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Some Remarks on Multivalent Functions of Higher-order Derivatives

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ABSTRACT: In this paper we give necessary conditions for a suitably normalized multivalent function $f(z)$ to be in the class $G_{p,q}(\beta)$ of p-valently starlike functions of higher-order derivatives. Also we drive some properties of functions belonging to the class $J_{p,q}(\alpha,\beta,f(z))$ which consisting of multivalent $\alpha-$convex functions of higher-order derivatives in the unit disc.

Key Words: $p-$valent functions, Higher-order derivatives, α-convex functions.

Contents

1. Introduction

Let $\mathbb{U}=\{z:|z|<1\}$ be the open unit disc of the complex plane $\mathbb{C}$ and let $\mathcal{A}_p$ denote the class of analytic and $p-$valent functions in $\mathbb{U}$ of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k \qquad (p\in\mathbb{N}=\{1,2,\dots\}) \tag{1.1}$$

Also let $\mathcal{A}_1:=\mathcal{A}$. For two functions f , g, we say that the function f is subordinate to g in $\mathbb{U}$, written as $f(z)\prec g(z)$, (or simply $f\prec g$) if there exists a Schwarz function ω analytic $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$, such that $f(z)=g(\omega(z))$. If the function g is univalent in $\mathbb{U}$, the subordination is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ (see [8]). For $0\le\beta<p-q, p>q, p\in\mathbb{N}$ and $q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$,we say that $f(z)\in\mathcal{A}_p$ is in the class $S^*_{p,q}(\beta)$ if it satisfies the following inequality

$$\Re\left\{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right\}>\beta \qquad (z\in\mathbb{U}). \tag{1.2}$$

Also, for $0\le\beta<p-q, p>q, p\in\mathbb{N}$ and $q\in\mathbb{N}_0$,we say that $f(z)\in\mathcal{A}_p$ is in the class $K_{p,q}(\beta)$ if it satisfies the following inequality

$$\Re\left\{1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right\}>\beta \qquad (z\in\mathbb{U}). \tag{1.3}$$

It follows from (1.2) and (1.3) that

$$f(z)\in K_{p,q}(\beta)\Longleftrightarrow F(z)\in S^*_{p,q}(\beta),$$

where $F\in\mathcal{A}_p$, such that $F^{(q)}(z)=\frac{zf^{(q+1)}(z)}{p-q}\,(z\in\mathbb{U})$. The classes $S^*_{p,q}(\beta)$ and $K_{p,q}(\beta)$ were introduced and studied by Aouf [2,3,4]. We note that $S^*_{p,0}(\beta)\cong S^*_p(\beta)$ and $K_{p,0}(\beta)\cong K_p(\beta)$ are, respectively, the class of $p-$valently starlike functions of order β and the class of $p-$valently convex functions of order $\beta($ $0\le\beta<p)$ see Owa [12] and Aouf [1].

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Let $G_{p,q}(\beta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec (p-q)+(p-q-\beta)z \qquad (0 \leq \beta < p-q, p > q). \tag{1.4}$$

It is clear that (1.4) is equivalent to

$$\left|\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right| < (p-q-\beta) \;\; (z \in \mathbb{U}). \tag{1.5}$$

Therefore $G_{p,q}(\beta)$ is a subclass of the class $S^*_{p,q}(\beta)$.

A function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently $\alpha-$convex functions of higher order derivatives of order β if it satisfies

$$\Re\left\{(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha(1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)})\right\} > \beta \tag{1.6}$$

for some $\alpha(\alpha \geq 0), \beta(0 \leq \beta < \delta(p,q))$ and for all $(z \in \mathbb{U})$, where

$$\delta(p,q) = \frac{p!}{(p-q)!} \;\; (p > q).$$

Denoting by $J_{p,q}(\alpha,\beta,f(z))$ the subclass of $\mathcal{A}_p$ consisting of all such functions. We note that $J_{p,q}(0,\beta,f(z)) \cong S^*_{p,q}(\beta)$ and $J_{p,q}(1,\beta,f(z)) \cong K_{p,q}(\beta)$. Also we note that $J_{p,1-p}(\alpha,0,f(z)) \cong A(p,\alpha)$ $(p \in \mathbb{N}, \alpha \geq 1)$ was introduced and studied by Nunokawa [9], Saitoh et al. [14] and Nishimoto and Owa [11] and $J_{p,0}(\alpha,\beta,f(z)) \cong M(p,1,\alpha,\beta)$ was introduced and studied by Owa [13].

