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On the Extremal Solutions of Superlinear Helmholtz Problems

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ABSTRACT: In this note, we deal with the Helmholtz equation $-\Delta u + cu = \lambda f(u)$ with Dirichlet boundary condition in a smooth bounded domain Ω of $\mathbb{R}^n$, $n > 1$. The nonlinearity is superlinear that is $\lim_{t\longrightarrow\infty} \frac{f(t)}{t} = \infty$ and f is a positive, convexe and C^1 function defined on $[0,\infty)$. We establish existence of regular solutions for λ small enough and the bifurcation phenomena. We prove the existence of critical value λ^* such that the problem does not have solution for $\lambda > \lambda^*$ even in the weak sense.
We also prove the existence of a type of stable solutions u^* called extremal solutions. We prove that for $f(t) = e^t$, $\Omega = B_1$ and $n \leq 9$, u^* is regular.

Key Words: Extremal solution, Stable minimal solution, Regularity, Super-nonlinearity.

Contents

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$, $c > 0$ a positive real parameter and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. The semilinear elliptic equation

$$\begin{cases} -\Delta u + cu = g(x,u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

has by now been widely investigated under various assumption on the nonlinearity g.
In this paper, we will suppose that

$$g(x,t) = \lambda f(t), \tag{1.2}$$

where f is C^1, positive, nondecreasing and convex function on $[0,+\infty)$ satisfying

$$\lim_{t\longrightarrow\infty} \frac{f(t)}{t} = \infty. \tag{1.3}$$

The condition (1.3) means that f is a superlinear function and the choice of the function g is motivated by the role of bifurcation problem in applied mathematics and which has been synthesized by Kielhöfer [6]. We say that a problem has a bifurcation if any change of its parameters cause a sudden change of regime and this is occur in nonlinear physics where the phenominon usually depends on a number of parameters, that control the evolution of the system.
If $g(x,t) = \lambda f(t)$ and f is asymptotically linear that is $\lim_{t\longrightarrow\infty} \frac{f(t)}{t} = a < \infty$, the problem

$$(P_{\lambda,c}) \qquad \begin{cases} -\Delta u + cu = \lambda f(t) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

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was treated by Dammak et al. in [1] where the hypothese $f(0) > 0$ was fondamental. The authors prove the existence of a critical value λ^* such that for $\lambda < \lambda^*$, the problem (1.4) has at least one solution, for $\lambda > \lambda^*$ the problem (1.4) has no solution and for $\lambda = \lambda^*$, the existence of a solution, named extremal solution depends of the signe of $\lim_{t \longrightarrow \infty} (f(t) - at)$.

If $c \equiv 0$ and $g(x,t) = \lambda f(t)$, the problem (1.4) has been treated by many authors. For the superlinear case, we can cite [3] and for the asymptotically linear and $f(0) > 0$, we can see [8] and their references.
In this work, we take the following definition of a weak solution.

Definition 1.1. *A weak solution of (1.4) is a function $u \in L^1(\Omega)$, $u \geq 0$ such that $f(u) \in L^1(\Omega)$, and*

$$- \int_\Omega u \Delta \zeta + c \int_\Omega \zeta u = \lambda \int_\Omega f(u)\zeta, \tag{1.5}$$

for all $\zeta \in C^2(\overline{\Omega})$ and $\zeta = 0$ on $\partial\Omega$.
Moreover, we say that u is weak super solution of (1.4) if the " $=$ " is replaced by " $\geq$ " for all functions $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$ and $\zeta \geq 0$.

If a weak solution $u \in L^\infty(\Omega)$, we say that u is regular while if $u \notin L^\infty(\Omega)$, we say that u is singular. We say that a solution u of (1.4) is minimal if $u \leq v$ in Ω for any solution v of problem (1.4).

Remark 1.2. *If u is a regular solution of (1.4), then by standard bootstrap argument and elliptic regularity, u is a classical solution.*

For regular solution, we will study the stability properties.
Let

$$I(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + c\, u^2) dx - \lambda \int_\Omega F(u) dx, \tag{1.6}$$

for $u \in H^1_0(\Omega)$ and where

$$F(u) = \int_0^u f(s) ds. \tag{1.7}$$

u is a solution of (1.4) if it is a critical point of the fonction I. The second variation of the energy is given by

$$Q(\varphi) = \int_\Omega |\nabla \varphi|^2 + \int_\Omega (c - f'(u))\varphi^2, \tag{1.8}$$

for all $\varphi \in H^1_0(\Omega)$.

