Derivation on Amalgamated Duplication of a Ring Along an Ideal

Khalid Louartiti, Abdellah Mamouni and Mohammed Tamekkante

ABSTRACT: Let \( A \) be a ring and \( I \) be an ideal of \( A \). The amalgamated duplication of \( A \) along \( I \) is the subring of \( A \times A \) defined by \( A \bowtie I := \{ (a, a + i) \mid a \in A, \ i \in I \} \). In this paper, we characterize \( A \bowtie I \) over which any (resp. minimal) prime ideal is invariant under any derivation provided that \( A \) is semiprime. When \( A \) is noncommutative prime, then \( A \bowtie I \) is noncommutative semiprime (but not prime except if \( I = (0) \)). In this case, we prove that any map of \( A \bowtie I \) which is both Jordan and Jordan triple derivation is a derivation.

Key Words: (Jordan) derivation, Prime and semiprime rings, Extension of a ring by an ideal.

Contents

1 Introduction 1

2 Semiprime amalgamated duplication of a ring along an ideal with prime ideals invariant under derivations 2

3 (Jordan) derivations on amalgamated duplication of a ring along an ideal 7

1. Introduction

Throughout, \( A \) will represent an associative ring with center \( Z(A) \). By an ideal \( I \) in \( A \), we shall always mean a two-sided ideal of \( A \). An ideal \( P \) of \( A \) is said to be prime if \( P \neq A \) and, for \( a, b \in A \), \( aAb \subseteq P \) implies that \( a \in P \) or \( b \in P \). The ring \( A \) is called a prime ring if \( (0) \) is a prime ideal of \( A \). A ring \( A \) is called a semiprime ring if \( aAa = (0) \) implies \( a = 0 \). A ring \( A \) is said to be 2-torsion free, if whenever \( 2a = 0 \), with \( a \in A \), then \( a = 0 \). The Jordan product of two elements \( x \) and \( y \) of \( A \) is \( x \circ y = xy + yx \).

By a derivation of \( A \), we mean an additive map \( d : A \rightarrow A \) satisfying \( d(xy) = d(x)y + xd(y) \) for all pairs \( x, y \in A \). Given a derivation \( d \) of \( A \), an ideal \( I \) of \( A \) is said to be invariant under \( d \) (or \( d \)-invariant for short) if \( d(I) \subseteq I \). It is well known that every minimal prime ideal of a torsion-free semiprime ring is invariant under all derivations \([11]\). Herstein raised the following problem:

**Problem.** Given a semiprime ring \( A \), does \( d(P) \subseteq P \) hold for any minimal prime ideal \( P \) of \( A \) and for any derivation \( d \) of \( A \)?

This problem has been often mentioned in the literature (see, for example, \([3,13]\)). The best result of the conjecture is the following: A ring \( A \) is said to be of bounded index \( m \) if \( m \) is a positive integer such that \( x^m = 0 \) for all nilpotent elements \( x \in A \). Beidar and Mikhalev proved the theorem: Let \( A \) be a ring of bounded index \( m \) such that the additive order of every nonzero torsion element of \( A \), if any, is strictly larger than \( m \). Then all minimal prime ideals of \( A \) are invariant under all derivations of \( A \) (see \([1]\) or \([2,\ \text{Theorem 8.16}]\)). As a special case of this, every minimal prime ideal of a reduced ring is invariant under derivations of the ring (See \([7,\ \text{p. 614}]\)). Unfortunately, this problem turns out to be false in general. Chuang and Lee \([7]\) constructed a semiprime ring \( A \) which possesses a minimal prime ideal not invariant under a derivation of the ring.

Let \( A \) be a ring and \( I \) be an ideal of \( A \). The subset of \( A \times A \) defined by:

\[ A \bowtie I := \{ (a, a + i) \mid a \in A, \ i \in I \} \]

is clearly a subring of \( A \times A \), called the amalgamated duplication of \( A \) along \( I \). The construction \( A \bowtie I \) (in the commutative case) was introduced and its basic properties were studied by D’Anna and...
Fontana (2007) in [9,10], and then it was investigated by D’Anna in [8] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [8, Theorem 14 and Corollary 17]. The aim of Section 2 of this paper is to characterize when the amalgamated duplication of a semiprime ring along an ideal satisfies the Herstein’s Problem. Hence, Theorem 2.6 states that if \( A \) is a semiprime ring and \( I \) is an ideal of \( A \). Then, the following are equivalent:

1. \( d(P) \subseteq P \) holds for any (resp. minimal) prime ideal \( P \) of \( A \bowtie I \) and for any derivation \( d \) of \( A \bowtie I \).
2. \( \delta(p) \subseteq p \) holds for any (resp. minimal) prime ideal \( p \) of \( A \) and for any derivation \( \delta \) of \( A \) keeping \( I \) invariant.

An additive map \( d : A \to A \) is called a Jordan derivation if \( d(x^2) = d(x)x + xd(x) \) for all \( x \in A \), and \( d \) is called a Jordan triple derivation if \( d(xy) = d(x)yx + xd(y)x + xyd(x) \) for all \( x, y \in A \). If \( A \) is 2-torsion-free, then every Jordan derivation is a Jordan triple derivation ([4, Proposition 2]). Obviously, every derivation is a Jordan (resp. triple) derivation. The converse is in general not true. In [5, Theorem 4.3], Brešar proved that if \( A \) is 2-torsion free semiprime then every Jordan triple derivation is a derivation. Which means that derivations, Jordan derivations, and Jordan triple derivations of a 2-torsion-free semiprime ring are the same. The case when the ring is of characteristic 2 is due to Herstein who proved (in [12, Theorem 4.1]) that over a noncommutative ring any map which is both Jordan derivation and Jordan triple derivation becomes a derivation. In Section 3, we extend the Herstein’s result to semiprime rings with form \( A \bowtie I \) where \( A \) is a prime noncommutative ring.

