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On a New Class of Double Integrals Involving Gauss's ${}_2F_1$ Hypergeometric Function

Sungtae Jun

Abstract: In this paper, one hundred interesting double integrals involving Gauss's hypergeometric function in the form of four general integrals (twenty five each) have been evaluated in terms of gamma function. More than two hundred special cases have also been given.

Key Words: Gauss's hypergeometric function, Generalized hypergeometric function, Watson summation theorem, Edwards's double integral.

Contents

1. Introduction

In order to justify our doing, we must quote Sylvester [9]:

" It seems to be expected of every pilgrim up the slope of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock"

We begin by calling the following Edwards's double integral [2] viz.

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.1}$$

provided $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

Very recently, a natural generalization of the Edwards's integral (1.1) has been given by Kim *et al.* [4] in the following form

$$\begin{aligned}\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}dxdy \\ = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\end{aligned} \tag{1.2}$$

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provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\delta) > 0$.

For $\gamma = \delta = 1$, (1.2) immediately reduces to (1.1).

Recently, good deal of progress has been done in applying Edwards's integral (1.1) in evaluating several double integrals involving Gauss's hypergeometric function ${}_2F_1$ and generalized hypergeometric functions ${}_3F_2$. For detail, we refer the papers [1,7].

In the same paper, Kim *et al.* [4] have evaluated a large number of double integrals by employing classical summation theorems such as those of Watson, Dixon and Whipple for the series ${}_3F_2$ of unit argument. We, however, in our present investigation, mention the following two integrals viz.

$$\begin{aligned}\int_0^1\int_0^1 & x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \\ & \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1)\end{matrix}; xy\right] dxdy \\ = \, & k\,\Omega\end{aligned} \tag{1.3}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(c) > 0$, and

$$\begin{aligned}\int_0^1\int_0^1 & x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \\ & \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1)\end{matrix}; 1-xy\right] dxdy \\ = \, & k\,\Omega\end{aligned} \tag{1.4}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(c) > 0$.

In both the integrals,

$$k = \frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.5}$$

and

$$\Omega = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})}. \tag{1.6}$$

Inspired by the double integrals (1.3) and (1.4), our aim, in this paper, is to evaluate the following double integrals

$$\begin{aligned}\int_0^1\int_0^1 & x^{c-1}y^{\alpha+c-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+j-\alpha-\beta} \\ & \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+i+1)\end{matrix}; xy\right] dxdy\end{aligned} \tag{i}$$

$$\int_0^1\int_0^1 x^{c+j-1}y^{\alpha+c+j-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times {}_2F_1\left[\begin{matrix}a, \quad b\\ \frac{1}{2}(a+b+i+1)\end{matrix}; xy\right]dxdy \tag{ii}$$

$$\int_0^1\int_0^1 x^{c-1}y^{\alpha+c-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+j-\alpha-\beta} \times {}_2F_1\left[\begin{matrix}a, \quad b\\ \frac{1}{2}(a+b+i+1)\end{matrix}; 1-xy\right]dxdy \tag{iii}$$

and

$$\int_0^1\int_0^1 x^{c+j-1}y^{\alpha+c+j-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times {}_2F_1\left[\begin{matrix}a, \quad b\\ \frac{1}{2}(a+b+i+1)\end{matrix}; 1-xy\right]dxdy \tag{iv}$$

each for $i, j = 0, \pm1, \pm2$.

In order to evaluate in all 100 double integrals involving Gauss's hypergeometric function in the form of four master formulas, we need the following generalizations of classical Watson's summation theorem for the series ${}_3F_2$ obtained earlier by Lavoie *et al.* [5] viz.

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}a, \quad b, \quad c\\ \frac{1}{2}(a+b+i+1), \quad 2c+j\end{matrix}; 1\right] \qquad (1.7)\\
&= A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{1}{2}(a+b+i+1))\Gamma(c+\left[\frac{j}{2}\right]+\frac{1}{2})\Gamma(c-\frac{1}{2}(a+b+|i+j|-j-1)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}\\
&\times\Big\{B_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{1}{2}b+\frac{1}{2}+\left[\frac{j}{2}\right])}\\
&+C_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\left[\frac{j+1}{2}\right]+\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{1}{2}b+\left[\frac{j+1}{2}\right])}\Big\}\\
&= \Omega_1
\end{aligned}$$

for $i, j = 0, \pm1, \pm2$.

