Theorems on Analogous of Ramanujan’s Remarkable Product of Theta-Function and Their Explicit Evaluations

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Abstract: In this article, we define $E_{m,n}$ for any positive real numbers $m$ and $n$ involving Ramanujan’s product of theta-functions $\psi(-q)$ and $f(q)$, which is analogous to Ramanujan’s remarkable product of theta-functions and establish its several properties by Ramanujan. We establish general theorems for the explicit evaluations of $E_{m,n}$ and its explicit values.

Key Words: Class invariant, Modular equation, Theta-function, Cubic continued fraction.

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1. Introduction

Ramanujan’s general theta-function \cite{15} $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n+1)/2}, \quad |ab| < 1,$$

$$\quad = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \quad (1.2)$$

Three special cases of $f(a, b)$ are as follows:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q)_{\infty}}{(q; -q)_{\infty}}, \quad (1.3)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (1.4)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q; q)_{\infty}, \quad (1.5)$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

On page 338 in his first notebook \cite{4,15}, Ramanujan defines

$$a_{m,n} = \frac{ne^{-\pi \sqrt{m/n}} \psi^2(e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}{\psi^2(e^{-\pi \sqrt{m/n}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}. \quad (1.6)$$

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He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [5], M. S. Mahadeva Naika and B. N. Dharmendra [7], also established some general theorems for explicit evaluations of the product of \( a_{m,n} \) and found some new explicit values from it. Further results on \( a_{m,n} \) are found by Mahadeva Naika, Dharmendra and K. Shivashankara [9], and Mahadeva Naika and M. C. Mahesh Kumar [10]. Recently Nipen Saikia [13] established new properties of \( a_{m,n} \).

In [12], Mahadeva Naika et al. defined the product

\[
b_{m,n} = \frac{ne^{-\frac{(n-1)\pi}{4} \sqrt{mn}} \psi^2(-e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}{\psi^2(-e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}.
\]

They established general theorems for explicit evaluation of \( b_{m,n} \) and obtained some particular values. Mahadeva Naika et al. [11] established general formulas for explicit values of Ramanujan’s cubic continued fraction \( V(q) \) in terms of the products of \( a_{m,n} \) and \( b_{m,n} \) defined above, where

\[
V(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \cdots, \quad |q| < 1,
\]

and found some particular values of \( V(q) \)

In this paper, we define

\[
E_{m,n} = \frac{f(e^{-\pi \sqrt{mn}}) \psi(-e^{-\pi \sqrt{mn}})}{e^{-\pi (m+1)/12} \sqrt{m} f(e^{-\pi \sqrt{mn}}) \psi(-e^{-\pi \sqrt{mn}})},
\]

where \( m \) and \( n \) are positive real numbers.

Let \( K, K', L \) and \( L' \) denote the complete elliptic integrals of the first kind associated with the moduli \( k, k' := \sqrt{1-k^2}, l \) and \( l' := \sqrt{1-l^2} \) respectively, where \( 0 < k, l < 1 \). For a fixed positive integer \( n \), suppose that

\[
n K' = \frac{L'}{L}.
\]

Then a modular equation of degree \( n \) is a relation between \( k \) and \( l \) induced by (1.5). Following Ramanujan, set \( \alpha = k^2 \) and \( \beta = l^2 \). Then we say \( \beta \) is of degree \( n \) over \( \alpha \).

Define

\[
\chi(q) := (-q; q^2)_\infty,
\]

and

\[
G_n := 2^{-\frac{1}{4}} q^{-\frac{n}{4}} \chi(q),
\]

where

\[
q = e^{-\pi \sqrt{r}}.
\]

Moreover, if \( q = e^{-\pi \sqrt{m}} \) and \( \beta \) has degree \( n \) over \( \alpha \), then

\[
G_{\infty} = (4\alpha(1-\alpha))^{-\frac{1}{4}}
\]

and

\[
G_{nm} = (4\beta(1-\beta))^{-\frac{1}{4}}.
\]

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan’s product of theta-function \( E_{m,n} \) and also some new explicit evaluations from it.
2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

**Lemma 2.1.** [2, Ch. 17, Entry 11(ii) and Entry 12(i), pp. 123–124] We have,

\[ 2^{1/2} e^{-y/8} \psi(-e^{-y}) = \sqrt{\frac{1}{4}} \{ \alpha(1-\alpha) \}^{1/8}, \]  
(2.1)

\[ 2^{1/2} e^{-my/8} \psi(-e^{-my}) = \sqrt{\frac{1}{4m}} \{ \beta(1-\beta) \}^{1/8}, \]  
(2.2)

\[ 2^{1/6} e^{-y/24} f(e^{-y}) = \sqrt{\frac{1}{24}} \{ \alpha(1-\alpha) \}^{1/24}, \]  
(2.3)

\[ 2^{1/6} e^{-my/24} f(e^{-my}) = \sqrt{\frac{1}{24m}} \{ \beta(1-\beta) \}^{1/24}. \]  
(2.4)