Let’s adopt the following notations:

**Notations.** Let \( A \) be a ring and \( I \) be an ideal of \( A \). By \( \pi_1 \) and \( \pi_2 \) we denote the naturel surjections of \( A \bowtie I \) into \( A \) defined by

\[
\pi_1(a, a + i) = a \quad \text{and} \quad \pi_2(a, a + i) = a + i \quad \text{for all} \quad a \in A, \ i \in I.
\]

For an additive map \( d : A \bowtie I \to A \bowtie I \), we consider the maps \( d_{i=1,2} : A \to A \) and \( s_{i=1,2} : I \to A \) defined by

\[
d_1(a) = \pi_1 \circ d(a), \quad d_2(a) = \pi_2 \circ d(a), \quad s_1(i) = \pi_1 \circ d(0, i), \quad s_2(i) = \pi_2 \circ d(i, 0)
\]

for all \( a \in A \) and \( i \in I \). It is clear that \( d_1, d_2, s_1, \) and \( s_2 \) are all additive.

### 2. Semiprime amalgamated duplication of a ring along an ideal with prime ideals invariant under derivations

In this section, we characterize the derivations of \( A \bowtie I \), specially when \( A \) is a semiprime ring. Our aim is to see when every (minimal) prime ideal of \( A \bowtie I \) is invariant under any derivation on \( A \bowtie I \).

Let \( R \) and \( T \) be rings and let \( \theta \) and \( \phi \) be homomorphisms of \( T \) into \( R \). Let \( X \) be an \( R \)-bimodule. Following [6], an additive mapping \( d : T \to X \) is called a \((\theta, \phi)\)-derivation (resp. \( \theta \) Jordan \((\theta, \phi)\)-derivation) if \( d(xy) = d(x)\phi(y) + \theta(x)d(y) \), for all \( x, y \in T \) (resp. if \( d(x^2) = d(x)\phi(x) + \theta(x)d(x) \), for all \( x \in T \)).

Suppose that \( d : T \to T \) is a (resp. Jordan) derivation. Then, \( \theta \circ d \) is a (resp. Jordan) \((\theta, \theta)\)-derivation. Indeed, \( \theta \circ d \) is clearly additive, and for all \( x, y \in T \), we have

\[
\theta \circ d(xy) = \theta(d(x)y + xd(y)) = \theta \circ d(x)\theta(y) + \theta(x)\theta \circ d(y)
\]

(resp. \( \theta \circ d(x^2) = \theta(d(x)x + xd(x)) = \theta \circ d(x)\theta(x) + \theta(x)\theta \circ d(x) \)).

We start with the following lemma.

**Lemma 2.1.** Let \( A \) be a ring and \( I \) be an ideal of \( A \). A map \( d : A \bowtie I \to A \bowtie I \) is a (resp. Jordan) derivation if and only if \( \pi_1 \circ d \) is a (resp. Jordan) \((\pi_1, \pi_1)\)-derivation and \( \pi_2 \circ d \) is a (resp. Jordan) \((\pi_2, \pi_2)\)-derivation.
Proof: (⇒) Clear.
(⇐) It is clear that, for all \( a \in A \) and \( i \in I \), we have
\[
d(a, a + i) = (\pi_1 \circ d(a, a + i), \pi_2 \circ d(a, a + i)).
\]
Hence, if \( \pi_1 \circ d \) and \( \pi_2 \circ d \) are additive then so is \( d \).
Suppose that \( \pi_1 \circ d \) is a \((\pi_1, \pi_1)\)-derivation and \( \pi_2 \circ d \) is a \((\pi_2, \pi_2)\)-derivation. For all \( a, b \in A \) and \( i, j \in I \) we have
\[
d((a, a + i)(b, b + j)) = \pi_1 \circ d((a, a + i)(b, b + j)), \pi_2 \circ d((a, a + i)(b, b + j))
\]
\[
= (\pi_1 \circ d(a, a + i) \pi_1 (b, b + j) + \pi_1 (a, a + i) \pi_1 \circ d(b, b + j), \pi_2 \circ d(a, a + i) \pi_2 (b, b + j) + \pi_2 (a, a + i) \pi_2 \circ d(b, b + j))
\]
\[
= (\pi_1 \circ d(a, a + i) b + a \pi_1 \circ d(b, b + j), \pi_2 \circ d(a, a + i) (b, b + j) + (a, a + i) \pi_2 \circ d(b, b + j))
\]
\[
= d(a, a + i) (b, b + j) + (a, a + i)d(b, b + j).
\]
Hence, \( d \) is a derivation.
By the same way, we show that if \( \pi_1 \circ d \) is a Jordan \((\pi_1, \pi_1)\)-derivation and \( \pi_2 \circ d \) is a Jordan \((\pi_2, \pi_2)\)-derivation then \( d \) is a Jordan derivation.

The next result gives the necessary and sufficient conditions for an additive map \( d \) from \( A \bowtie I \) into itself to be a derivation.

**Proposition 2.2.** Let \( A \) be a ring, \( I \) be an ideal of \( A \), and \( d : A \bowtie I \to A \bowtie I \) be an additive map. Then, \( d \) is a derivation if and only if

1. \( d_1 \) and \( d_2 \) are derivations.
2. \( s_k(ai) = as_k(i), s_k(ia) = s_k(i)a, \) and \( s_k(ij) = 0 \) for \( k = 1, 2 \) and for all \( a \in A \) and \( i, j \in I \).

