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On Several New Laplace Transforms of Generalized Hypergeometric Functions ${}_2F_2(x)$

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ABSTRACT: By employing generalizations of Gauss's second, Bailey's and Kummer's summation theorems obtained earlier by Rakha and Rathie, we aim to establish presumably new Laplace transforms of six rather general formulas of generalized hypergeometric function ${}_2F_2\,[a, b; c, d; x]$.

The results obtained in this paper are simple, interesting, easily established and may be useful in theoretical physics, engineering and mathematics. Results presented here are pointed out to reduce to yield some known results.

Key Words: Hypergeometric Summation Theorems, Laplace Transform.

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1. Introduction

The Laplace transform has many applications in physics and engineering (including mechanical and electronics). It is also used in process control. It has the usual property that many relationships and operations over the original function $f(t)$ correspond to simpler relationships and operation over image $g(s)$. It is named after Pierre-Simon Laplace, who introduced the transform in his work of probability theory. The Laplace transform is a linear operator that switches a function $f(t)$ to $F(s)$, goes from time argument with real input to a complex angular frequency input which is complex, so we define the Laplace transform or (direct Laplace transformation) of $f(t)$, where $f(t)$ is defined for $t \geq 0$, as the following integral:

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$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt \tag{1.1}$$
$$= \lim_{\tau\to\infty} \int_0^\tau e^{-s\tau} f(\tau) d\tau$$

whenever the limit exists (as a finite number). When it does, the integral (1.1) is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for $f(t)$. The notation $\mathcal{L}\{f(t)\}$ used to denote the Laplace transform of $f(t)$, and the integral is the ordinary Riemann (improper) integral. The parameter s belongs to some domain on the real line or in the complex plane. In a mathematical and technical sense, the domain of s is quite important such that for any given signal the Laplace transform converges for a range of values of s. This range is referred to as the region of convergence (ROC) and plays an important role in specifying the Laplace transform associated with a given signal. For more details we refer for examples to [2,3,11]. Further, if we put $f(t) = t^{\alpha-1}$ in (1.1), we get the known formula :

$$\int_0^\infty e^{-st} t^{\alpha-1}\, dt = \Gamma(\alpha) s^{-\alpha}, \tag{1.2}$$

which is valid when $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(\alpha) > 0$ and also consider as a relation between Laplace transformation and Gamma function.

If we perform with generalized hypergeometric function

$${}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ & & \\ b_1, & \ldots, & b_q \end{matrix} ; \ z\right] = \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}, \ (p \le q),$$

the Laplace transform of a generalized hypergeometric function ${}_pF_q$ is given as

$$\int_0^\infty e^{-st} t^{\upsilon-1}\ {}_pF_q\left[\begin{matrix} a_1, & ..., & a_p \\ & & \\ b_1, & ..., & b_q \end{matrix} ; \ \omega t\right] dt$$
$$= \Gamma(\upsilon)\ s^{-\upsilon}\ {}_{p+1}F_q\left[\begin{matrix} \upsilon, a_1, & ..., & a_p \\ & & \\ b_1, & ..., & b_q \end{matrix} ; \ \tfrac{\omega}{s}\right] \tag{1.3}$$

provided that $p < q,\ \mathrm{Re}(\upsilon) > 0,\ \mathrm{Re}(s) > 0$ and ω arbitrary, or when $p = q > 0,\ \mathrm{Re}(\upsilon) > 0$ and $\mathrm{Re}(s) > Re(\omega)$.

- When $p = q = 1$, for generalized hypergeometric function, we define its Laplace transform as

$$\int_0^\infty e^{-st} t^{d-1}\ {}_1F_1\left[\begin{matrix} a, \\ \\ c \end{matrix} ;\omega t\right] dt = \Gamma(d)\ s^{-d}\ {}_2F_1\left[\begin{matrix} a, & d \\ & \\ c & \end{matrix} ; \tfrac{\omega}{s}\right] \tag{1.4}$$

where $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(s) > \max\{\mathrm{Re}(\omega), 0\}$.