2. Main Results

In order to prove our results we need the following lemmas.

Lemma 2.1. *[6]Let $\omega(z)$ be regular in $\mathbb{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, we have $z_0\omega(z_0) = m\omega(z_0)$, where $m \geq 1$.*

Lemma 2.2. *[7]Let $\phi(z)$ be a complex valued function*

$$\phi : D \to C,\; D \subset C \times C \;\; (C \text{ is the complex plane}).$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in D;
(ii) $(1,0) \in D$ and $\Re\{\phi(1,0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$, $\Re\{\phi(iu_2,v_1)\} \leq 0$.

Let $p(z) = 1 + p_1z + p_2z^2 + ...$ be regular in the unit disc $\mathbb{U}$, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\Re\{\phi(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\Re\{p(z)\} > 0 \;\; (z \in \mathbb{U})$.

Theorem 2.3. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\begin{aligned} &\left|\lambda\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right) + (1-\lambda)\left(\frac{z^2f^{(q+2)}(z)}{f^{(q)}(z)} - (p-q)(p-q-1)\right)\right| \\ < \;\; &(p-q-\beta)[\lambda + (1-\lambda)(p-q+\beta)\;] \;\; (z \in \mathbb{U}), \end{aligned} \tag{2.1}$$

for some $(0 \leq \beta < p-q, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $0 \leq \lambda < 1)$, then $f(z) \in G_{p,q}(\beta)$.

Proof: Define the function $\omega(z)$ by

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = (p-q) + (p-q-\beta)\omega(z). \tag{2.2}$$

Then, $\omega(z)$ is regular in $\mathbb{U}$ and $\omega(0)=0$.Differentiating (2.2) logarithmically with respect to z, we obtain

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - (p-q) = (p-q-\beta)\omega(z) + \frac{(p-q-\beta)z\omega'(z)}{(p-q)+(p-q-\beta)\omega(z)} \tag{2.3}$$

From (2.2) and (2.3), we have

$$\begin{aligned} \frac{z^2 f^{(q+2)}(z)}{f^{(q)}(z)} - (p-q)(p-q-1) &= (p-q-\beta)\omega(z)[2(p-q)-1 \\ &\quad + (p-q-\beta)\omega(z) + \frac{z\omega'(z)}{\omega(z)}]. \end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we have

$$\begin{aligned} &\lambda\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right) + (1-\lambda)\left(\frac{z^2 f^{(q+2)}(z)}{f^{(q)}(z)} - (p-q)(p-q-1)\right) \\ = \;&(p-q-\beta)\omega(z) \\ &\times\left\{\lambda + (1-\lambda)[2(p-q)-1+(p-q-\beta)\omega(z) + \frac{z\omega'(z)}{\omega(z)}]\right\}. \end{aligned} \tag{2.5}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1.$$

Then by using Lemma 2.1, and letting $\omega(z_0) = e^{i\theta}$, we get

$$\begin{aligned} &\left|\lambda\left(\frac{zf^{(q+1)}(z_0)}{f^{(q)}(z_0)} - (p-q)\right) + (1-\lambda)\left(\frac{z^2 f^{(q+2)}(z_0)}{f^{(q)}(z_0)} - (p-q)(p-q-1)\right)\right| \\ = \;&|(p-q-\beta)\omega(z_0) \\ &\left\{\lambda + (1-\lambda)[2(p-q)-1+\frac{z\omega'(z_0)}{\omega(z_0)}] + (1-\lambda)(p-q-\beta)\omega(z_0\right\}\Bigg| \\ = \;&(p-q-\beta)\left|\lambda + (1-\lambda)[2(p-q)-1+k] + (1-\lambda)(p-q-\beta)e^{i\theta}\right| \\ \ge \;&(p-q-\beta)[\lambda + (1-\lambda)(p-q+\beta)\ ]. \end{aligned}$$