Definition 1.3. *We say that a regular solution u of (1.4) is stable if the second variation of energy Q, satisfies $Q(\varphi) \geq 0$ for all $\varphi \in H^1_0(\Omega)$. Otherwise, we say that u is unstable.*

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ a smooth bounded domain and assume that f is a function satisfying (1.3). Then there exists a critical value $\lambda^* \in (0, \infty)$ such that*

1. *For any $\lambda \in (0, \lambda^*)$, problem (1.4) has a minimal solution u_λ, which is regular and the map $\lambda \longmapsto u_\lambda$ is increasing.*
   Moreover, u_λ is the unique stable solution of (1.4).
2. *For $\lambda = \lambda^*$, the problem (1.4) admits a unique weak solution u^*, $u^* = \lim_{\lambda \longrightarrow \lambda^*} u_\lambda$, called the extremal solution.*
3. *For $\lambda > \lambda^*$, (1.4) admits no weak solution.*

Theorem 1.4 applies to the existence of stable solution for all $\lambda < \lambda^*$. For the case $\lambda = \lambda^*$, we prove the following result.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain and assume that f satisfies condition (1.3). Let $v \in H_0^1(\Omega)$ be a singular weak solution of (1.4). Then, the following facts are equivalent:*
(i)

$$\int_\Omega |\nabla\varphi|^2 + c\int_\Omega \varphi^2 dx \geq \lambda \int_\Omega f'(v)\varphi^2 dx \quad \forall \varphi \in C_0^1(\Omega) \tag{1.9}$$

(ii) $v = u^*$ and $\lambda = \lambda^*$.

As consequence if the problem (1.4) has a singular solution that is "stable" then necessary $\lambda = \lambda^*$ the extremal value for which the problem has solution.
In the case $c \equiv 0$, we prove the following result which assert that u^* is regular for $n \leq 9$.

Theorem 1.6. *Assume that $\Omega = B_1$, $n \geq 2$, and that $f(u) = e^u$. Then $u^* \in L^\infty(\Omega)$, for all $n \leq 9$ and so it is a regular solution.*

For $n \geq 10$ and $c = 0$, u^* is a singular solution of (1.4) [4,5] but for $c \neq 0$ the problem still an open one and this is due to the missing of an adequate Hardy ineguality.

2. Technical Lemmas

In all this section, we suppose that Ω is a smooth bounded subset of $\mathbb{R}^n$, $n \geq 2$. For proving our first theorem, we need to prove auxiliary results.

Lemma 2.1. *Given $g \in L^1(\Omega)$, there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} -\Delta v + cv &= g \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

in the sense that

$$\int_\Omega v(-\Delta\zeta + c\zeta) = \int_\Omega g\zeta, \quad \text{for all} \quad \zeta \in C^2(\overline{\Omega}) \text{ and } \zeta = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Moreover

$$\|v\|_{L^1(\Omega)} \leq c_0 \|g\|_{L^1(\Omega,\,\delta(x)dx)} \tag{2.3}$$

for some constant $c_0 > 0$ independent of g. In addition, if $g \geq 0$ in Ω, then $v \geq 0$ in Ω.

Proof. The uniqueness. Let v_1 and v_2 be two solutions of problem (2.1), then $v = v_1 - v_2$ satisfies

$$\int_\Omega v(-\Delta\zeta + c\zeta) = 0, \quad \forall \zeta \in C^2(\overline{\Omega}) \text{ and } \zeta = 0 \text{ on } \partial\Omega. \tag{2.4}$$

Given any $\varphi \in \mathcal{D}(\Omega)$, let ζ be solution of

$$\begin{cases} -\Delta\zeta + c\zeta &= \varphi \quad \text{in} \quad \Omega \\ \zeta &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

$\zeta \in C^2(\overline{\Omega})$ and $\zeta = 0$ on $\partial\Omega$. It follows that

$$\int_\Omega v\varphi = \int_\Omega v(-\Delta\zeta + c\zeta) = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.
The existence. We assume that $g \geq 0$, if not we write $g = g^+ - g^-$.
Given an integer $k \geq 0$, and set $g_k(x) = \min\{g(x), k\}$. By the monotone convergence theorem, we have $g_k \xrightarrow[k\to\infty]{} g$ in $L^1(\Omega)$. Since g_k is in $L^2(\Omega)$, the following problem

