Some Classes of 3-Dimensional Trans-Sasakian Manifolds with Respect to Semi-Symmetric Metric Connection

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ABSTRACT: The object of the present paper is to study semi-symmetric metric connection on a 3-dimensional trans-Sasakian manifold. We found the necessary condition under which a vector field on a 3-dimensional trans-Sasakian manifold will be a strict contact vector field. Then, we obtained extended generalized $\phi$-recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Next, a 3-dimensional trans-Sasakian manifold satisfies the condition $\tilde{L} \cdot \tilde{S} = 0$ with respect to semi-symmetric metric connection is studied.

Key Words: Semi-symmetric metric connection, Trans-Sasakian manifold, Extended generalized $\phi$–recurrent trans-Sasakian manifold, Conharmonically curvature tensor.

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1. Introduction

Let $(M, \phi, \xi, \eta, g)$ be a $(2m + 1)$–dimensional almost contact metric manifold. Then the product $\tilde{M} = M \times R$ has a natural almost complex structure $J$ with the product metric $G$ being Hermitian metric. The geometry of the almost Hermitian manifold $(\tilde{M}, J, G)$ gives the geometry of the almost contact metric manifold $(M, \phi, \xi, \eta, g)$. Sixteen different types of structures on $M$ like Sasakian manifold, Kenmotsu manifold, and etc., are given by the almost Hermitian manifold $(\tilde{M}, J, G)$. The notion of trans-Sasakian manifolds was introduced by Oubina [11] in 1985. Then, J. C. Marrero [6] have studied the local structure of trans-Sasakian manifolds. In general, a trans-Sasakian manifold $(\tilde{M}, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$. Trans-Sasakian manifold of type $(0, 0), (\alpha, 0), (0, \beta)$ are called cosymplectic, $\alpha$–Sasakian and $\beta$–Kenmotsu manifold, respectively. Marrero has proved that trans-Sasakian structures are generalized quasi-Sasakian structure. He has also proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha$–Sasakian or $\beta$–Kenmotsu manifold. So, we have considered here three dimensional trans-Sasakian manifold.

The notion of a semi-symmetric linear connection on a differential manifold has been first introduced...
trans-Sasakian manifold if it satisfies the following condition
\[ \alpha \] for some smooth functions
\[ \phi \]
there exists a (1,1) tensor field
\[ n \] 
An (\alpha, \beta) tensor field \( n \) on \( M \) is defined by
\[
(nX)\xi = \alpha g(X,\xi) + \beta \phi g(\phi X,\phi Y),
\]
for any vector fields \( X, Y \) on \( M \). An odd dimensional almost contact metric manifold \( M \) is called a trans-Sasakian manifold if it satisfies the following condition
\[
(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\},
\]
for some smooth functions \( \alpha, \beta \) on \( M \) and we say that the trans-Sasakian structure is of type \((\alpha, \beta)\). For an \( n \)-dimensional trans-Sasakian manifold [9], from (2.7) we have
\[
\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi),
\]
(2.8) \[
(\nabla_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).
\]
(2.9)
In an \( n \)-dimensional trans-Sasakian manifold, we have
\[
R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X),
\]
(2.10) \[
2\alpha\beta + \xi\alpha = 0,
\]
(2.11) \[
S(X, \xi) = [(n - 1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) \]
\[
- ((\phi X)\alpha - (n - 2)(X\beta)).
\]
(2.12)
For \( \alpha, \beta = \text{constants} \) then the above equations reduce to
\[
R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(Y)X),
\]
(2.13) \[
R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y),
\]
(2.14) \[
S(X, Y) = \left[ \frac{r}{2} - (\alpha^2 - \beta^2) \right] g(X, Y) - \left[ \frac{r}{2} - 3(\alpha^2 - \beta^2) \right] \eta(X)\eta(Y),
\]
(2.15) \[
S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X),
\]
(2.16) \[
S(\phi X, \phi Y) = \left[ \frac{r}{2} - (\alpha^2 - \beta^2) \right] g(X, Y),
\]
(2.17) \[
S(\phi X, Y) = -S(X, \phi Y).
\]
(2.18)
Definition 2.1. A trans-Sasakian manifold $M^n$ is said to be $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form
\[ S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \]
where $a, b$ are smooth functions.