Proof: (⇒) From Lemma 2.1, \( \pi_1 \circ d \) is a \((\pi_1, \pi_1)\)-derivation and \( \pi_2 \circ d \) is a \((\pi_2, \pi_2)\)-derivation. Hence, for all \( a \in A \), we have
\[
d_1(ab) = \pi_1 \circ d(ab, ab)
\]
\[
= \pi_1 \circ d((a, a)(b, b))
\]
\[
= \pi_1 \circ d(a, a) b + a \pi_1 \circ d(b, b)
\]
\[
= d_1(a) b + ad_1(b).
\]
Hence, \( d_1 \) is derivation. Similarly, we obtain that \( d_2 \) is a derivation.
Let \( a \in A \) and \( i \in I \). We have
\[
s_1(ai) = \pi_1 \circ d(0, ai)
\]
\[
= \pi_1 \circ d((a, a)(0, i))
\]
\[
= \pi_1 \circ d(a, a) \pi_1 (0, i) + \pi_1 (a, a) \pi_1 \circ d(0, i)
\]
\[
= as_1(i).
\]
Similarly, \( s_1(ia) = s_1(i)a \). Now, for all \( i, j \in I \), we have
\[
s_1(ij) = \pi_1 \circ d(0, ij)
\]
\[
= \pi_1 \circ d((0, i)(0, j))
\]
\[
= \pi_1 \circ d(0, i) \pi_1 (0, j) + \pi_1 (0, i) \pi_1 \circ d(0, j)
\]
\[
= 0.
\]
By the same argument, we prove that $s_2$ satisfies the same conditions.

$(\Leftarrow)$ For all $a \in A$ and $i \in I$ we have

$$
\pi_1 \circ d(a, a + i) = \pi_1 \circ d(a, a) + \pi_1 \circ d(0, i) = d_1(a) + s_1(i).
$$

and

$$
\pi_2 \circ d(a, a + i) = \pi_2 \circ d(a + i, a + i) - \pi_2 \circ d(i, 0) = d_2(a + i) - s_2(i).
$$

By Lemma 2.1, we have to prove that $\pi_1 \circ d$ is a $(\pi_1, \pi_1)$-derivation and $\pi_2 \circ d$ is a $(\pi_2, \pi_2)$-derivation. Let $a, b \in A$ and $i, j \in I$. We have

$$
\pi_1 \circ d((a, a + i)(b, b + j)) = \pi_1 \circ d(ab, ab + aj + ib + ij)
$$

$$
= d_1(ab) + s_1(aj + ib + ij)
$$

$$
= d_1(a)b + ad_1(b) + as_1(j) + s_1(i)b
$$

$$
= (d_1(a) + s_1(i))b + a(d_1(b) + s_1(j))
$$

$$
= \pi_1 \circ d(a, a + i)\pi_1(b, b + j)
$$

$$
+ \pi_1(a, a + i)\pi_1 \circ d(b, b + j)
$$

and, since $0 = s_2(ij) = s_2(i)j = is_2(j)$, we get

$$
\pi_2 \circ d((a, a + i)(b, b + j)) = \pi_2 \circ d(ab, ab + aj + ib + ij)
$$

$$
= d_2((a + i)(b + j)) - s_2(aj + ib + ij)
$$

$$
= d_2(a + i)(b + j) + (a + i)d_2(b + j)
$$

$$
- as_2(j) - s_2(i)b
$$

$$
= (d_2(a + i) - s_2(i))(b + j)
$$

$$
+ (a + i)(d_2(b + j) - s_2(j))
$$

$$
= \pi_2 \circ d(a, a + i)\pi_2(b, b + j)
$$

$$
+ \pi_2(a, a + i)\pi_2 \circ d(b, b + j)
$$

Hence, we have the desired result. \hfill \Box

Next, we characterize the derivations of $A \bowtie I$ when $A$ is a semiprime ring.

**Proposition 2.3.** Let $A$ be a semiprime ring, $I$ be an ideal of $A$, and $d : A \bowtie I \rightarrow A \bowtie I$ be an additive map. Then, the following are equivalent:

1. $d$ is a derivation

2. $d_1$ and $d_2$ are derivations and the ideals $0 \times I$ and $I \times 0$ of $A \bowtie I$ are $d$-invariant.

3. there exist a derivation $\delta_1 : A \rightarrow A$ keeping $I$ invariant and a derivation $\delta_2 : A \rightarrow I$ such that

$$
d(a, a + i) = (\delta_1(a), \delta_1(a + i) + \delta_2(a + i)) \quad \text{for all } a \in A, i \in I.
$$

**Proof:** (1) $\Rightarrow$ (2) From Proposition 2.2, we have $s_k(ai) = as_k(i)$, $s_k(ia) = s_k(i)a$, and $s_k(ij) = 0$ for $k = 1, 2$ and for all $a \in A$ and $i, j \in I$. Then, for any $i \in I$, we have

$$
s_k(i)as_k(i) = s_k(i)s_k(ai) = s_k(is_k(ai)) = s_k(s_k(iai)) = 0 \quad \text{for all } a \in A.
$$