- When $p = q = 2$, we have

$$\int_0^\infty e^{-st} t^{d-1}\ {}_2F_2\left[\begin{matrix} a, & b \\ & & ;\omega t \\ c, & d \end{matrix}\right] dt = \Gamma(d)\ s^{-d}\ {}_2F_1\left[\begin{matrix} a, & b \\ & & ;\frac{\omega}{s} \\ c & \end{matrix}\right] \tag{1.5}$$

where $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(s) > \max\{\mathrm{Re}(\omega), 0\}$.

Similarly, we can write Laplace transforms for the function ${}_3F_3$.

On the other hand, the theory of hypergeometric and generalized hypergeometric functions are fundamental in the field of mathematics, mathematical physics, engineering and statistics.

Lately, a good advance has been done in finding generalizations and extensions of the Gauss, Gauss's second, Kummer, and Bailey for the series ${}_2F_1$, Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ classical summation theorems. For generalizations, we refer, for example, to [4,5,6,9,10].

In 2011, Rakha and Rathie [10] further generalized these summation theorems in the most general form. However in our present investigations, we shall mention the generalization of Gauss's second summation theorem, Bailey summation theorem and Kummer summation theorem.

- **Generalizations of Gauss's second summation theorem** [10]

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ \frac{1}{2} \\ \frac{1}{2}(a+b+i+1) & \end{matrix}\right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \frac{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}-\frac{1}{2}i)}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})} \\ &\times \sum_{r=0}^{i} \binom{i}{r} \frac{(-1)^r\,\Gamma(\frac{1}{2}b+\frac{1}{2}r)}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})},\ \ i = 0, 1, \ldots \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a, & b \\ & & ;\ \frac{1}{2} \\ \frac{1}{2}(a+b-i+1) & \end{matrix}\right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \\ &\times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\frac{1}{2}b+\frac{1}{2}r)}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})},\ \ i = 0, 1, \ldots \end{aligned} \tag{1.7}$$

- **Generalizations of Bailey's summation theorem** [10]

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad 1-a+i \\ c \end{matrix} ; \frac{1}{2}\right] \\
&= \frac{2^{1+i-c}\,\Gamma(\frac{1}{2})\Gamma(c)\Gamma(a-i)}{\Gamma(a)\Gamma(\frac{1}{2}c-\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} \\
&\times \sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}r-i)},\ i=0,1,\ldots
\end{aligned}
\tag{1.8}
$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad 1-a-i \\ c \end{matrix} ; \frac{1}{2}\right] \\
&= \frac{2^{1-i-c}\,\Gamma(\frac{1}{2})\Gamma(c)}{\Gamma(\frac{1}{2}c-\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} \\
&\times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}r)},\ i=0,1,\ldots
\end{aligned}
\tag{1.9}
$$

- **Generalizations of Kummer's summation theorem** [10]

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad b \\ 1+a-b+i \end{matrix} ; -1\right] \\
&= \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(b-i)\Gamma(1+a-b+i)}{\Gamma(b)\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)} \\
&\times \sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})},\ i=0,1,\ldots
\end{aligned}
\tag{1.10}
$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad b \\ 1+a-b-i \end{matrix} ; -1\right] \\
&= \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-i)}{\Gamma(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-b-\frac{1}{2}i+1)} \\
&\times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}r-\frac{1}{2}i+\frac{1}{2})},\ i=0,1,\ldots
\end{aligned}
\tag{1.11}
$$

Several mathematicians have obtained Laplace transforms of generalized hypergeometric functions by employing the above mentioned classical summation theorems. For example, Kim et al. [7,8] have obtained a few results on Laplace transforms for the functions ${}_1F_1$ and ${}_2F_2$, for some values of i.

In our present investigations, we propose to study systematically and investigate six master formulas for the Laplace transforms for the generalized hypergeometric function ${}_2F_2$ from which we can obtain as many as formulas by specializing the parameters.

In the next section, we shall mention six presumably new Laplace transforms.