Definition 2.2. A vector field $X$ on a 3-dimensional trans-Sasakian manifold $(M^3, \phi, \eta, \xi, g)$ is said to be a contact vector field if
\[ (\mathcal{L}_X \eta)(Y) = \sigma \eta(Y), \]
where $\sigma$ is scalar function on $M^3$ and $\mathcal{L}_X$ denotes the Lie derivative along $X$. $X$ is called a strict contact vector field if $\sigma = 0$.

Let $(M^n, g)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. A linear connection $\tilde{\nabla}$ on $(M^n, g)$ is said to be semi-symmetric ([15], [18]) if its torsion tensor $T$ can be written as
\[ T(X, Y) = \pi(Y) X - \pi(X) Y, \]
where $\pi$ is an 1-form on $M^n$ and the associated vector field $\rho$ defined by $\pi(X) = g(X, \rho)$, for all vector fields $X \in \chi(M^n)$.

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if $\tilde{\nabla} g = 0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form $\pi$ of the above equation with the contact 1-form $\eta$, i.e., by setting [15]
\[ T(X, Y) = \eta(Y) X - \eta(X) Y, \]
with
\[ g(X, \rho) = \eta(X), \forall X \in \chi(M^n). \]

K. Yano has obtained the relation between semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ of $M^n$ in [18] and it is given by
\[ \tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) X - g(X, Y) \xi, \]
where $g(X, \xi) = \eta(X)$.

Further, a relation between the curvature tensors $R$ and $\hat{R}$ of type (1,3) of the connections $\nabla$ and $\tilde{\nabla}$, respectively is given by [18],
\[ \hat{R}(X, Y) Z = R(X, Y) Z - K(Y, Z) X + K(X, Z) Y - g(Y, Z) F X + g(X, Z) F X, \]
where $K$ is a tensor field of type $(0,2)$ and $F$ is a $(1,1)$ tensor field defined by
\[ K(Y, Z) = g(F Y, Z) = (\nabla_Y \eta)(Z) - \eta(X) \eta(Z) + \frac{1}{2} \eta(\xi) g(Y, Z). \]

In this paper, we have considered that $M^3$ is 3-dimensional trans-Sasakian manifold. So, using (2.9), (2.19), (2.23) it follows that
\[ K(Y, Z) = - \alpha g(\phi Y, Z) - (\beta + 1) \eta(Y) \eta(Z) + (\beta + \frac{1}{2}) g(Y, Z). \]
Using (2.22), from above equation we get

\[ FY = -\alpha \phi Y - (\beta + 1)\eta(Y)\xi + (\beta + \frac{1}{2})Y. \tag{2.25} \]

Now, by using above two equations we get

\[
\tilde{R}(X, Y)Z = R(X, Y)Z - \alpha(g(\phi X, Z)Y - g(\phi Y, Z)X) - \alpha(g(X, Z)\phi Y - g(Y, Z)\phi X) \\
-(\beta + 1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\
-(\beta + 1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi \\
+(2\beta + 1)(g(X, Z)Y - g(Y, Z)X). \tag{2.26}
\]

In the view of (2.26) we get

\[
\tilde{S}(Y, Z) = S(Y, Z) + \alpha g(\phi Y, Z) + (\beta + 1)\eta(Y)\eta(Z) - (3\beta + 1)g(Y, Z), \tag{2.27}
\]

where \(\tilde{S}\) and \(S\) are Ricci tensors of \(M^3\) with respect to semi-symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\nabla\), respectively.

From above, we have

\[
\tilde{r} = r - 8\beta - 2, \tag{2.28}
\]

where \(\tilde{r}\) and \(r\) are scalar curvature of \(M^3\) with respect to semi-symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\nabla\), respectively.

We obtain from (2.15) and (2.27) that

\[
\tilde{Q}\xi = 2(\alpha^2 - \beta^2 - \beta)\xi, \tag{2.29}
\]

where \(\tilde{Q}\) is the Ricci operator with respect to semi-symmetric metric connection \(\tilde{\nabla}\).