$$\begin{cases} -\Delta v_k + cv_k &= g_k \quad \text{in} \quad \Omega \\ v_k &= 0 \quad \text{on} \quad \partial\Omega \\ v_k &> 0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.6}$$

admits a unique solution v_k.
The sequence (g_k) is nondecreasing, then (v_k) is nondecreasing sequence also. Let $k > l > 0$ two integers and ζ_0 the solution of

$$\begin{cases} -\Delta\zeta_0 + c\zeta_0 &= 1 \quad \text{in} \quad \Omega \\ \zeta_0 &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.7}$$

we have

$$\int_\Omega (v_k - v_l) = \int_\Omega (g_k - g_l)\zeta_0,$$

hence

$$|\int_\Omega (v_k - v_l)| = \int_\Omega |v_k - v_l| \leq C \int_\Omega |g_k - g_l| dx.$$

Since $g_k \xrightarrow[k\longrightarrow\infty]{} g$ in $L^1(\Omega)$, the sequence (v_k) is a Cauchy sequence in the Banach space $L^1(\Omega)$ then (v_k) converges in $L^1(\Omega)$, denote by v its limit. Passing to the limit in (2.6), we oblain (2.2). So v is a weak solution of the equation (2.1). Finally, taking $\zeta = \zeta_0$ in (2.2), we obtaine (2.3). □ □

Lemma 2.2. *Suppose that f is a function satisfies (1.3) and let $\overline{u}$ be a weak super solution of (1.4), then there exists a weak solution u of the problem (1.4) with $0 \leq u \leq \overline{u}$.*

Proof. We use a standard monotone iteration argument. Let $u_1 = 0$ and let $(u_n)_n$ the sequences defined by:

$$\begin{cases} -\Delta u_n + cu_n &= \lambda f(u_{n-1}) \quad \text{in} \quad \Omega \\ u_n &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.8}$$

By maximum principle we have $u_1 = 0 \leq u_2 \leq ... \leq u_n \leq u_{n+1} \leq ... \leq \overline{u}$. Since the sequence u_n is nondecreasing, it converges to a limit $u \in L^1(\Omega)$, which is clearly a weak solution of (1.4). Moreover u is independent of the choice of the super solution $\overline{u}$. □

Next, let φ_1 the positive normalized eigenfunction associated to the first eigenvalue of $-\Delta + c$ in Ω with Dirichlet boundary condition, λ_1, that is

$$\begin{cases} -\Delta\varphi_1 + c\varphi_1 &= \lambda_1\varphi_1 \quad \text{in} \quad \Omega \\ \varphi_1 &= 0 \quad \text{on} \quad \partial\Omega \\ ||\varphi_1||_2 &= 1, \end{cases} \tag{2.9}$$

and let $r_0 = \inf\limits_{t>0} \dfrac{f(t)}{t}$, we have the following result.

Lemma 2.3. *Let f be a function satisfying (1.3), problem (1.4) has no solution for any $\lambda > \frac{\lambda_1}{r_0}$ but has solution provided λ is positive and small enough.*

Proof. Let $\xi \in C^2(\overline{\Omega})$ satisfying $-\Delta\xi + c\xi = 1$ in Ω and $\xi = 0$ on $\partial\Omega$. For $\lambda \leq \frac{1}{f(||\xi||_\infty)}$, ξ is a super solution of (1.4), so from Lemma 2, equation (1.4) has a weak solution u such that $0 \leq u \leq \xi$. Also u is regular then classical solution of (1.4) and from the maximum principle, we have $u > 0$ in Ω.
Now, if (1.4) has a solution u for some $\lambda > 0$, take φ_1 a test function, we have

$$\int_\Omega (-\Delta\varphi_1 + c\varphi_1)u = \lambda \int_\Omega f(u)\varphi_1$$

$$\int_\Omega \lambda_1\varphi_1 u = \lambda \int_\Omega f(u)\varphi_1$$

$$\int_\Omega \lambda_1\varphi_1 u \geq r_0\lambda \int_\Omega \varphi_1 u$$

since $\varphi_1 > 0$ and $u > 0$ we have $\lambda \leq \frac{\lambda_1}{r_0}$, this complete the proof. □

We define now

$$\Lambda = \{\lambda > 0 \quad \text{such} \quad \text{that} \quad \text{problem} \quad (P_{\lambda,c}) \quad \text{has} \quad a \quad \text{solution}\},$$

and

$$\lambda^* = \sup \Lambda.$$

From Lemma 2.3 we know that $\lambda^* < \infty$ and we have the following result.