This contradicts the condition (2.1). Therefore $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. This implies that

$$\left|\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right| < p-q-\beta \quad (z \in \mathbb{U}),$$

that is $f(z) \in G_{p,q}(\beta)$.This completes the proof of Theorem 2.3 □

Taking $q=0$ in Theorem 2.3 , we have

Corollary 2.4. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\begin{aligned} &\left|\lambda\left(\frac{zf^{'}(z)}{f(z)}-p\right)+(1-\lambda)\left(1+\frac{z^2f^{''}(z)}{f(z)}-p(p-1)\right)\right| \\ < \;& (p-\beta)[\lambda+(1-\lambda)(p+\beta)\;] \;\; (z\in\mathbb{U}), \end{aligned}$$

then $f(z) \in G_p(\beta) := \left\{ f(z)\in\mathcal{A}_p : \left|\frac{zf^{'}(z)}{f(z)}-p\right| < p-\beta \quad (z\in\mathbb{U})\right\}.$

Putting $p = 1$ in Corollary 2.4, we have

Corollary 2.5. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\lambda\left(\frac{zf^{'}(z)}{f(z)}-1\right)+(1-\lambda)\frac{z^2f^{''}(z)}{f(z)}\right| < (1-\beta)[\lambda+(1-\lambda)(1+\beta)\;] \;\; (z\in\mathbb{U}),$$

then $f(z) \in G(\beta) := \left\{ f(z)\in\mathcal{A} : \left|\frac{zf^{'}(z)}{f(z)}-1\right| < 1-\beta \quad (z\in\mathbb{U})\right\}.$

This corollary is an improvement of the results obtained by Fukui [5, Theorem1] and Nunokawa and Hoshino [10, Theorem 1].

Putting $\lambda = 0$ in Theorem 2.3, we have

Corollary 2.6. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\frac{z^2f^{(q+2)}(z)}{f^{(q)}(z)}-(p-q)(p-q-1)\right| < (p-q-\beta)(p-q+\beta) \quad (z\in\mathbb{U}),$$

for some $(0\le\beta<p-q, p>q, p\in\mathbb{N}$ *and* $q\in\mathbb{N}_0)$, *then* $f(z)\in G_{p,q}(\beta)$.

Putting $q = 0$ in Corollary 2.6, we obtain the following corollary

Corollary 2.7. *if $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\frac{z^2f^{''}(z)}{f(z)}-p(p-1)\right| < (p^2-\beta^2) \quad (z\in\mathbb{U}),$$

for some $(0\le\beta<p$, *then* $f(z)\in G_p(\beta)$.

Remark 2.8. *Our result in Corollary 2.7 when $p = 1$ is an improvement of the results obtained by Fukui [5, Corollary1] and Nunokawa and Hoshino [10, Corollary 2].*

Putting $\lambda = \frac{1}{2}$ in Corollary 2.4, we obtain

Corollary 2.9. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf^{'}(z)+z^2f^{''}(z)}{f(z)}-p^2\right| < (p-\beta)\;(p+1+\beta) \;\; (z\in U),$$

for some $(0\le\beta<p$, *then* $f(z)\in G_p(\beta)$.

Remark 2.10. *Our result in Corollary 2.9 when $p = 1$ is an improvement of the results obtained by Fukui [5, Corollary2] and Nunokawa and Hoshino [10, Corollary 3].*

Theorem 2.11. *Let the function $f(z)$ defined by (1.1) belongs to the class $J_{p,q}(\alpha, f(z))$ with $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $\alpha \geq 1$, then*

$$\Re\left\{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right\} > \beta = \frac{-\alpha + \sqrt{\alpha(\alpha + 8\delta(p,q))}}{4}. \tag{2.6}$$

Proof: Define the function $g(z)$ by

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \beta + [\delta(p,q) - \beta]g(z), \quad 0 \leq \beta < \delta(p,q) \tag{2.7}$$

for $f(z) \in J_{p,q}(\alpha, f(z))$, where

$$\beta = \frac{-\alpha + \sqrt{\alpha(\alpha + 8\delta(p,q))}}{4}. \tag{2.8}$$

It follows from (2.7) that $g(z)$ is regular in $\mathbb{U}$ and that $g(z) = 1 + g_1 z + g_2 z^2 + \ldots$. Differentiating (2.7) logarithmically with respect to z, we obtain