Thus, since $A$ is semiprime, we have that $s_k(i) = 0$ for all $i \in I$. Hence, for all $i \in I$, we have $\pi_1 \circ d(0, i) = 0$ and $\pi_2 \circ d(i, 0) = 0$. So, $d(0, i) = (0, r) \in A \bowtie I$ and $d(i, 0) = (r', 0) \in A \bowtie I$. Consequently, $r, r' \in I$, $d(0, i) \in 0 \times I$, and $d(i, 0) \in I \times 0$. 
(2) ⇒ (3) Since \(0 \times I\) and \(I \times 0\) are \(d\)-invariant, we have clearly \(s_1 = s_2 = 0\). Thus, for all \(a \in A\) and \(i \in I\), we have
\[
d(a, a + i) = (d_1(a), d_2(a + i)).
\]
Set \(\delta_1 = d_1\) and \(\delta_2 = d_2 - d_1\). For all \(i \in I\), we have \(\delta_1(i) = \pi_1 \circ d(i, i) = \pi_1 \circ d(i, 0) + \pi_1 \circ d(0, i) = \pi_1 \circ d(i, 0) \in I\). Hence, \(I\) is \(\delta_1\)-invariant. Set \(d(a, a) = (b, b + j)\) for some \(b \in A\) and \(j \in I\). We have
\[
d_2(a) = d_2(a) - d_1(a) = \pi_2 \circ d(a, a) - \pi_1 \circ d(a, a) = (b + j) - b = j \in I.
\]
Then, \(\delta_2(A) \subseteq I\).

(3) ⇒ (1) Suppose that
\[
d(a, a + i) = (d_1(a), \delta_1(a + i) + \delta_2(a + i)) \quad \text{for all } a \in A, i \in I
\]
with \(\delta : A \to A\) is a derivation keeping \(I\) invariant and \(\delta_2 : A \to I\) is a derivation. Firstly, \(d\) is well defined. Indeed, for all \(a \in A\) and \(i \in I\), we have
\[
(\delta_1(a + i) + \delta_2(a + i)) - \delta_1(a) = \delta_1(i) + \delta_2(a + i) \in I.
\]
A simple check shows that such \(d\) is a derivation. \(\Box\)

We need the following lemmas.

**Lemma 2.4.** Let \(p\) be a prime ideal of \(A\). Then,
\[
p \bowtie I := \{(a, a + i) \mid a \in p, i \in I\}
\]
and
\[
\overline{p} := \{(a + i, a) \mid a \in p, i \in I\}
\]
are prime ideals of \(A \bowtie I\).

**Proof:** Clearly \(p \bowtie I\) and \(\overline{p}\) are ideals of \(A \bowtie I\). Moreover, the mappings \(\psi : \frac{A \bowtie I}{p \bowtie I} \to \frac{A}{p}\) and \(\varphi : \frac{A \bowtie I}{\overline{p}} \to \frac{A}{p}\)
defined by \(\psi \left(\frac{(a, a + i)}{p \bowtie I}\right) = \frac{a}{p}\) and \(\varphi \left(\frac{(a, a + i)}{\overline{p}}\right) = \frac{a + i}{p}\) are well defined isomorphisms of rings. Then, since \(p\) is prime, \(\frac{A}{p}\) is a prime ring and so are \(\frac{A \bowtie I}{p \bowtie I}\) and \(\frac{A \bowtie I}{\overline{p}}\). Then, \(p \bowtie I\) and \(\overline{p}\) are prime ideals of \(A \bowtie I\). \(\Box\)

**Lemma 2.5.** Let \(P\) be a prime ideal of \(A \bowtie I\). Then, \(0 \times I \subseteq P\) or \(I \times 0 \subseteq P\). Moreover,

1. If \(0 \times I \subseteq P\) then there exists a prime ideal \(p\) of \(A\) such that
\[
P = p \bowtie I := \{(a, a + i) \mid a \in p, i \in I\}.
\]
2. If \(I \times 0 \subseteq P\) then there exists a prime ideal \(p\) of \(A\) such that
\[
P = \overline{p} := \{(a + i, a) \mid a \in p, i \in I\}.
\]

In the both cases, \(P\) is minimal if and only if \(p\) is minimal.

**Proof:** Suppose that \(0 \times I \not\subseteq P\). Then, there exists \(i_0 \in I\) such that \((0, i_0) \notin P\). However, for any \(i, j \in I\) and \(a \in A\), we have \((i, 0)(a, a + j)(0, i_0) = (0, 0) \in P\). Hence, \((i, 0) \in P\) for all \(i \in I\). Thus, \(I \times 0 \subseteq P\).

(1) Set \(p = \pi_1(P)\). It is clear that \(p\) is an ideal of \(A\) (since \(\pi_1\) is surjective). Let \(a, b \in A\) with \(arb \in p\) for all \(r \in A\). Then, for each \(r\) there exists \(i_r \in I\) such that \((arb, arb + i_r) \in P\). Then, for all \(j \in I\), \((arb, a(r + j)b) = (arb, arb + i_r) + (0, ajb - i_r) \in P\) since \(0 \times I \subseteq P\). Thus, \((a, a)(r, r + j)(b, b) \in P\). Hence, \((a, a) \in P\) or \((b, b) \in P\). Then, \(a \in p\) or \(b \in p\). So, \(p\) is prime.

Clearly, we have \(P \subseteq p \bowtie I\). For the reverse inclusion, let \(a \in p\). There exists \(i \in I\) such that \((a, a + i) \in P\).
Hence, for all $j \in I$, we have $(a, a + j) = (a, a + i) + (0, j - i) \in P$. Then, $P = p \triangleright I$.