Lemma 2.4. *Let f a reaction term satisfying (1.3), if the problem $(P_{\lambda,c})$ has a solution for some λ. Then*

(i) There exists a minimal solution denoted by u_λ for $(P_{\lambda,c})$.

(ii) For any $\lambda' \in (0, \lambda)$, the problem $(P_{\lambda',c})$ has a solution.

Proof. (i) Let v be a solution of $(P_{\lambda,c})$, by lemma 2 and since v is regular solution, there exist a solution u such that $0 < u \leq v$ and by construction u is independent of the choice of v (see the proof of Lemma 2). We denote by u_λ this solution. u_λ is a minimal solution.

(ii) For any $\lambda' \in (0, \lambda)$, u_λ is a super solution of $(P_{\lambda',c})$. By Lemma 2, $(P_{\lambda',c})$ has a weak solution $u_{\lambda'}$ such that $0 \leq u_{\lambda'} \leq u_\lambda$ and so $u_{\lambda'}$ is a regular solution for $(P_{\lambda',c})$.

□

3. Proof of Theorem 1.4

(i) By lemma 2.3 and lemma 2.4, Λ is an interval. Then, by definition of λ^*, if $\lambda \in (0, \lambda^*)$, the problem (1.4) has a minimal solution u_λ and the map $\lambda \longmapsto u_\lambda$ is increasing.
To prove that u_λ is stable, we suppose that the first eigenvalue $\eta_1 = \eta_1(c, \lambda, u_\lambda)$ of the operator $-\Delta + c - \lambda f'(u_\lambda)$ is negative. We define $\psi \in H_0^1(\Omega)$ a positive eigenfunction associate to η_1 with Dirichlet boundary condition.

Consider $u^\varepsilon = u_\lambda - \varepsilon\psi$, $\varepsilon > 0$, so

$$\begin{aligned} -\Delta u^\varepsilon + cu^\varepsilon - \lambda f(u^\varepsilon) &= -\varepsilon\eta_1\psi - \lambda[f(u_\lambda - \varepsilon\psi) - f(u_\lambda) + \varepsilon f'(u_\lambda)\psi] \\ &= -\varepsilon\psi[-\eta_1 + \theta_\varepsilon(1)]. \end{aligned}$$

Since $\eta_1 < 0$, then $-\Delta u^\varepsilon + cu^\varepsilon - \lambda f(u^\varepsilon) \geq 0$ in Ω for ε small enough, and by Hopf's Lemma, $u^\varepsilon \geq 0$, so u^ε is a super solution of (1.4) for ε small enough, then from Lemma 2 we can get a solution u of (1.4) such that $u \leq u^\varepsilon$ in Ω. So we have $0 \leq u \leq u^\varepsilon < u_\lambda$ and this contradicts the minimality of u_λ and hence $\eta_1 \geq 0$.

To prove that u_λ is the unique stable solution of (1.4), we suppose that there exists another stable solution $v \neq u_\lambda$ and we denote $\varphi = v - u_\lambda$.
We get from the stability properties

$$\begin{aligned} \lambda \int_\Omega f'(v)\varphi^2 &\leq -\int_\Omega \varphi\Delta\varphi + c\int_\Omega \varphi^2 \\ &\leq \int_\Omega (-\Delta\varphi + c\varphi)\varphi \\ &\leq \int_\Omega \lambda(f(v) - f(u_\lambda)\varphi. \end{aligned} \tag{3.1}$$

So

$$\int_\Omega [f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)]\varphi \geq 0. \tag{3.2}$$

We know that $\varphi > 0$ by maximum principle and by convexity of f, we have

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) \leq 0. \tag{3.3}$$

From (3.2) and (3.3), we have

$$f(v) - f(u_\lambda) = f'(v)(v - u_\lambda)$$

this means that f is affine over $[u_\lambda(x), v(x)]$ thus $f(x) = ax + b$ in $[0, \max_\Omega v]$ and we get two solutions u_λ and v of

$$\begin{cases} -\Delta w + cw = & \lambda(aw + b) \quad \text{in} \quad \Omega \\ w = & 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

This implies

$$0 = \int_\Omega (u_\lambda \Delta v - v\Delta u_\lambda)dx = \lambda b \int_\Omega (v - u_\lambda)dx = \lambda b \int_\Omega \varphi(x)dx, \tag{3.4}$$

which implies $b = f(0) = 0$, this is impossible since $f(0) > 0$. So u_λ is the unique stable solution of $(P_{\lambda,c})$. □

(ii) We denote by u^* the limit $u^* = \lim\limits_{\lambda \longrightarrow \lambda^*} u_\lambda$ and in this step We use a technical proceeding inspired from [3].