### 3. Geometric Vector Fields on 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Suppose that a contact vector field \(X\) on a 3-dimensional trans-Sasakian manifold leaves the Ricci tensor with respect to semi-symmetric metric connection invariant, i.e.

\[
(\mathcal{L}_X \tilde{S})(Y, Z) = 0. \tag{3.1}
\]

It follows from (3.1) we get

\[
\mathcal{L}_X \tilde{S}(Y, Z) = \tilde{S}(\mathcal{L}_X Y, Z) + \tilde{S}(Y, \mathcal{L}_X Z).
\]

Putting \(Z = \xi\) we have

\[
\mathcal{L}_X \tilde{S}(Y, \xi) = \tilde{S}(\mathcal{L}_X Y, \xi) + \tilde{S}(Y, \mathcal{L}_X \xi). \tag{3.2}
\]

From the equations (2.15) and (2.27) we have

\[
\tilde{S}(Y, \xi) = 2(\alpha^2 - \beta^2 - \beta)\eta(Y). \tag{3.3}
\]
From the equation (3.3) we get
\[ 2(\alpha^2 - \beta^2 - \beta)(\mathcal{L}_X\eta)(Y) = \bar{S}(Y, \mathcal{L}_X\xi). \]

Hence we have
\[ 2(\alpha^2 - \beta^2 - \beta)\eta(Y) = \bar{S}(Y, \mathcal{L}_X\xi). \] (3.4)

Taking \( Y = \xi \) in (3.4) we obtain
\[ \sigma = \eta(\mathcal{L}_X\xi). \] (3.5)

Again, putting \( Y = \xi \) in (2.19) one can get
\[ \sigma = -\eta(\mathcal{L}_X\xi). \] (3.6)

Hence, we have \( \sigma = 0 \).

Therefore, we state the following theorem:

**Theorem 3.1.** Every contact vector field on a 3-dimensional trans-Sasakian manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.

4. On Extended Generalized \( \phi \)-Recurrent 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Connection

**Definition 4.1.** A 3-dimensional trans-Sasakian manifold with respect to semi-symmetric connection is said to be a \( \phi \)-recurrent manifold if \( \exists \) a nonzero 1-form \( B \) such that
\[ \phi^2((\nabla_W R)(X, Y)Z) = B(W)R(X, Y)Z, \]
for arbitrary vector fields \( X, Y, Z, W \).

**Definition 4.2.** A Riemannian manifold \((M^3, g)\) is called \( \phi \)-generalized recurrent [2], if its curvature tensor \( R \) satisfies the condition
\[ \phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y], \]
where \( A \) and \( B \) are two 1-forms, \( B \) is non zero and these are defined by \( g(W, \rho_1) = A(W) \) and \( g(W, \rho_2) = B(W), \forall W \in \chi(M) \).

\( \rho_1 \) and \( \rho_2 \) being the vector fields associated to the 1-form \( A \) and \( B \) respectively.

**Definition 4.3.** A three-dimensional trans-Sasakian manifold is said to be an extended generalized \( \phi \)-recurrent trans-Sasakian manifold if its curvature tensor \( R \) satisfies the relation
\[ \phi^2((\nabla_W R)(X, Y)Z) = A(W)\phi^2(R(X, Y)Z) + B(W)\phi^2([g(Y, Z)X - g(X, Z)Y]), \]
for all \( X, Y, Z, W \in \chi(M) \), where \( A \) and \( B \) are two non-vanishing 1-forms such that \( g(W, \rho_1) = A(W) \) and \( g(W, \rho_2) = B(W), \forall W \in \chi(M) \), with \( \rho_1 \) and \( \rho_2 \) being the vector fields associated to the 1-form \( A \) and \( B \), respectively [16].
In this connection, we mention the works of Prakasha [7] on Sasakian manifolds.

First suppose that $M^3$ is an $\eta-$ Einstein trans-Sasakian manifold, i.e.,

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad (4.1)$$

where, $a, b$ are smooth function on $M^3$. Putting $X = Y = \xi$ in the above equation we get

$$a + b = 2(\alpha^2 - \beta^2). \quad (4.2)$$

In a local coordinate, (4.1) can be written as

$$R_{ij} = ag_{ij} + b\eta_i\eta_j, \quad (4.3)$$

which implies that

$$r = 3a + b. \quad (4.4)$$

Taking covariant derivative with respect to $k$ from the equation (4.3), we have

$$R_{ij,k} = a_{,k}g_{ij} + b_{,k}\eta_i\eta_j + b\eta_{i,k}\eta_j + b\eta_i\eta_{j,k}. \quad (4.5)$$