Here, as usual, $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes its modulus. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given in Tables 1, 2 and 3 at the end of this paper.

Clearly, for $i = j = 0$, (1.7) reduces at once to the following classical Watson's summation theorem [1] for the series ${}_3F_2$ viz.

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}; 1\right] \tag{1.8}
$$
$$
= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))\Gamma(c-\frac{1}{2}a+\frac{1}{2}))\Gamma(c-\frac{1}{2}b+\frac{1}{2}))}
$$

provided $\mathrm{Re}(2c-a-b) > -1$.

2. Main Results

In this section, we shall evaluate one hundred double integrals involving Gauss's hypergeometric function in the form of four general integrals asserted in the following four theorems.

Theorem 2.1. *For $i, j = 0, \pm 1, \pm 2$, the following general result holds true.*

$$
\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \tag{2.1}
$$
$$
\times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; xy\right] dxdy
$$
$$
= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \Omega_1
$$

provided $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(c) > 0$ and $Re(2c-a-b+2j+i+1) > 0$ for $i, j = 0, \pm 1, \pm 2$. Also Ω_1 is the same as given in (1.7).

Theorem 2.2. *For $i, j = 0, \pm 1, \pm 2$, the following general result holds true.*

$$
\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \tag{2.2}
$$
$$
\times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; xy\right] dxdy
$$
$$
= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \Omega_1
$$

provided $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(c+j) > 0$ for $j = 0, \pm 1, \pm 2$ and $Re(2c-a-b+2j+i+1) > 0$ for $i, j = 0, \pm 1, \pm 2$. Also Ω_1 is the same as given in (1.7).

Theorem 2.3. *For $i, j = 0, \pm 1, \pm 2$, the following general result holds true.*

$$\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; 1-xy\right] dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \Omega_1 \tag{2.3}$$

provided $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(c) > 0$ *and* $Re(2c-a-b+2j+i+1) > 0$ *for* $i, j = 0, \pm 1, \pm 2$. *Also* Ω_1 *is the same as given in* (1.7).

Theorem 2.4. *For* $i, j = 0, \pm 1, \pm 2$, *the following general result holds true.*

$$\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; 1-xy\right] dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \Omega_1 \tag{2.4}$$

provided $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(c+j) > 0$ *for* $j = 0, \pm 1, \pm 2$ *and* $Re(2c-a-b+2j+i+1) > 0$ *for* $i, j = 0, \pm 1, \pm 2$. *Also* Ω_1 *is the same as given in* (1.7).

Proof: In order to establish the general result (2.1) given in Theorem 1, we proceed as follows. Denoting the left-hand side of (2.1) by I, we have

$$I = \int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; xy\right] dxdy.$$

Now, expressing ${}_2F_1$ as a series, we have

$$I = \int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \times \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(\frac{1}{2}(a+b+i+1))_n \, n!} x^n y^n \, dxdy.$$

Changing the order of integration and summation which is clearly seen to be justified by the fact that the series involved in the process is uniformly convergent, we have after some algebra

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(\frac{1}{2}(a+b+i+1))_n n!}$$
$$\times \int_0^1 \int_0^1 x^{c+n-1} y^{\alpha+c+n-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} dxdy.$$

Now, evaluating the double integral with help of the known results (1.2), we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(\frac{1}{2}(a+b+i+1))_n \, n!} \times \frac{\Gamma(c+n)\Gamma(c+j)}{\Gamma(2c+j+n)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Using the result

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)},$$

we have after some calculation

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \times \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(\frac{1}{2}(a+b+i+1))_n (2c+j)_n \, n!}.$$

Summing up the series, we have

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \times {}_3F_2 \left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix} ; 1 \right].$$

Now, it is seen that the ${}_3F_2$ appearing can be evaluated with the help of known result (1.7) and we easily arrive at the right-hand side of (2.1) asserted in the Theorem 1.

In exactly the same manner, the other general results (2.2) to (2.4) asserted in the Theorem 2.2 to 2.4 can be proven.

□

3. Special Cases

In this section we shall mention more than two hundred interesting special cases of our main findings.