For any $\lambda \in [\frac{\lambda^*}{2}, \lambda^*)$, taking φ_1 defined by (2.9) as a test function, we obtain

$$\begin{aligned} \lambda_1 \int_\Omega u_\lambda \varphi_1 &= \int_\Omega (-\Delta\varphi_1 + c\varphi_1)u_\lambda \\ &= \int_\Omega (-\Delta u_\lambda + cu_\lambda)\varphi_1 \\ &= \lambda \int_\Omega f(u_\lambda)\varphi_1 \\ &\geq \frac{\lambda^*}{2} \int_\Omega f(u_\lambda)\varphi_1. \end{aligned} \tag{3.5}$$

Since f is super linear, there exists $c_1 > 0$ such that $\lambda_1 t \leq \frac{\lambda^*}{4} f(t) + c_1$ in $\mathbf{R}_+$. Using (3.5), we get

$$\begin{aligned} \frac{\lambda^*}{2} \int_\Omega \varphi_1 f(u_\lambda)dx - \frac{\lambda^*}{4} \int_\Omega \varphi_1 f(u_\lambda)dx & \\ \leq \lambda_1 \int_\Omega \varphi_1 u_\lambda dx - \frac{\lambda^*}{4} \int_\Omega \varphi_1 u_\lambda dx & \\ \leq \int_\Omega c_1 \varphi_1 dx \leq c_1. & \end{aligned} \tag{3.6}$$

So (3.6) yields

$$\int_\Omega f(u_\lambda)\varphi_1 dx \leq c_2. \tag{3.7}$$

Where $c_2 \geq 0$ is a constant. Let ζ_0 the function given by (2.7), we have

$$\begin{aligned} \int_\Omega u_\lambda dx = \int_\Omega u_\lambda.1dx &= \int_\Omega u_\lambda(-\Delta\zeta_0 + c\zeta_0)dx \\ &= \int_\Omega (-\Delta u_\lambda + cu_\lambda)\zeta_0 dx \\ &= \lambda \int_\Omega f(u_\lambda)\zeta_0 dx. \end{aligned}$$

Using the Hopf's Lemma we deduce that $\zeta_0 \leq c_3\varphi_1$ and (3.7) implies

$$\int_\Omega u_\lambda dx \leq c_3 \int_\Omega \varphi_1 f(u_\lambda) \leq c_4. \tag{3.8}$$

By (3.7) and (3.8), we deduce by passing to the limit that $u^* \in L^1(\Omega)$ and $f(u^*) \in L^1(\Omega)$ and u^* satisfy $(P_{\lambda^*,c})$ and hence u^* is a weak solution of $(P_{\lambda^*,c})$.

Now to prove the uniqueness of u^*, we can use the following result due to Martel [7] and the proof is not changed in our case, so we omit it.

Proposition 3.1. *[7] Let $v \in L^1(\Omega)$ be a weak super solution of equation $(P_{\lambda^*,c})$, then $v = u^*$.*

4. Proof of Theorem 1.5

Recall that the extremal solution u^* is the increasing limit of classical stable solutions u_λ and we have

$$\lambda \int_\Omega f'(u_\lambda)\varphi^2 dx \le \int_\Omega |\nabla\varphi|^2 dx + c\int_\Omega \varphi^2 dx, \quad \forall \varphi \in C_0^1(\Omega)$$

and so by passing to the limit, we obtain

$$\lambda \int_\Omega f'(u^*)\varphi^2 dx \le \int_\Omega |\nabla\varphi|^2 dx + c\int_\Omega \varphi^2 dx, \quad \forall \varphi \in C_0^1(\Omega).$$

Conversely, if we have a singular solution v satisfying (1.9) for some $\lambda > 0$ and we shoud prove that $\lambda = \lambda^*$ and this solution is the extremal one u^*. We argue by contradiction, suppose that $\lambda < \lambda^*$. We take $\varphi = v - u_\lambda$ as test function in (1.9) where u_λ is the minimal solution. Exploiting the boundary conditions, we get

$$\begin{aligned}
\lambda \int_\Omega (v-u_\lambda)(f(v)-f(u_\lambda))dx &= \int_\Omega (v-u_\lambda)(-\Delta(v-u_\lambda)+c(v-u_\lambda))dx \\
&= \int_\Omega |\nabla(v-u_\lambda)|^2 + \int_\Omega c(v-u_\lambda)^2 \\
&\ge \lambda \int_\Omega f'(v)(v-u_\lambda)^2 dx.
\end{aligned}$$