2. Laplace Transforms of Certain Special ${}_2F_2(x)$

In this section, we will point out six Laplace transforms of ${}_2F_2(x)$.

$$\begin{aligned}&\int_0^\infty e^{-st}t^{d-1}\,{}_2F_2\left[\begin{matrix}a, & b \\ d, & \frac{1}{2}(a+b+i+1)\end{matrix};\frac{1}{2}st\right]dt\\ &=\frac{\Gamma(d)}{s^d}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)}\\ &\times\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}\end{aligned} \tag{2.1}$$

for $i=0,1,2,\ldots$.

$$\begin{aligned}&\int_0^\infty e^{-st}t^{d-1}\,{}_2F_2\left[\begin{matrix}a, & b \\ d, & \frac{1}{2}(a+b-i+1)\end{matrix};\frac{1}{2}st\right]dt\\ &=\frac{\Gamma(d)}{s^d}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\\ &\times\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}\end{aligned} \tag{2.2}$$

for $i=0,1,2,\ldots$.

$$\begin{aligned}&\int_0^\infty e^{-st}t^{d-1}\,{}_2F_2\left[\begin{matrix}a, & 1-a+i \\ d, & c\end{matrix};\frac{1}{2}st\right]dt\\ &=\frac{\Gamma(d)}{s^d}\frac{2^{1+i-c}\,\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(a-i)}{\Gamma(a)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}\\ &\times\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}r-i\right)}\end{aligned} \tag{2.3}$$

for $i = 0, 1, 2, \ldots$.

$$\int_0^\infty e^{-st} t^{d-1} \, {}_2F_2\left[\begin{matrix} a, & 1-a-i \\ & \\ d, & c \end{matrix} ; \tfrac{1}{2}st\right] dt$$
$$= \frac{\Gamma(d)}{s^d} \frac{2^{1+i-c}\,\Gamma\left(\frac{1}{2}\right)\Gamma(c)}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}$$
$$\times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}r\right)} \tag{2.4}$$

for $i = 0, 1, 2, \ldots$

$$\int_0^\infty e^{-st} t^{d-1} \, {}_2F_2\left[\begin{matrix} a, & b \\ & \\ d, & 1+a-b+i \end{matrix} ; -st\right] dt$$
$$= \frac{\Gamma(d)}{s^d} \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\Gamma(b-i)\,\Gamma(1+a-b+i)}{\Gamma(b)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)}$$
$$\times \sum_{r=0}^{i} \binom{i}{r} \frac{(-1)^r\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}r-\frac{1}{2}i+\frac{1}{2}\right)} \tag{2.5}$$

for $i = 0, 1, 2, \ldots$.

$$\int_0^\infty e^{-st} t^{d-1} \, {}_2F_2\left[\begin{matrix} a, & b \\ & \\ d, & 1+a-b-i \end{matrix} ; -st\right] dt$$
$$= \frac{\Gamma(d)}{s^d} \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\Gamma(1+a-b-i)}{\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}i+1\right)}$$
$$\times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}r-\frac{1}{2}i+\frac{1}{2}\right)} \tag{2.6}$$

for $i = 0, 1, 2, \ldots$.

2.1. Proofs

Proof of the results (2.1) to (2.6) is quite straight forward. For this if we set $\omega = \frac{1}{2}s$ in (1.5) and taking

1. $c = \frac{1}{2}(a+b+i+1)$,
2. $c = \frac{1}{2}(a+b-i+1)$,
3. $b = 1-a+i$, and

4. $b = 1 - a - i$,

each for $i = 0, 1, 2, \ldots$, then the resulting series ${}_2F_1\left(\frac{1}{2}\right)$ appearing on the right-hand side of (1.5) can be summed by using corresponding summation formulas (1.6) to (1.9) and we respectively get the results (2.1) to (2.4). Similarly, if we set $\omega = -s$ in (1.5) and taking

1. $c = 1 + a - b + i$, and

2. $c = 1 + a - b - i$

each for $i = 0, 1, 2, \ldots$, then the resulting series ${}_2F_1(-1)$ appearing on the right-hand side of (1.4) can be summed by using corresponding summation formulas (1.10) to (1.11) and we respectively get the results (2.5) to (2.6).

3. Special Cases

In this section, we shall mention a large number of special cases, which are presumably new.