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} = \beta + [\delta(p,q) - \beta]g(z) + \frac{[\delta(p,q) - \beta]zg'(z)}{\beta + [\delta(p,q) - \beta]g(z)}. \tag{2.9}$$

From (2.7) and (2.9), we have

$$\begin{aligned} &\Re\left\{(1-\alpha)\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right)\right\} \\ = \ &\Re\left\{\beta + [\delta(p,q) - \beta]g(z) + \frac{\alpha[\delta(p,q) - \beta]zg'(z)}{\beta + [\delta(p,q) - \beta]g(z)}\right\} > 0. \end{aligned} \tag{2.10}$$

Letting $u = u_1 + iu_2$, $v = v_1 + iv_2$ and

$$\phi(u,v) = \beta + [\delta(p,q) - \beta]u + \frac{\alpha[\delta(p,q) - \beta]v}{\beta + [\delta(p,q) - \beta]u}, \tag{2.11}$$

we know that

(i) $\phi(u,v)$ is continuous in $D = \left(C - \frac{\beta}{\beta - \delta(p,q)}\right) \times C$;

(ii) $(1,0) \in D$ and $\Re\{\phi(1,0)\} = \delta(p,q) > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned} \Re\{\phi(iu_2, v_1)\} &= \beta + \frac{\alpha[\delta(p,q) - \beta]v_1}{\beta^2 + [\delta(p,q) - \beta]^2 u_2^2} \\ &\leq \beta - \frac{\alpha\beta[\delta(p,q) - \beta](1 + u_2^2)}{2\left(\beta^2 + [\delta(p,q) - \beta]^2 u_2^2\right)} \leq 0. \end{aligned}$$

Therefore, the function $\phi(u,v)$ defined by (2.11) satisfies the conditions of Lemma 2.2. It follows from this fact that $\Re\{g(z)\} > 0$, that is that

$$\Re\left\{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right\} > \beta = \frac{-\alpha + \sqrt{\alpha(\alpha + 8\delta(p,q))}}{4}.$$

This completes the proof of Theorem 2.11. □

Putting $q = 1 - 1(p \in \mathbb{N})$ in Theorem 2.11, we obtain the following corollary

Corollary 2.12. *Let the function $f(z)$ defined by (1.1) belongs to the class*

$$J_{p,1-p}(\alpha, f(z)) = J_p(\alpha, f(z))$$

with $p \in \mathbb{N}$ and $\alpha \geq 1$, then

$$\Re\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \beta = \frac{-\alpha + \sqrt{\alpha(\alpha + 8p!)}}{4}.$$

From the definition of the class $J_p(\alpha, f(z))$ and Theorem 2.11, we have

Corollary 2.13. *Let the function $f(z)$ defined by (1.1) belongs to the class $J_{p,q}(\alpha, f(z))$ with $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $\alpha \geq 1$, then*

$$\Re\left\{1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}\right\} > \beta = \frac{(\alpha - 1)(-\alpha + \sqrt{\alpha(\alpha + 8\delta(p, q)})}{4\alpha}.$$

Putting $q = p - 1 (p \in \mathbb{N})$ in Corollary 2.13, we obtain the following corollary

Corollary 2.14. *Let the function $f(z)$ defined by (1.1) belongs to the class $J_p(\alpha, f(z))$ with $p \in \mathbb{N}$ and $\alpha \geq 1$, then*

$$\Re\left\{1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > \beta = \frac{(\alpha - 1) + \sqrt{\alpha(\alpha + 8p!)}}{4\alpha}.$$

Remark 2.15. *Our result in Corollary 2.14 is an improvement of the result obtained by Saitoh et al. [14, Corollary 1].*

Putting $\alpha = 1$ in Corollary 2.12, we have

Corollary 2.16. *Let the function $f(z)$ defined by (1.1) be in the class K_p , then*

$$\Re\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \beta = \frac{-1 + \sqrt{1 + 8p!}}{4}.$$

Remark 2.17. *Putting $p = 1$ in Corollary 2.16, then if the function $f(z) \in A$ is convex in $\mathbb{U}$, then $f(z)$ is starlike of order $\frac{1}{2}$ in $\mathbb{U}$ (see also Saitoh et al. [14, Corollary 2]).*

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