Then, by convexity of the function f, we have $v = u_\lambda$. But u_λ is regular, and this contradicts the fact that v is singular. So $\lambda = \lambda^*$ and by uniqueness of the solutions of problem $(P_{\lambda^*,c})$, $v = u^*$. □

5. Proof of Theorem 1.6

For every $\lambda \in (0, \lambda^*)$, we know that the minimal solution u_λ satisfies the equation

$$\int_\Omega \nabla u_\lambda \nabla v dx + c\int_\Omega u_\lambda v dx = \lambda \int_\Omega f(u_\lambda) v dx = \lambda \int_\Omega e^{u_\lambda} v dx; \tag{5.1}$$

for all $v \in H^1(\Omega)$.

Also u_λ satisfies the stability condition

$$\int_\Omega |\nabla w|^2 dx + c\int_\Omega w^2 dx \ge \lambda \int_\Omega f'(u_\lambda) w^2 dx = \lambda \int_\Omega e_\lambda^u w^2 dx, \tag{5.2}$$

for all $w \in C_0^1(\Omega)$.

To prove the regularity of u^* for $n \le 9$, we generalise the idea of [2].

In (5.1) we take $v = e^{(q-1)u_\lambda}$ as a test function and $w = e^{\frac{q-1}{2}u_\lambda}$, where $q > 1$, we obtain

$$(q-1)\int_\Omega e^{(q-1)u_\lambda}|\nabla u_\lambda|^2 dx + c\int_\Omega e^{(q-1)u_\lambda} dx = \lambda \int_\Omega e^{qu_\lambda} dx \tag{5.3}$$

and

$$\frac{(q-1)^2}{4}\int_\Omega |\nabla u_\lambda|^2 e^{(q-1)u_\lambda} dx + c\int_\Omega e^{(q-1)u_\lambda} dx \ge \lambda \int_\Omega e^{qu_\lambda} dx \tag{5.4}$$

By multiplying (5.4) with $\frac{4}{q-1}$ and putting together these inequalities, we obtain

$$\frac{4c}{q-1}\int_\Omega e^{(q-1)u_\lambda}dx - c\int_\Omega u_\lambda e^{(q-1)u_\lambda}dx \geq \lambda(\frac{4}{q-1}-1)\int_\Omega e^{qu_\lambda}dx$$

Now assume that $1 < q < 5$, so that $\frac{4}{q-1} > 1$. As $\lambda \longrightarrow \lambda^*$, the left hand side cannot blow-up since the leading term is $u_\lambda e^{(q-1)u_\lambda}$ and the right hand side remains bounded, this means that e^{u_λ} is uniformly bounded in $L^q(\Omega)$, since u_λ solves the equation, by elliptic regularity this means that u_λ is uniformly bounded in $W^{2,q}(\Omega)$ for all $1 < q < 5$. Since $n \leq 9$, by Sobolev embedding, u_λ is uniformly bounded in $L^\infty(\Omega)$ so that $u^* \in L^\infty(\Omega)$. □

References

1. I. Abid, M. Dammak and I. Douchich, *Stable solutions and bifurcation problem for asymptotically linear Helmholtz equations*, Nonl. Funct. Anal. and Appl, 21 (2016), 15-31.
2. E Berchio, F. Gazzola, D. Pierotti, *Gelfand type elliptic problems under Steklov boundary problem,* Ann. I. H. Poincaré -AN. Vol. 27 (2010), 315-335.
3. H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for* $u_t - \Delta u = g(u)$ revisited, Adv. Diff. Equa. 1 (1996), 73-90.
4. H. Brezis, J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems,* Rev. Mat. Univ. Complut. Madrid 10 (1997), 443-469.
5. M. G. Crandall, P. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems,* Arch. Rat. Mech. Anal. 58 (1975), 207-218.
6. H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations* (Springer-Verlag), Berlin, 2003.
7. Y. Martel *Uniqueness of weak extremal solutions of nonlinear elliptic problems* Houston J. Math. 23 (1997), 161-168.
8. P. Mironescu and V. Rădulescu, *The study of a bifurcation problem associated to an asymtotically linear function*, Nonlinear Anal. 26 (1996), 857-875.

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