Contracting (4.5) with $g^{ik}$, we get

$$R^k_{j,k} = a_{,j} + b_{,k}\xi^k\eta_j + b\eta_{i,k}g^{ik}\eta_j + b\eta_i\eta_{j,k}g^{ik}. \quad (4.6)$$

We also have

$$R^k_{j,k} = \frac{1}{2}r_{,j}. \quad (4.7)$$

Hence, we obtain

$$r_{,j} = 2[a_{,j} + b_{,k}\xi^k\eta_j + b\eta_{i,k}g^{ik}\eta_j + b\eta_i\eta_{j,k}g^{ik}]. \quad (4.8)$$

Since

$$\eta_{i,k} = -\alpha g_{ih}\phi^h_k + \beta(g_{ik} - \eta_i\eta_k),$$

we have

$$\eta_{i,k}g^{ik} = -\alpha g_{ih}\phi^h_k g^{ik} + 2\beta. \quad (4.9)$$

From the equation (4.8), we get

$$r_{,j} = 2[a_{,j} + b_{,k}\xi^k\eta_j + b(-\alpha\phi^h_k + 2\beta)\eta_j]. \quad (4.10)$$

Again,

$$a_{,k} + b_{,k} = 4[\alpha\alpha_{,k} - \beta\beta_{,k}]. \quad (4.11)$$
And also,

\[ r_{,j} = 2a_{,j} + 4[\alpha \alpha_{,j} - \beta \beta_{,j}]. \] (4.12)

From the equations (4.10) and (4.12) we get

\[ 2a_{,j} + 4[\alpha \alpha_{,j} - \beta \beta_{,j}] = 2[a_{,j} + b_{,k} \xi^k \eta_j + b(-\alpha \phi_h^h + 2\beta) \eta_j]. \] (4.13)

Contracting (4.13) with \( \xi^j \) and using (4.2), we have

\[ 2b_{,k} \xi^k + 2b(-\alpha \phi_h^h + 2\beta) = a_{,k} + b_{,k}. \] (4.14)

If \( b \) and \( a \) are constant functions, then the equation (4.14) implies that either \( b = 0 \) or \( \alpha \) and \( \beta \) are related by

\[ -\alpha \phi_h^h + 2\beta = 0. \] (4.15)

Again, contracting (4.15) with \( \xi^h \) we get \( \beta = 0 \).

So, we have the following theorem:

**Theorem 4.4.** Suppose \( M^3 \) is an \( \eta \)-Einstein trans-Sasakian manifold. If \( b \) and \( a \) are constant functions, then either \( M^3 \) is an Einstein manifold or \( M^3 \) is an \( \alpha \)-Sasakian manifold.

Now, we prove the following result:

**Theorem 4.5.** An extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( (M^3, g) \) with respect to semi-symmetric metric connection is an \( \eta \)-Einstein manifold and more over, the 1-forms \( A \) and \( B \) are related as \( A(W)[\alpha^2 - \beta^2 - \beta] + B(W) = 0 \).

**Proof:** Let us assume an extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( (M^3, \phi, \eta, \xi, g) \) with respect to semi-symmetric connection. Then we have

\[ \phi^2((\tilde{\nabla}_W \tilde{\nabla})(X, Y)Z) = A(W)\phi^2(\tilde{R}(X, Y)Z) + B(W)\phi^2([g(Y, Z)X - g(X, Z)Y]). \]

From above we get

\[ - (\tilde{\nabla}_W \tilde{\nabla})(X, Y)Z + \eta((\tilde{\nabla}_W \tilde{\nabla})(X, Y)Z)\xi \]

\[ = A(W)\tilde{R}(X, Y)Z + \eta(\tilde{R}(X, Y)Z)\xi + B(W)[-g(Y, Z)X + g(X, Z)Y \]

\[ + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]. \] (4.16)