(1) In (2.1), let $b = -2n$ and change a to $a + 2n$ or let $b = -2n - 1$ and change a to $a + 2n + 1$, where n is zero or a positive integer. In both the such cases, we notice that one of the two terms appearing on the right-hand side of (2.1) will vanish and hence we get fifty interesting and new special cases given in the following two corollaries.

Corollary 3.1. *For $i, j = 0, \pm 1, \pm 2$, the following general result (containing 25 results) holds true.*

$$\begin{aligned}
&\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \qquad (3.1)\\
&\quad \times {}_2F_1 \left[\begin{matrix} -2n, \quad a+2n \\ \frac{1}{2}(a+i+1) \end{matrix} ; xy \right] dxdy \\
&= D_{i,j} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \\
&\quad \times \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + \frac{3}{4} - \frac{(-1)^i}{4} \left[\frac{1}{2}j + \frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j}{2}\right]\right)_n \left(\frac{1}{2}a + \frac{1}{4}(1+(-1)^i)\right)_n} \\
&= \Omega_2,
\end{aligned}$$

where the coefficients $D_{i,j}$ are given in Table 4 at the end of this paper.

Corollary 3.2. *For $i, j = 0, \pm 1, \pm 2$, the following general result (containing 25 results) holds true.*

$$\begin{aligned}
&\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \qquad (3.2)\\
&\quad \times {}_2F_1 \left[\begin{matrix} -2n-1, \quad a+2n+1 \\ \frac{1}{2}(a+i+1) \end{matrix} ; xy \right] dxdy \\
&= E_{i,j} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)} \\
&\quad \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + \frac{5}{4} - \frac{(-1)^i}{4} - \left[\frac{1}{2}j + \frac{1}{4}(1+(-1)^i\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j+1}{2}\right]\right)_n \left(\frac{1}{2}a + \frac{1}{4}(3-(-1)^i)\right)_n} \\
&= \Omega_3,
\end{aligned}$$

where the coefficients $E_{i,j}$ are given in Table 5 at the end of this paper.

(2) In (2.2) to (2.4), if we apply the same conditions mentioned in special case **(1)**, we get two general results containing 150 results given in the following six corollaries.

Corollary 3.3. *For $i, j = 0, \pm 1, \pm 2$, the following general result (containing 25 results) holds true.*

$$\begin{aligned}
&\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \qquad (3.3)\\
&\quad \times {}_2F_1 \left[\begin{matrix} -2n, \quad a+2n \\ \frac{1}{2}(a+i+1) \end{matrix} ; xy \right] dxdy \\
&= \Omega_2,
\end{aligned}$$

where Ω_2 *is the same as in* (3.1).

Corollary 3.4. *For* $i, j = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} -2n-1, \quad a+2n+1 \\ \frac{1}{2}(a+i+1) \end{matrix} ; xy \right] dxdy = \Omega_3, \tag{3.4}$$

where Ω_3 *is the same as in* (3.2).

Corollary 3.5. *For* $i, j = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} -2n, \quad a+2n \\ \frac{1}{2}(a+i+1) \end{matrix} ; 1-xy \right] dxdy = \Omega_2, \tag{3.5}$$

where Ω_2 *is the same as in* (3.1).

Corollary 3.6. *For* $i, j = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+j-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} -2n-1, \quad a+2n+1 \\ \frac{1}{2}(a+i+1) \end{matrix} ; 1-xy \right] dxdy = \Omega_3, \tag{3.6}$$

where Ω_3 *is the same as in* (3.2).

Corollary 3.7. *For* $i, j = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} -2n, \quad a+2n \\ \frac{1}{2}(a+i+1) \end{matrix} ; 1-xy \right] dxdy = \Omega_2, \tag{3.7}$$

where Ω_2 *is the same as in* (3.1).

Corollary 3.8. *For $i, j = 0, \pm 1, \pm 2$, the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{c+j-1} y^{\alpha+c+j-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} -2n-1, \quad a+2n+1 \\ \frac{1}{2}(a+i+1) \end{matrix} ; 1-xy \right] dxdy = \Omega_3, \tag{3.8}$$

where Ω_3 is the same as in (3.2).

(3) In (2.1), if we set $i = j = 0$, we get the results asserted in the following result.

$$\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \times {}_2F_1 \left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; xy \right] dxdy = \pi 2^{1-2c} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} \tag{3.9}$$

provided $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > -1$.