1. In (2.1) and (2.2), if we take $d = b$ we get two results on Laplace transforms for Kummer's confluent hypergeometric function ${}_1F_1$ obtained recently by Choi and Rathie [1].

2. In (2.3) and (2.4), if we take $d = 1 - a + i$ and $d = 1 - a - i$, for $i = 0, 1, 2, \ldots$ we get two results on Laplace transforms for Kummer's ${}_1F_1$ obtained recently by Choi and Rathie [1].

3. In (2.5) and (2.6), if we take $d = b$ or $d = a$, we get four results on Laplace transforms for Kummer's ${}_1F_1$ obtained recently by Choi and Rathie [1].

4. For $i = 0, \pm 1, \pm 2 \ldots, \pm 9$, the main results (2.1) and (2.6) can be written in the following compact forms

$$\int_0^\infty e^{-st} t^{d-1} {}_2F_2\left[\begin{matrix} a, & b \\ d, & \frac{1}{2}(a+b+i+1) \end{matrix} ; \frac{1}{2}st\right] dt$$
$$= \frac{s^{-d}\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}i + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}|i| + \frac{1}{2}\right)}$$
$$\times \left\{ \frac{C_i(a,b)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} - \left[\frac{1+i}{2}\right]\right)} + \frac{D_i(a,b)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}i - \left[\frac{i}{2}\right]\right)} \right\} \tag{3.1}$$

$$\int_0^\infty e^{-st}t^{d-1}\ {}_2F_2\left[\begin{matrix} a, & 1-a+i \\ & & ;\tfrac{1}{2}st \\ d, & c \end{matrix}\right]dt$$
$$= \frac{s^{-d}\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma(c)\Gamma(1-a)}{2^{c-i-1}\ \Gamma\left(1-a-\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \frac{E_i(a,c)}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\left[\frac{1+i}{2}\right]\right)}$$
$$+ \frac{s^{-d}\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma(c)\Gamma(1-a)}{2^{c-i-1}\ \Gamma\left(1-a-\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \frac{F_i(a,c)}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}\left[\frac{i}{2}\right]\right)} \tag{3.2}$$

and

$$\int_0^\infty e^{-st}t^{d-1}\ {}_2F_2\left[\begin{matrix} a, & b \\ & & ;-st \\ d, & 1+a-b+i \end{matrix}\right]dt$$
$$= \frac{s^{-d}2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \frac{A_i(a,b)}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)}$$
$$+ \frac{s^{-d}2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \frac{B_i(a,b)}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}. \tag{3.3}$$

Here $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $A_i(a,b)$, $B_i(a,b)$, $C_i(a,c)$, $D_i(a,c)$, $E_i(a,b)$ and $F_i(a,b)$ are given in the tables provided at the end of the paper.

5. In (3.1) to (3.3), if we set

   (a) $d=b$,

   (b) $d=1-a+i$,and

   (c) $d=b$

   respectively for $i=0,\pm1,\pm2\ldots,\pm5$, we get known results obtained by Kim et al. [7,8]. Similarly, other results can also be obtained.

4. Tables

Table 1: Table for the coefficients $\mathbf{A}_i$ and $\mathbf{B}_i, i = 0 \pm 1, \cdots \pm 9$