From the equation (4.16), we have

\[ -g((\tilde{\nabla}_W \tilde{\nabla})(X, Y)Z, U) + \eta((\tilde{\nabla}_W \tilde{\nabla})(X, Y)Z)\eta(U) \]

\[ = A(W)[-g(\tilde{R}(X, Y)Z, U) + \eta(\tilde{R}(X, Y)Z)\eta(U)] + B(W)[-g(Y, Z)g(X, U) \]

\[ + g(X, Z)g(Y, U) + g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U)]. \] (4.17)
Let \( \{e_1, e_2, e_3\} \) be an orthonormal basis for the tangent space of \( M^3 \) at a point \( p \in M^3 \). Putting \( X = U = e_i \) in (4.17) and taking summation over \( i \), we get

\[
-(\tilde{\nabla}_W\tilde{S})(Y, Z) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W\tilde{R})(e_i, Y)Z)\eta(e_i)
= A(W)[-\tilde{S}(Y, Z) + \eta(\tilde{R}(\xi, Y)Z)] + B(W)[-g(Y, Z) - \eta(Y)\eta(Z)]. \tag{4.18}
\]

Putting \( Z = \xi \), we have

\[
-(\tilde{\nabla}_W\tilde{S})(Y, \xi) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W\tilde{R})(e_i, Y)\xi)\eta(e_i)
= A(W)[-\tilde{S}(Y, \xi) + \eta(\tilde{R}(\xi, Y)\xi)] - 2B(W)\eta(Y). \tag{4.19}
\]

Now,

\[
g((\tilde{\nabla}_W\tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W\tilde{R})(e_i, Y)\xi, \xi) + \eta(\tilde{R}(e_i, Y)\xi)g(W, \xi) - g(W, \tilde{R}(e_i, Y)\xi)g(\xi, \xi). \tag{4.20}
\]

We have

\[
g((\nabla_W\tilde{R})(e_i, Y)\xi, \xi) = g(\nabla_W\tilde{R}(e_i, Y)\xi, \xi) - g(\tilde{R}(\nabla_W e_i, Y)\xi, \xi) - g(\tilde{R}(e_i, \nabla_W Y)\xi, \xi) - g(\tilde{R}(e_i, Y)\nabla_W \xi, \xi). \tag{4.21}
\]

at \( p \in M^3 \). Since \( e_i \) is an orthonormal basis, so \( \nabla_W e_i = 0 \) at \( p \).

Also,

\[
g(\tilde{R}(e_i, Y)\xi, \xi) = -g(\tilde{R}(\xi, Y)e_i, e_i) = 0. \tag{4.22}
\]

Since \( \nabla_W g = 0 \), we obtain

\[
g((\nabla_W\tilde{R})(e_i, Y)\xi, \xi) + g(\tilde{R}(e_i, Y)\xi, \nabla_W \xi) = 0, \tag{4.23}
\]

which implies that

\[
g((\nabla_W\tilde{R})(e_i, Y)\xi, \xi) = 0. \tag{4.24}
\]

Since \( \eta(\tilde{R}(e_i, Y)\xi) = 0 \), we have from (4.20) that

\[
g((\tilde{\nabla}_W\tilde{R})(e_i, Y)\xi, \xi) = -g(W, \tilde{R}(e_i, Y)\xi). \tag{4.25}
\]

Therefore,

\[
\sum_{i=1}^{3} \eta((\tilde{\nabla}_W\tilde{R})(e_i, Y)Z)\eta(e_i) = \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y). \tag{4.26}
\]

Again, from (4.20) and (4.26) in (4.18) we have

\[
-(\tilde{\nabla}_W\tilde{S})(Y, \xi) + \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y)
= A(W)[-\tilde{S}(Y, \xi) + \eta(\tilde{R}(\xi, Y)\xi)] - 2B(W)\eta(Y). \tag{4.27}
\]
Now,
\[(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi).\]

After brief calculations the equation (4.27) gives
\[
(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W) + 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) \\
+ \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - \alpha(3\beta + 1)g(\phi Y, W) - (\beta + 1)S(Y, W) \\
- \alpha(\beta + 1)g(\phi Y, W) - (\beta + 1)\eta(Y)\eta(W) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) \\
+ (3\beta + 1)(\beta + 1)g(Y, W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W).\quad(4.28)
\]

From the equation (4.28) we get
\[
\alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y) - [2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W) \\
+ 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) + \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - \alpha(3\beta + 1)g(\phi Y, W) \\
- \alpha(\beta + 1)g(\phi Y, W) - (\beta + 1)\eta(Y)\eta(W) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) \\
- (\beta + 1)S(Y, W) + (3\beta + 1)(\beta + 1)g(Y, W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W)] = 0. \quad(4.29)
\]