Let us consider some very interesting special cases of (3.9).

(a) In (3.9), if we set $a = b = \frac{1}{2}$ and use the result [6, p.473, Eq.(75)]

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} ; z \right] = \frac{2}{\pi} K(\sqrt{z}),$$

where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 sin^2 t}},$$

we get the following result

$$\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} K\left(\sqrt{xy}\right) dxdy = \pi^2 \, 2^{-2c} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c)}{\Gamma^2(\frac{3}{4})\Gamma(c+\frac{1}{4})\Gamma(c+\frac{1}{4})} \tag{3.10}$$

provided $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$ and $\text{Re}(c) > 0$.

(b) In (3.9), if we set $a = b = 1$ and use the result [6, p.476, Eq.(147)]

$$ {}_2F_1\left[\begin{matrix}1, & 1 \\ & \frac{3}{2}\end{matrix}; z\right] = \frac{\sin^{-1}(\sqrt{z})}{\sqrt{z(1-z)}}, $$

we get the following result

$$ \begin{aligned} &\int_0^1 \int_0^1 x^{c-\frac{3}{2}} y^{\alpha+c-\frac{3}{2}} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta-\frac{1}{2}} \\ &\qquad \times \sin^{-1}\left(\sqrt{xy}\right) dxdy \\ &= \pi^{\frac{3}{2}} 2^{-2c} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(c-\frac{1}{2})}{\Gamma(\alpha+\beta)\Gamma(c)} \end{aligned} \tag{3.11} $$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(c) > \frac{1}{2}$.

(c) In (3.9), if we take $b = -a$ and use the result [6, p.459, Eq.(83)]

$$ {}_2F_1\left[\begin{matrix}a, & -a \\ & \frac{1}{2}\end{matrix}; z\right] = \cos(2a \sin^{-1}(\sqrt{z})), $$

we get the following result

$$ \begin{aligned} &\int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c-\alpha-\beta} \\ &\qquad \times \cos\left[2a \sin^{-1} \sqrt{xy}\right] dxdy \\ &= \pi^{\frac{3}{2}} 2^{1-2c} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(c+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}-\frac{1}{2}a)} \end{aligned} \tag{3.12} $$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(c) > 0$.

We conclude this section by remarking that similar results can also be obtained from the results (3.2) to (3.8).

Remark 3.9. *For other double integrals of this type, see [3,8].*

i\j	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{(c+1)}$

Table 1: Table for $A_{i,j}$

i\j	-2	-1	0	1	2
2	$c(a+b-1)-$ $(a+1)(b+1)+2$	$a+b-1$	$a(2c-a)+$ $b(2c-b)-2c+1$	$2c(a+b-1)-$ $(a-b)^2+1$	$B_{2,2}$
1	$c-b-1$	1	1	$2c-a+b$	$2c(c+1)-$ $(a-b)(c-b+1)$
0	$(c-a-1)(c-b-1)$ $+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)$ $+c(c+1)$
-1	$2(c-1)(c-2)-$ $(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$B_{-2,-2}$	$B_{-2,-1}$	$a(2c-a)+$ $b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)-$ $(a-1)(b-1)$

Table 2: Table for $B_{i,j}$

$$
\begin{aligned}
B_{-2,-1} &= 2(c-1)(a+b-1)-(a-b)^2+1 \\
B_{2,2} &= 2c(c+1)[(2c+1)(a+b-1)-a(a-1)-b(b-1)] \\
&\quad -(a-b-1)(a-b+1)[(c+1)(2c-a-b+1)+ab] \\
B_{-2,-2} &= 2(c-1)(c-2)[(2c-1)(a+b-1)-a(a+1)-b(b+1)+2] \\
&\quad -(a-b-1)(a-b+1)[(c-1)(2c-a-b-3)+ab]
\end{aligned}
$$

i\j	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-[8c^2-2c(a+b-1)$ $-(a-b)^2+1]$	$-4(2c+a-b+1)$ $\times(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-[2c(c+1)+$ $(a-b)(c-a+1)]$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)+$ $(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)$ $\times(2c+a-b-3)$	$C_{-2,-1}$	8	$4c-a-b+1$	4