i	$\mathbf{A}_i$	$\mathbf{B}_i$
9	$-16b^4+72a^3b-108a^2b^2+60ab^3+23b^4-328a^3+972a^2b-792ab^2+150b^3-2240a^2+3612ab-999b^2-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4+248a^3-516a^2b+240ab^2-10b^3+1160a^2-1028ab+35b^2+1576a-50b+24$
8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4+128a^3-312a^2b+172ab^2-10b^3+624a^2-672ab+35b^2+896a-50b-24$	$8b^3-40ab^2+48a^2b-16a^3-192a^2+312ab-88b^2-640a+352b-512$
7	$7b^3-28ab^2+28a^2b-8a^3-100a^2+196ab-70b^2-352a+245b-302$	$8a^3-20a^2b+12ab^2-b^3+68a^2-76ab+6b^2+128a-11b+6$
6	$4a^3-12a^2b+9ab^2-b^3+36a^2-51ab+6b^2+74a-11b+6$	$16ab-8a^2-6b^2-48a+34b-52$
5	$10ab-4a^2-5b^2-26a+25b-32$	$4a^2-6ab+b^2+14a-3b+2$
4	$2a^2-4ab+b^2+8a-3b+2$	$4(b-a-2)$
3	$3b-2a-5$	-2
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2a^2-4ab+b^2-8a+5b+6$	$4(a-b-2)$
-5	$4a^2-10ab+5b^2-24a+25b+32$	$4a^2-6ab+b^2-16a+7b+12$
-6	$4a^3-12a^2b+9ab^2-b^3-36a^2+57ab-12b^2+92a-47b-60$	$8a^2-16ab+6b^2-48a+38b+64$
-7	$8a^3-28a^2b+28ab^2-7b^3-96a^2+196ab-77b^2+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3-72a^2+92ab-15b^2+184a-74b-120$
-8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4-128a^3+328a^2b-208ab^2+22b^3+688a^2-928ab+179b^2-1408a+638b+840$	$16a^3-48a^2b+40ab^2-8b^3-192a^2+328ab-104b^2+704a-480b-768$
-9	$16a^4-72a^3b+108a^2b^2-60ab^3+9b^4-320a^3+972a^2b-828ab^2+174b^3+2240a^2-3936ab+1323b^2-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4-256a^3+564a^2b-300ab^2+26b^3+1376a^2-1568ab+251b^2-2816a+1066b+1680$

Table 2: Table for the coefficients $\mathbf{C}_i$ and $\mathbf{D}_i, i = 0 \pm 1, \cdots \pm 9$