(2) Set $p = \pi_2(P)$. It is clear that $p$ is an ideal of $A$ (since $\pi_2$ is surjective). Let $a, b \in A$ with $arb \in p$ for all $r \in A$. Then, for each $r$ there exists $i_r \in I$ such that $(arb + i_r, arb) \in P$. Then, for all $j \in I$, $(arb, a(r + j)b) = (a(r + j)b + i_r + j, a(r + j)b) = (i_r + j, a + j, 0) \in P$ since $I \times 0 \subseteq P$. Thus, $(a, a)(r, r + j)(b, b) \in P$. Hence, $(a, a) \in P$ or $(b, b) \in P$. Hence, $a \in p$ or $b \in p$. So, $p$ is prime.

Clearly, $P \subseteq \overline{p}$. Now, let $a \in p$. There exists $i \in i$ such that $(a + i, a) \in P$. Hence, for all $j \in I$, we have $(a + j, a) = (a + i + a) + (j - i, 0) \in P$. Then, $P = \overline{p}$.

For the last statement, let $p$ be a prime ideal of $A$.

Suppose that $P = p \triangleright I$ is minimal prime and let $q$ be a prime ideal of $A$ with $q \subseteq p$. Easily, we can see that $q \triangleright I \subseteq p \triangleright I = P$. Since $q \triangleright I$ is prime (by Lemma 2.4), we have $P = q \triangleright I$, and so $p = \pi_1(q \triangleright I) = q$.

Conversely, suppose that $p$ is minimal prime, and let $Q$ be a prime ideal of $A \triangleright I$ with $Q \subseteq P$. If $0 \times I \subseteq Q$ then $Q = q \triangleright I$ for some prime ideal $q$ of $A$, and so we get $q \subseteq p$ which means that $q = p$, and then $Q = P$. Now, if $I \times 0 \subseteq Q$ then $Q = \overline{q}$ for some prime ideal $q$ of $A$. Hence, $q \subseteq q + I = \pi_1(Q) \subseteq \pi_1(P) = p$, and then $I \subseteq q = p$. Hence,

$$Q = \overline{q} = \{(a + i, a) \mid a \in q, i \in I\} = \{(a + i + a) \mid a \in q, i \in I\} = q \triangleright I = P.$$}

Now, suppose that $P = \overline{q}$ is minimal prime, and let $q$ be a prime ideal of $A$ such that $q \subseteq p$. Then, $\overline{q} \subseteq \overline{p} = P$. Then, $\overline{q} = \overline{p}$. Hence, $p = q$. Therefore, $p$ is minimal.

Conversely, suppose that $p$ is minimal and let $Q$ be a prime ideal of $A \triangleright I$ with $Q \subseteq P = \overline{q}$. If $0 \times I \subseteq Q$ then $Q = q \triangleright I$ for some prime ideal $q$ of $A$. Then, $\pi_2(Q) \subseteq \pi_2(P)$ means that $q + I \subseteq p$. Hence, $I \subseteq q = p$. So,

$$Q = \{(a, a + i) \mid a \in q, i \in I\} = \{(a + i, a) \mid a \in q, i \in I\} = \overline{q} = P.$$}

If $I \times 0 \subseteq Q$ then $Q = \overline{q}$ for some prime ideal $q$ of $A$. Hence, $q \subseteq p$, and so $q = p$. Then, $Q = P$. \hfill \square

The main result of this section is as follows:

**Theorem 2.6.** Let $A$ be a semiprime ring and $I$ be an ideal of $A$. The following are equivalent:

1. $d(P) \subseteq P$ holds for any (resp. minimal) prime ideal $P$ of $A \triangleright I$ and for any derivation $d$ of $A \triangleright I$.

2. $\delta(p) \subseteq p$ holds for any (resp. minimal) prime ideal $P$ of $A$ and for any derivation $\delta$ of $A$ keeping $I$ invariant.

**Proof:** ($\Rightarrow$) Let $\delta$ be a derivation on $A$ with $\delta(I) \subseteq I$. Then, by Proposition 2.3, the additive map $d : A \triangleright I \to A \triangleright I$ defined by $d(a, a + i) = (\delta(a), \delta(a + i))$ is a derivation on $A \triangleright I$. Let $p$ be a (resp. minimal) prime ideal of $A$. By Lemmas 2.4 and 2.5, $p \triangleright I$ is a (resp. minimal) prime ideal of $A \triangleright I$. Hence, $d(p \triangleright I) \subseteq p \triangleright I$. Let $a \in p$. Then, $(a, a) \in p \triangleright I$. Thus, $(\delta(a), \delta(a)) = d(a, a) \in p \triangleright I$, and so $\delta(a) \in p$. Hence, $\delta(p) \subseteq p$.

($\Leftarrow$) Let $d$ be a derivation on $A \triangleright I$. Following Proposition 2.3, there exist a derivation $\delta_1 : A \to A$ keeping $I$ invariant and a derivation $\delta_2 : A \to I$ such that

$$d(a, a + i) = (\delta_1(a), \delta_1(a + i) + \delta_2(a + i))$$

for all $a \in A, i \in I$.

Let $P$ be a (resp. minimal) prime ideal of $A \triangleright I$. Then, using Lemma 2.5, $P = p \triangleright I$ or $P = \overline{p}$ for some (resp. minimal) prime ideal $p$ of $A$. By hypothesis, $\delta_1(p) \subseteq p$ and $\delta_2(p) \subseteq p$ (see that $I$ is also invariant under $\delta_2$).