Table 3: Table for $C_{i,j}$

$$C_{-2,-1} = 8c^2-2(c-1)(a+b+7)-(a-b)^2-7$$

i\j	-2	-1	0
2	$\frac{[(c-1)(a-1)+2n(a+2n)]}{(a+4n-1)(a+4n+1)} \times \frac{(a+1)}{(c-1)}$	$\frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}$	$\frac{[(a-1)(2c-a-1)-4n(a+2n)]}{(a+4n+1)(a+4n-1)} \times \frac{(a+1)}{(2c-a-1)}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$
0	$1-\frac{2n(a+2n)}{(c-1)(2c-a-3)}$	1	1
-1	$1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)}$	$1-\frac{4n}{(2c-a-2)}$	1
-2	$D_{-2,-2}$	$1-\frac{8n(a+2n)}{(a-1)(2c-a-3)}$	$1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}$

i\j	1	2
2	$\frac{[(a-1)(2c-a-1)-8n(a+2n)]}{(a+4n+1)(a+4n-1)} \times \frac{(a+1)}{(2c-a-1)}$	$D_{2,2}$
1	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\frac{[(c+1)(2c-a)-2n(2c+a+4n+2)]}{(2c-a)(a+4n)} \times \frac{a}{(c+1)}$
0	1	$1-\frac{2n(a+2n)}{(c+1)(2c-a+1)}$
-1	1	$1+\frac{2n}{(c+1)}$
-2	1	$1+\frac{2n(a+2n)}{(c+1)(a-1)}$

Table 4: Table for $D_{i,j}$

$$
\begin{aligned}
D_{2,2} &= \frac{(a+1)[(a-1)(c+1)(2c-a+1)(2c-a-1)}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)} \\
&\quad + \frac{-2an(6c+a+5)(2c-a+1)}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)} \\
&\quad + \frac{4n^2(5a^2+4a-5-4c(3c-a+4))+64n^3(a+n)]}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)} \\
D_{-2,-2} &= 1-\frac{2an(6c+a-7)(2c-a-3)}{(c-1)(a-1)(2c-a-3)(2c-a-5)} \\
&\quad + \frac{-4n^2[5a^2-4a-21-4c(3c-a-8)]-64n^3(a+n)}{(c-1)(a-1)(2c-a-3)(2c-a-5)}
\end{aligned}
$$

i\j	-2	-1	0
2	$\frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)}$	$\frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}$	$\frac{2(a+1)}{(a+4n+1)(a+4n+3)}$
1	$\frac{(c-a-2n-2)}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(a+4n+2)(2c-1)}$	$\frac{1}{a+4n+2}$
0	$\frac{-1}{(c-1)}$	$\frac{-1}{(2c-1)}$	0
-1	$E_{-1,-2}$	$\frac{-(2c+a+4n)}{a(2c-1)}$	$\frac{-1}{a}$
-2	$\frac{-(2c+a+4n-1)(2c-a-4n-5)}{(a-1)(c-1)(2c-a-5)}$	$E_{-2,-1}$	$\frac{-2}{(a-1)}$

i\j	1	2
2	$E_{2,1}$	$\frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(c+1)(2c-a-1)(a+4n+1)(a+4n+3)}$
1	$\frac{(2c+a+4n+2)}{(2c+1)(a+4n+2)}$	$\frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)}$
0	$\frac{1}{(2c+1)}$	$\frac{1}{(c+1)}$
-1	$\frac{-(2c-a)}{a(2c+1)}$	$\frac{-(c-a-2n)}{a(c+1)}$
-2	$\frac{-(4c-a+1)}{(a-1)(2c+1)}$	$\frac{-(2c-a+1)}{(a-1)(c+1)}$

Table 5: Table for $E_{i,j}$

$$E_{2,1} = \frac{(a+1)[(4c+a+3)(2c-a-1)-8n(a+2n+2)]}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)}$$

$$E_{-2,-1} = -\frac{[(4c+a-1)(2c-a-3)-8n(a+2n+2)]}{(a-1)(2c-1)(2c-a-3)}$$

$$E_{-1,-2} = -\frac{[(c+a)(2c-a-4)-2n(3a-2c+4n+6)]}{a(c-1)(2c-a-4)}$$

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Sungtae Jun
General Education Institute,
Konkuk University, Chungju 380-701,
Republic of Korea.
E-mail address: sjun@kku.ac.kr