$, we have

$$f(t,x) \geq mx \text{ for all } 0 \leq x \leq R_3. \tag{3.9}$$

Let

$$\Omega_3 = \{u \in E : ||u|| < R_3\}.$$

Then, using (3.9) and Lemma 2.6, we obtain for $u \in K \cap \partial\Omega_3$ and $t \in [0,1]$

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \int_0^1 \sigma k(s)mu(s)ds \\ &\geq m\sigma \int_0^1 k(s)\sigma\|u\|ds \\ &\geq m\sigma^2||u|| \int_0^1 k(s)ds = ||u||. \end{aligned}$$

So,

$$||Tu|| \geq ||u||, \ \forall u \in K \cap \partial\Omega_3. \tag{3.10}$$

Now, Since $0 \leq f^{\infty} \leq M$, there exists $R > 0$ such that if $x \geq R$, then $f(t,x) \leq Mx$.
Let $R_4 = max\{2R_3, R\}$, and put

$$\Omega_4 = \{u \in E : ||u|| < R_4\}.$$

Then, we obtain for any $u \in K \cap \partial\Omega_4$ and $t \in [0,1]$:

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq M\int_0^1 k(s)u(s)ds \\ &\leq M\|u\| \int_0^1 k(s)ds = \|u\| \end{aligned}$$

So,

$$||Tu|| \leq ||u||, \ \ \forall u \in K \cap \partial\Omega_4. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain from the second part of Theorem 2.8 that the operator T has a fixed point in $K \cap (\overline{\Omega_4} \backslash \Omega_3)$. This completes the proof.

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