Replacing Y = \xi in (4.29) we obtain
\[
A(W)[\alpha^2 - \beta^2 - \beta] + B(W) = 0. \quad(4.30)
\]

From, (4.29) we have
\[
\alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y) - [2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W) \\
+ 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) + \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - \alpha(3\beta + 1)g(\phi Y, W) \\
- \alpha(\beta + 1)g(\phi Y, W) - (\beta + 1)\eta(Y)\eta(W) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) \\
- (\beta + 1)S(Y, W) + (3\beta + 1)(\beta + 1)g(Y, W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W)] = 0. \quad(4.31)
\]

Interchanging Y and W and then adding we get
\[
S(Y, W) = a g(Y, W) + b\eta(Y)\eta(W),
\]
where \(a = \frac{1}{(\beta + 1)}[\alpha^2 + (3\beta + 1)(\beta + 1) + (\alpha^2 - \beta^2 - \beta)(2\beta + 3)]\) and \(b = \frac{1}{(\beta + 1)}[2(\beta + 1)(\alpha^2 - \beta^2 - \beta) - (\beta + 1)^2 - (\alpha^2 - \beta^2 - \beta)(2\beta + 3) - \alpha^2]\) with \(\beta \neq -1\). Hence \(M^3\) is an \(\eta\)-Einstein manifold with \(\beta \neq -1\). \(\square\)

5. Conharmonic Curvature Tensor on a 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

A Conharmonic curvature tensor has been studied by Ozgur [12], Siddiqui and Ahsan [14], Tarafdar and Bhattacharyya [17] and many other authors. In almost contact manifold \(M\) of dimension \(n \geq 3\), the conharmonic curvature tensor \(\tilde{L}\) with respect to semi-symmetric connection \(\tilde{\nabla}\) is given by
\[
\tilde{L}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{n - 2}S(Y, Z)X - \tilde{S}(X, Z)Y + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y, \quad(5.1)
\]
for \(X, Y, Z \in \chi(M)\), where \(\tilde{R}, \tilde{S}, \tilde{Q}\) are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection \(\tilde{\nabla}\), respectively.

A conharmonic curvature tensor \(\tilde{L}\) with respect to semi-symmetric connection \(\tilde{\nabla}\) is said to flat if it vanishes identically with respect to the connection \(\tilde{\nabla}\).
Theorem 5.1. If a 3-dimensional trans-Sasakian manifold with semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci tensor satisfies \( \tilde{L}(X, Y) \cdot \tilde{S} = 0 \), then the non-zero eigen values of the endomorphism \( \tilde{Q} \) of the tangent space corresponding to \( \tilde{S} \) are \( 2(\alpha^2 - \beta^2 - \beta) \) and \( (\beta^2 - \alpha^2 + \beta) \), where \( \alpha, \beta \) are smooth functions on \( M^3 \).

Proof: Consider, a 3-dimensional trans-Sasakian manifold with respect to a semi-symmetric metric connection satisfying the condition \( \tilde{L}(X, Y) \cdot \tilde{S} = 0 \).

Thus, we have

\[
\tilde{S}(\tilde{L}(\xi, X)Y, Z) + \tilde{S}(Y, \tilde{L}(\xi, X)Z) = 0.
\]  

(5.2)

Hence, from the above equation we get

\[
(\alpha^2 - \beta^2 - \beta)g(X, Y)\tilde{S}(\xi, Z) - (\alpha^2 - \beta^2 - \beta)\eta(Y)\tilde{S}(X, Z) + \alpha g(\phi X, Y)\tilde{S}(\xi, Z) \\
- \alpha\eta(Y)\tilde{S}(\phi X, Z) - \tilde{S}(X, Y)\tilde{S}(\xi, Z) + \tilde{S}(\xi, Y)\tilde{S}(X, Z) - g(X, Y)\tilde{S}(\tilde{Q}\xi, Z) \\
+ g(\xi, Y)\tilde{S}(\tilde{Q}X, Z) = 0
\]

(5.3)

Let \( \tilde{\lambda} \) be the eigenvalue of the endomorphism \( \tilde{Q} \) corresponding to an eigenvector \( X \). Then

\[
\tilde{Q}X = \tilde{\lambda}X.
\]  