Suppose that $P = p \triangleright I$. Then, the elements of $P$ have the form $(a, a + i)$ with $a \in p$ and $i \in I$, and we have

$$d(a, a + i) = (\delta_1(a), \delta_1(a + i) + \delta_2(a + i)) = (\delta_1(a), \delta_1(a) + (\delta_1(i) + \delta_2(a + i))) \in P$$

since $\delta_1(a) \in p$ and $\delta_1(i) + \delta_2(a + i) \in I$. Thus, $d(P) \subseteq P$.

Now, suppose that $P = \overline{p}$. The elements of $P$ in this case have the form $(a + i, a)$ with $a \in p$ and $i \in I$,
and we have
\[
d(a + i, a) = (\delta_1(a + i), \delta_1(a) + \delta_2(a)) = (\delta_1(a) + \delta_2(a) + (\delta_1(i) - \delta_2(a)), \delta_1(a) + \delta_2(a)) \in P
\]
since \(\delta_1(a) + \delta_2(a) \in p\) and \(\delta_1(i) - \delta_2(a) \in I\). Again, \(d(P) \subseteq P\).

As consequences of the above theorem, we have the following corollaries.

**Corollary 2.7.** Let \(A\) be a semiprime ring and \(I\) be a prime ideal of \(A\). The following are equivalent:

1. \(d(P) \subseteq P\) holds for any prime ideal \(P\) of \(A \triangleright I\) and for any derivation \(d\) of \(A \triangleright I\).
2. \(\delta(p) \subseteq p\) holds for any prime ideal \(p\) of \(A\) and for any derivation \(\delta\).

**Corollary 2.8.** Let \(A\) be a semiprime ring and \(I\) be a minimal prime ideal of \(A\). The following are equivalent:

1. \(d(P) \subseteq P\) holds for any minimal prime ideal \(P\) of \(A \triangleright I\) and for any derivation \(d\) of \(A \triangleright I\).
2. \(\delta(p) \subseteq p\) holds for any minimal prime ideal \(p\) of \(A\) and for any derivation \(\delta\).

**Corollary 2.9.** Let \(A\) be a prime ring and \(I\) an ideal of \(A\). Then, \(d(P) \subseteq P\) holds for any minimal prime ideal \(P\) of \(A \triangleright I\) and for any derivation \(d\) of \(A \triangleright I\).

**Proof:** Follows immediately from Theorem 2.6 since the only minimal prime ideal of \(A\) is \((0)\) which is always invariant under any derivation on \(A\) (in particular under those keeping \(I\) invariant).

3. **(Jordan) derivations on amalgamated duplication of a ring along an ideal**

**Proposition 3.1.** Let \(A\) be a ring, \(I\) be an ideal of \(A\), and \(d : A \triangleright I \rightarrow A \triangleright I\) be an additive map. Then, \(d\) is a Jordan derivation if and only if

1. \(d_1\) and \(d_2\) are Jordan derivations.
2. \(s_k(a \circ i) = a \circ s_k(i)\) and \(s_k(i^2) = 0\) for all \(k = 1, 2, a \in A\) and \(i, j \in I\).

**Proof:** Let \(R\) and \(T\) be rings and let \(\theta\) be a homomorphism of \(T\) into \(R\). it’s easy to check that if \(d : T \rightarrow R\) is a Jordan \((\theta, \theta)\)-derivation then for all \(x, y \in T\) we have
\[
d(x \circ y) = d(x) \circ \theta(y) + \theta(x) \circ d(y).
\]

\((\Rightarrow)\) From Lemma 2.1, \(\pi_1 \circ d\) is a Jordan \((\pi_1, \pi_1)\)-derivation and \(\pi_2 \circ d\) is a Jordan \((\pi_2, \pi_2)\)-derivation. Hence, for all \(a \in A\), we have
\[
d_1(a^2) = \pi_1 \circ d(a^2, a^2) = \pi_1 \circ d((a, a)(a, a)) = \pi_1 \circ d(a, a)a + a\pi_1 \circ d(a, a) = d_1(a)a + ad_1(a).
\]
Hence, \(d_1\) is Jordan derivation. Similarly, we obtain that \(d_2\) is a Jordan derivation.

Let \(a \in A\) and \(i \in I\). We have
\[
s_1(a \circ i) = \pi_1(d(0, a \circ i)) = \pi_1(d((a, a) \circ (0, i))) = \pi_1(d(a, a)) \circ \pi_1(0, i) + \pi_1(a, a) \circ \pi_1(d(0, i)) = a \circ s_1(i).
\]
Moreover, for all \( i \in I \), we have
\[
s_1(i^2) = \pi_1 \circ d(0, i^2) = \pi_1 \circ d((0, i)(0, i)) = \pi_1 \circ d(0, i)\pi_1(0, i) + \pi_1(0, i)\pi_1 \circ d(0, i) = 0.
\]
Similarly, \( s_2 \) satisfies the same conditions.