i	$\mathbf{C}_i$	$\mathbf{D}_i$
9	$-16b^4+36b^3(b-a+10)-27b^2(b-a+10)^2+\frac{15}{2}b(b-a+10)^3+\frac{23}{16}b(b-a+10)^4-328b^3+486b^2(b-a+10)-198b(b-a+10)^2+\frac{75}{4}b(b-a+10)^3-2240b^2+1806b(b-a+10)-\frac{999}{4}(b-a+10)^2-5696b+1581(b-a+10)-3984$	$16b^4-28b^3(b-a+10)+15b^2(b-a+10)^2-\frac{5}{2}b(b-a+10)^3+\frac{1}{16}(b-a+10)^4+248b^3-258b^2(b-a+10)+60b(b-a+10)^2-\frac{5}{4}(b-a+10)^3+1160b^2-514b(b-a+10)+\frac{35}{4}(b-a+10)^2+1576b-25(b-a+10)-24$
8	$8b^4-16b^3(b-a+9)+10b^2(b-a+9)^2-2b(b-a+9)^3+\frac{1}{16}(b-a+9)^4+128b^3-156b^2(b-a+9)+44b(b-a+9)^2-\frac{5}{4}(b-a+9)^3+624b^2-336b(b-a+9)+\frac{35}{4}(b-a+9)^2+896b-25(b-a+9)+24$	$(b-a+9)^3-10b(b-a+9)^2+24b^2(b-a+9)-16b^3-192b^2+156b(b-a+9)-22(b-a+9)^2-640b+176(b-a+9)-512$
7	$\frac{7}{8}(b-a+8)^3--7b(b-a+8)^2+14b^2(b-a+8)-8b^3-100b^2+98b(b-a+8)-\frac{35}{2}(b-a+8)^2-352b+\frac{245}{2}(b-a+8)-302$	$8b^3-10b^2(b-a+8)+3b(b-a+8)^2-\frac{1}{8}(b-a+8)^3+68b^2-38b(b-a+8)+\frac{3}{2}(b-a+8)^2+128b-\frac{11}{2}(b-a+8)+6$
6	$4b^3-6b^2(b-a+7)+\frac{9}{4}b(b-a+7)^2-\frac{1}{8}(b-a+7)^3+36b^2-\frac{51}{2}b(b-a+7)+\frac{3}{2}(b-a+7)^2+74b-\frac{11}{2}(b-a+7)+6$	$25b-17a-18ab-\frac{3}{2}(b-a+7)^2+67$
5	$\frac{1}{4}(-8+10a-5a^2+6b-10ab-b^2)$	$\frac{1}{4}(8-6a+a^2-10b+10ab+5b^2)$
4	$\frac{1}{4}(3-4a+a^2-4b+6ab+b^2)$	$2(1-a-b)$
3	$\frac{1}{2}(2-3a-b)$	$\frac{1}{2}(a+3b-2)$
2	$\frac{1}{2}(a+b-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(a+b-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(a+3b-2)$
-4	$\frac{1}{4}(3-4a+a^2-4b+6ab+b^2)$	$2(a+b-1)$
-5	$\frac{1}{4}(8-10a+5a^2-6b+10ab+b^2)$	$\frac{1}{4}(8-6a+a^2-10b+10ab+5b^2)$
-6	$4b^3-6b^2(b-a-5)+\frac{9}{4}b(b-a-5)^2-\frac{1}{8}(b-a-5)^3-36b^2+\frac{57}{2}b(b-a-5)-3(b-a-5)^2+92b-\frac{47}{2}(b-a-5)-60$	$8ab+11b-19a-31+\frac{3}{2}(b-a-5)^2$
-7	$8b^3-14b^2(b-a-6)+7b(b-a-6)^2-\frac{7}{8}(b-a-6)^3-96b^2+98b(b-a-6)-\frac{77}{4}(b-a-6)^2+352b-147(b-a-6)-384$	$8b^3-10b^2(b-a-6)+3b(b-a-6)^2-\frac{1}{8}(b-a-6)^3-72b^2+46b(b-a-6)-\frac{15}{4}(b-a-6)^2+184b-37(b-a-6)-120$
-8	$8a^4-16b^3(b-a-7)+10b^2(b-a-7)^2-2b(b-a-7)^3+\frac{1}{16}(b-a-7)^4-128b^3+164b^2(b-a-7)-52b(b-a-7)^2+\frac{11}{4}(b-a-7)^3+688b^2-464b(b-a-7)+\frac{179}{4}(b-a-7)^2-1408b+319(b-a-7)+840$	$16b^3-24b^2(b-a-7)+10b(b-a-7)^2-(b-a-7)^3-192b^2+164b(b-a-7)-26(b-a-7)^2+704b-240(b-a-7)-768$

-9	$16b^4-36b^3(b-a-8)+27b^2(b-a-8)^2-\frac{15}{2}b(b-a-8)^3+\frac{9}{16}(b-a-8)^4-320b^3+468b^2(b-a-8)-207b(b-a-8)^2+\frac{87}{4}(b-a-8)^3+2240b^2-1968b(b-a-8)+\frac{1323}{4}(b-a-8)^2-6400b+2307(b-a-8)+6144$	$16b^4-28b^3(b-a-8)+15b^2(b-a-8)^2-\frac{5}{2}b(b-a-8)^3+\frac{1}{16}(b-a-8)^4-256b^3+282b^2(b-a-8)-75b(b-a-8)^2+\frac{13}{4}(b-a-8)^3+1376b^2-784b(b-a-8)+\frac{251}{4}(b-a-8)^2-2816b+533(b-a-8)+1680$

Table 3: Table for the coefficients $\mathbf{E}_i$ and $\mathbf{F}_i, i=0\pm1,\cdots\pm9$