(5.4)

And also, we have

\[
g(\tilde{Q}X, Y) = \tilde{S}(X, Y) = \tilde{\lambda}g(X, Y).
\]

From the equation (2.27) we get

\[
\tilde{S}(Z, Y) = \tilde{S}(Y, Z) + 2\alpha g(\phi Z, Y).
\]  

(5.5)

Now, putting \( Y = Z = \xi \) and using the equations (5.4) and (5.5) in the equation (5.3) we get

\[
\tilde{\lambda} = 2(\alpha^2 - \beta^2 - \beta), (\beta^2 - \alpha^2 + \beta).
\]  

(5.6)

Hence the theorem is proved. \( \square \)

6. Example of 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Example 6.1: Let \( M = \{(x, y, z) \in R^3 : y, z \neq 0\} \) where \( (x, y, z) \) are the standard coordinates in \( R^3 \). The linearly independent vector fields are given by

\[
e_1 = \frac{\partial}{\partial y}, e_2 = \left( \frac{\partial}{\partial z} + 2y \frac{\partial}{\partial x} \right), e_3 = \frac{\partial}{\partial x}
\]

. Let \( g \) be the Riemannian metric defined by

\[
g_{ij} = \begin{cases} 
1 & \text{for } i = j, \\
0 & \text{for } i \neq j.
\end{cases}
\]
Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M^3) \). Let \( \phi \) be the \((1, 1)\) tensor field defined by \( \phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0 \). Then using the linearity property of \( \phi \) and \( g \) we have,

\[
\eta(e_3) = 1, \phi^2(Z) = -Z + \eta(Z)e_3, g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]

for any \( Z, W \in \chi(M^3) \). Thus for \( e_3 = \xi, (\phi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \). Now, after some calculations we have,

\[
[e_1, e_3] = 0, [e_1, e_2] = 2e_3, [e_2, e_3] = 0.
\]

By Koszul’s formula we get,

\[
\nabla_{e_1} e_1 = 0, \nabla_{e_2} e_1 = -e_3, \nabla_{e_3} e_1 = -e_2, \nabla_{e_1} e_2 = e_3, \nabla_{e_3} e_2 = 0,
\]

\[
\nabla_{e_3} e_2 = e_1, \nabla_{e_1} e_3 = -e_2, \nabla_{e_2} e_3 = e_1, \nabla_{e_3} e_3 = 0.
\]

From the above it can be easily shown that \( M^3(\phi, \xi, \eta, g) \) is a trans-Sasakian manifold of type \((1, 0)\). Now we consider a linear connection \( \tilde{\nabla} \) such that

\[
\tilde{\nabla}_{e_1} e_j = \nabla_{e_1} e_j + \eta(e_j)e_i - g(e_i, e_j)e_3, \forall i, j = 1, 2, 3.
\]

It is easily seen that

\[
\tilde{\nabla}_{e_1} e_1 = -e_3, \tilde{\nabla}_{e_2} e_1 = -e_3, \tilde{\nabla}_{e_3} e_1 = -e_2, \tilde{\nabla}_{e_1} e_2 = e_3, \tilde{\nabla}_{e_3} e_2 = -e_3,
\]

\[
\tilde{\nabla}_{e_3} e_2 = e_1, \tilde{\nabla}_{e_1} e_3 = -e_2 + e_1, \tilde{\nabla}_{e_2} e_3 = e_1 + e_2, \tilde{\nabla}_{e_3} e_3 = 0.
\]

If \( \tilde{T} \) is the torsion tensor of the connection \( \tilde{\nabla} \), then we have \( \tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y \) and \( (\tilde{\nabla}_X, g)(Y, Z) = 0 \), which implies that \( \tilde{\nabla} \) is a semi-symmetric metric connection on \( M \).

7. Conclusion

The notion of curvatures play an important role in the differential geometry and also in physics. According to Newton’s laws, the magnitude of a force required to move an object at constant speed along a curve path is a constant multiple of the curvature of the trajectory [10]. Here we study some curvature properties on 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. We construct that an extended generalized \( \phi \)-recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection becomes an \( \eta \)-Einstein manifold under some certain condition. We explain geometric properties of curvature tensors on 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection.

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