(\( \iff \)) As in the proof of Proposition 2.2, for all \( a \in A \) and \( i \in I \), we have
\[
\pi_1 \circ d(a, a + i) = d_1(a) + s_1(i) \quad \text{and} \quad \pi_2 \circ d(a, a + i) = d_2(a + i) - s_2(i).
\]
Using Lemma 2.1, we have to prove that \( \pi_1 \circ d \) is a Jordan \((\pi_1, \pi_1)\)-derivation and \( \pi_2 \circ d \) is a Jordan \((\pi_2, \pi_2)\)-derivation. Let \( a \in A \) and \( i \in I \). We have
\[
\pi_1 \circ d((a, a + i)^2) = \pi_1 \circ d(a^2, a^2 + ai + ia + i^2) = d_1(a)^2 + s_1(a \circ i + i^2) = d_1(a) \circ a + a \circ s_1(i) = (d_1(a) + s_1(i)) \circ a = \pi_1 \circ d(a, a + i)\pi_1(a, a + i) + \pi_1(a, a + i)\pi_1 \circ d(a, a + i).
\]
and, since \( 0 = 2s_2(i^2) = s_2(i \circ i) = s_2(i) \circ i = is_2(i) + s_2(i)i \), we get
\[
\pi_2 \circ d((a, a + i)^2) = \pi_2 \circ d(a^2, (a + i)^2) = d_2((a + i)^2) - s_2(a \circ i + i^2) = d_2(a + i)(a + i) + (a + i)d_2(a + i) - a \circ s_2(i) = (d_2(a + i) - s_2(i))(a + i) + (a + i)(d_2(a + i) - s_2(i)) = \pi_2 \circ d(a, a + i)\pi_2(a, a + i) + \pi_2(a, a + i)\pi_2 \circ d(a, a + i).
\]
Hence, we have the desired result. \( \square \)

**Lemma 3.2.** Let \( A \) be a ring and \( I \) be an ideal of \( A \). Then,

1. \( A \bowtie I \) is prime if and only if \( I = (0) \) and \( A \) is prime.
2. \( A \bowtie I \) is semiprime if and only if \( A \) is semiprime.
3. \( A \bowtie I \) is 2-torsion free if and only if \( A \) is 2-torsion free.

**Proof:** (1) Suppose that \( A \bowtie I \) is prime. Hence, \( \{(0, 0)\} \) is a prime ideal of \( A \bowtie I \). Thus, by Lemma 2.5, \( 0 \times I \subseteq \{(0, 0)\} \) or \( I \times 0 \subseteq \{(0, 0)\} \). In the both cases, \( I = (0) \). By Lemma 2.5, \( \{(0, 0)\} = p \bowtie (0) \) for some prime ideal of \( A \). Hence, \( p = (0) \) is a prime ideal of \( A \), and so \( A \) prime. Conversely, if \( I = (0) \) and \( A \) is prime then \( \{(0, 0)\} = (0) \bowtie (0) \) is a prime ideal of \( A \bowtie I \), and then \( A \bowtie I \) is prime.

(2) Suppose that \( A \bowtie I \) is semiprime and let \( a \in A \) with \( ara = 0 \) for all \( r \in A \). Then, \( (a, a)(r, r + j)(a, a) = (0, 0) \) for all \( r \in A \) and all \( j \in I \). Hence, \( (a, a) = (0, 0) \), and so \( a = 0 \). Thus, \( A \) is semiprime. Conversely, suppose that \( A \) is semiprime and let \( a \in A \) and \( i \in I \) with \( (a, a + i)(r, r + j)(a, a + i) = (0, 0) \) for all \( r \in A \) and \( j \in I \). Then, \( ara = 0 \) for all \( r \in A \), and then \( a = 0 \). Now, we have \( i(r + j)i = 0 \) for all \( r \in A \) and \( j \in I \). Which means that \( iri = 0 \) for all \( r \in A \). Then, \( i = 0 \). Hence, \( A \bowtie I \) is prime.

(3) Trivial. \( \square \)

**Corollary 3.3.** Let \( A \) be a 2-torsion free semiprime ring, \( I \) be an ideal of \( A \), and \( s : I \to A \) be an additive map. Then,
1. if \( s(a \circ i) = a \circ s(i) \) and \( s(i^2) = 0 \) for all \( a \in A \) and \( i \in I \) then \( s = 0 \).

2. if there exists a derivation \( d \) on \( A \) such that \( s(a \circ i) = d(a) \circ i + a \circ s(i) \) and \( s(i^2) = s(i) \circ i \) for all \( a \in A \) and \( i \in I \) then \( d \) and \( s \) coincide on \( I \).

**Proof:** (1) Consider the additive map \( d : A \bowtie I \to A \bowtie I \) defined by \( d(a, a + i) = (s(i), s(i)) \). For all \( a \in A \) and \( i \in I \), we have

\[
d((a, a + i)^2) = (s(a \circ i + i^2), s(a \circ i + i^2)) = (s(a \circ i), a \circ s(i)) = (s(i), s(i))(a, a + i) + (a, a + i)(s(i), s(i)) = (d(a, a + i)(a, a + i) + (a, a + i)d(a, a + i).
\]

since \( s(i) \circ i = s(2i^2) = 0 \). Hence, \( d \) is a Jordan derivation on \( A \bowtie I \). But \( A \bowtie I \) is a 2-torsion free semiprime ring (by Lemma 3.2). Thus, by [4, Theorem 1], \( d \) is a derivation. So, by Proposition 2.3, \( 0 \times I \) is \( d \)-invariant. Hence, for all \( i \in I \), \( d(0, i) = (s(i), s(i)) \) \( 0 \times I \), and so \( s = 0 \).

(2) Set \( s' := s - d : I \to A \). For all \( a \in A \) and \( i \in I \), we have

\[
s'(a \circ i) = s(a \circ i) - d(a \circ i) = d(a) \circ i + a \circ s(i) - d(a) \circ i - a \circ d(i) = a \circ s(i) - a \circ d(i) = a \circ s'(i)
\]

and

\[
s'(i^2) = s(i^2) - d(i^2) = s(2i^2) - s(i^2) - d(i^2) = s(i) - i \circ s(i) - d(i) \circ i = 0.
\]

Hence, from (1), \( s' = 0 \), and so \( s(i) = d(i) \) for all \( i \in I \). \( \square \)

**Theorem 3.4.** Let \( A \) be a non commutative prime ring and \( I \) be a nonzero ideal. If \( d \) is both a Jordan derivation and a Jordan triple derivation of \( A \bowtie I \) then \( d \) is a derivation.