i	$\mathbf{E}_i$	$\mathbf{F}_i$
9	$-16(c+a-10)^4+72a(c+a-10)^3-108a^2(c+a-10)^2+60a^3(c+a-10)+23a^4-328(c+a-10)^3+972a(c+a-10)^2-792a^2(c+a-10)+150a^3-2240(c+a-10)^2+3612a(c+a-10)-999a^2-5696(c+a-10)+3162a-3984$	$16(c+a-10)^4-56a(c+a-10)^3+60a^2(c+a-10)^2-20a^3(c+a-10)+a^4+248(c+a-10)^3-516a(c+a-10)^2+240a^2(c+a-10)-10a^3+1160(c+a-10)^2-1028a(c+a-10)+35a^2+1576(c+a-10)-50a-24$
8	$8(c+a-9)^4-32a(c+a-9)^3+40a^2(c+a-9)^2-16a^3(c+a-9)+a^4+128(c+a-9)^3-312a(c+a-9)^2+176a^2(c+a-9)-10a^3+624(c+a-9)^2-672a(c+a-9)+35a^2+896(c+a-9)-50a+24$	$8a^3-40a^2(c+a-9)+48a(c+a-9)^2-16(c+a-9)^3-192(c+a-9)^2+312a(c+a-9)-88a^2-640(c+a-9)+352a-512$
7	$7a^3-28a^2(c+a-8)+28a(c+a-8)^2-8(c+a-8)^3-100(c+a-8)^2+196a(c+a-8)-70a^2-352(c+a-8)+245a-302$	$8(c+a-8)^3-20a(c+a-8)^2+12a^2(c+a-8)-a^3+68(c+a-8)^2-76a(c+a-8)+6a^2+128(c+a-8)-11a+6$
6	$4(c+a-7)^3-12a(c+a-7)^2+9a^2(c+a-7)-a^3+36(c+a-7)^2-51a(c+a-7)+6a^2+74(c+a-7)-11a+6$	$16a(c+a-7)-8(c+a-7)^2-6a^2-48(c+a-7)+34a-52$
5	$-20-13a+a^2+22c+2ac-4c^2$	$62-a-a^2-34c+2ac+4c^2$
4	$12+5a-a^2-12c+2c^2$	$12-4c$
3	$3+a-2c$	$-7+a+2c$
2	$-2+c$	-2
1	-1	1
0	1	0
-1	1	1
-2	c	2
-3	$-a+2c$	$2+a+2c$

-4	$-3a-a^2+4c+2c^2$	$4+4c$
-5	$-7a-a^2+8c-2ac+4c^2$	$12-a-a^2+16c+2ac+4c^2$
-6	$4(c+a+5)^3-12a(c+a+5)^2+9a^2(c+a+5)-a^3-36(c+a+5)^2+57a(c+a+5)-12a^2+92(c+a+5)-47a-60$	$8(c+a+5)^2-16a(c+a+5)+6a^2-48(c+a+5)+38a+64$
-7	$8(c+a+6)^3-28a(c+a+6)^2+28a^2(c+a+6)-7a^3-96(c+a+6)^2+196a(c+a+6)-77a^2+352(c+a+6)-294a-384$	$8(c+a+6)^3-20a(c+a+6)^2+12a^2(c+a+6)-a^3-72(c+a+6)^2+92a(c+a+6)-15a^2+184(c+a+6)-74a-120$
-8	$8(c+a+7)^4-32a(c+a+7)^3+40a^2(c+a+7)^2-16a^3(c+a+7)+a^4-128(c+a+7)^3+328a(c+a+7)^2-208a^2(c+a+7)+22a^3+688(c+a+7)^2-928a(c+a+7)+179a^2-1408(c+a+7)+638a+840$	$16(c+a+7)^3-48a(c+a+7)^2+40a^2(c+a+7)-8a^3-192(c+a+7)^2+328a(c+a+7)-104a^2+704(c+a+7)-480a-768$
-9	$16(c+a+8)^4-72a(c+a+8)^3+108a^2(c+a+8)^2-60a^3(c+a+8)+9a^4-320(c+a+8)^3+972a(c+a+8)^2-828a^2(c+a+8)+174a^3+2240(c+a+8)^2-3936a(c+a+8)+1323a^2-6400(c+a+8)+4614a+6144$	$16(c+a+8)^4-56a(c+a+8)^3+60a^2(c+a+8)^2-20a^3(c+a+8)+a^4-256(c+a+8)^3+564a(c+a+8)^2-300a^2(c+a+8)+26a^3+1376(c+a+8)^2-1568a(c+a+8)+251a^2-2816(c+a+8)+1066a+1680$

5. Conflict of Interests

The authors declare that they have no conflict of interests.

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