**Proof:** When the characteristic of \( A \) is different of 2, the result follows from by [4, Theorem 1] and Lemma 3.2. Hence, suppose that \( A \) is of characteristic two. Also, if \( I = (0) \), then \( A \bowtie (0) \cong A \) (following the isomorphism \( (a, a) \mapsto a \)). In this case, the result follows immediately from [12, Theorem 4.1]. Thus, we may suppose \( I \neq (0) \). From Proposition 3.1, \( d_1 \) and \( d_2 \) are Jordan derivations and, for \( k = 1, 2 \) and for all \( a \in A \) and \( i \in I \), we have \( s_k(a \circ i) = a \circ s_k(i) \) and \( s_k(i^2) = 0 \). Now, let \( a, b \in A \), we have

\[
d_1(aba) = \pi_1 \circ d(aba, aba) = \pi_1 \circ d((a, a)(b, b)(a, a)) = \pi_1(d(a, a)(ba, ba) + (a, a)d(b, b)(a, a) + (a, a)(b, b)d(a, a)) = \pi_1 \circ (abc + a(ba) + a(ba) + bac + ba) = d_1(a)ba + a \circ d_1(b)a + abd_1(a).
\]

Hence, \( d_1 \) is also a Jordan triple derivation. Similarly, \( d_2 \) is a Jordan triple derivation. Hence, since \( A \) is non commutative prime, \( d_1 \) and \( d_2 \) are derivations (by [12, Theorem 4.1]). Let \( i, j \in I \). We have

\[
s_1(iji) = \pi_1 \circ d(0, iji) = \pi_1 \circ d(0, i)(0, j)(0, i) = \pi_1(d(0, i)(0, ji) + (0, i)d(0, j)(0, i) + (0, ij)d(0, i)) = 0
\]

Also,

\[
s_1(iji) = \pi_1 \circ d(0, iji) = \pi_1 \circ d((i, i)(0, j)(i, i)) = \pi_1(d(i, i)(0, ji) + (i, i)d(0, j)(i, i) + (0, ij)d(i, i)) = i\pi_1(d(0, j))i = is_1(j)i.
\]
Hence, \[ 0 = s_1(iji) = is_1(j)i \quad \text{for all } i, j \in I. \] (3.1)

By analogy \( s_2 \) satisfies the same condition.

For \( k = 1, 2 \), by linearizing the condition \( s_k(i^2) = 0 \) for all \( i \in I \), we obtain

\[ s_k(i \circ j) = 0 \quad \text{for all } i, j \in I. \] (3.2)

Hence, \[ ias_k(j) + s_k(j)ia = ia \circ s_k(j) = s_k(ia \circ j) = 0 \quad \text{for all } i, j \in I, a \in A. \] (3.3)

Then, \[ i(a \circ s_k(j)) + (i \circ s_k(j))a = 0 \quad \text{for all } i, j \in I, a \in A. \] (3.4)

But \( i \circ s_k(j) = s_k(i \circ j) = 0 \). Hence, \[ i(a \circ s_k(j)) = 0 \quad \text{for all } i, j \in I, a \in A. \] (3.5)

So, \[ ir(a \circ s_k(j)) = 0 \quad \text{for all } i, j \in I, a, r \in A. \] (3.6)

Since \( A \) is prime, we get that \[ i = 0 \quad \text{or } a \circ s_k(j) = 0 \quad \text{for all } i, j \in I, a \in A. \] (3.7)

But \( I \neq (0) \), and then \[ a \circ s_k(j) = 0 \quad \text{for all } j \in I, a \in A. \] (3.8)

which means that \( s(j) \in Z(A) \) for all \( j \in I \) since \( A \) is of characteristic two.

Thus, (3.1) means that \[ s_k(j)i^2 = 0 \quad \text{for all } i, j \in I. \] (3.9)

Thus, since \( s(j) \in Z(A) \) for all \( j \in I \), we have

\[ s_k(j) = 0 \quad \text{or } i^2 = 0 \quad \text{for all } i, j \in I. \] (3.10)

If \( i^2 = 0 \) for all \( i \in I \), then \( 0 = ij + ji = ij - ji \) for all \( i, j \in I \). Hence, for all \( i \in I \) and \( r \in A \), we have \( ir = (ri)i = (ri)i = ri^2 = 0 \). Thus, \( i = 0 \) for all \( i \in I \) since \( A \) is prime. But \( I \neq 0 \), and so there exists \( i \in I \) such that \( i^2 \neq 0 \). Consequently, by (3.10), \( s_k(j) = 0 \) for all \( j \in I \). Seen Proposition 2.2, \( d \) is a derivation.

\[ \square \]

Acknowledgments

The authors thank the referees for a careful reading of the manuscript.

References


K. Louartiti,
Department of Mathematics,
Faculty of Sciences Ben M’sik
Hassan II University,
P.O Box 7955 Sidi Othman, Casablanca,
Morocco.
E-mail address: khalid.louartiti@usmba.ac.ma

and

A. Mamouni,
Department of Mathematics,
Faculty of Science and Technology Errachidia,
Moulay Ismail University,
P.O Box 509 Boutalamine, Errachidia,
Morocco.
E-mail address: a.mamouni.fste@gmail.com

and

M. Tamekkante,
ATOA Team,
Faculty of Sciences,
Moulay Ismail University,
P.O Box 11201 Zitoune, Meknes,
Morocco.
E-mail address: tamekkante@yahoo.fr