**Lemma 2.2.** [2, Ch. 16, Entry 27(iii) and (iv), pp. 43] We have,

\[ e^{-\alpha/24} \sqrt[4]{\alpha} f(e^{-\alpha}) = e^{-\beta/24} \sqrt[4]{\beta} f(e^{-\beta}), \text{ if } \alpha \beta = \pi^2 \]  
(2.5)

\[ e^{-\alpha/12} \sqrt[3]{\alpha} f(e^{-2\alpha}) = e^{-\beta/12} \sqrt[3]{\beta} f(e^{-2\beta}), \text{ if } \alpha \beta = \pi^2. \]  
(2.6)

**Lemma 2.3.** [6, Theorem 2.1] We have,

\[ \frac{f^6(q)}{f^6(q^3)} = \frac{\psi(-q)}{\psi(-q^3)} \left\{ \frac{\psi^4(q) + 9q\psi^4(-q^3)}{\psi^4(-q) + q\psi^4(-q^3)} \right\}. \]  
(2.7)

**Lemma 2.4.** [16, p. 56] [14] We have,

\[ \frac{f^3(q)}{f^3(q^3)} = \frac{\psi(-q)}{\psi(-q^3)} \left\{ \frac{\psi^2(q) + 5q^2\psi^2(-q^5)}{\psi^2(-q) + q^2\psi^2(-q^5)} \right\}. \]  
(2.8)

**Lemma 2.5.** [6, Theorem 2.2] We have,

\[ \frac{f^3(q)}{f^3(q^9)} = \frac{\psi(-q)}{\psi(-q^9)} \left\{ \frac{\psi(-q) + 3q\psi(-q^9)}{\psi(-q) + q\psi(-q^9)} \right\}^2. \]  
(2.9)

**Lemma 2.6.** [2, Chapter 19, entry 5(xii), page 231] We have,

If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{1}{\alpha(1-\alpha)} \right\}^{1/4} \), then

\[ Q + \frac{1}{Q} = 2\sqrt{2} \left( \frac{1}{P} - P \right). \]  
(2.10)

**Lemma 2.7.** [2, Chapter 19, entry 13(xiv), page 282] We have,

If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \) and \( Q := \left\{ \frac{1}{\alpha(1-\alpha)} \right\}^{1/8} \), then

\[ Q + \frac{1}{Q} = 2 \left( \frac{1}{P} - P \right). \]  
(2.11)

**Lemma 2.8.** [2, Chapter 19, entry 19(ix), page 315] We have,

If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{1}{\alpha(1-\alpha)} \right\}^{1/6} \), then

\[ Q + \frac{1}{Q} = 7 + 2\sqrt{2} \left( P + \frac{1}{P} \right). \]  
(2.12)

**Lemma 2.9.** [1, Theorem 5.1] We have,

If \( P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \) and \( Q = \frac{\varphi(q)}{\varphi(q^3)} \), then

\[ Q^4 + P^4Q^4 = 9 + P^4. \]  
(2.13)
**Lemma 2.10.** [1, Theorem 5.1] We have,

If 

\[ P = \frac{\psi(-q)}{q^{1/2} \psi(-q^3)} \quad \text{and} \quad Q = \frac{\varphi(q)}{\varphi(q^3)}, \]

then

\[ Q^2 + P^2 Q^2 = 5 + P^2. \] \hspace{1cm} (2.14)

**Lemma 2.11.** [8, Theorem 3.2] We have,

If 

\[ P = \frac{\psi(-q)}{q \psi(-q^3)} \quad \text{and} \quad Q = \frac{\varphi(q)}{\varphi(q^3)}, \]

then

\[ Q + PQ = 3 + P. \] \hspace{1cm} (2.15)

3. Some Properties of \( E_{m,n} \)

In this section, we have established some properties of \( E_{m,n} \).

**Theorem 3.1.**

\[ E_{m,n} = E_{n,m}. \] \hspace{1cm} (3.1)

**Proof.** Employing the equation (2.5) and (2.6), we deduce that

\[ e^{-\alpha/8} \sqrt{\alpha} \psi(-e^{-\alpha}) = e^{-\beta/8} \sqrt{\beta} \psi(-e^{-\beta}), \quad \text{if} \quad \alpha \beta = \pi^2. \] \hspace{1cm} (3.2)

Using the equations (2.5) and (3.2) in (1.9), we obtain (3.1). \( \square \)

**Theorem 3.2.**

\[ E_{m,n} E_{m,1/n} = 1. \] \hspace{1cm} (3.3)

**Proof.** Using the equations (2.5) and (3.2) in (1.9), we obtain (3.3). \( \square \)

**Corollary 3.3.**

\[ E_{m,1} = 1. \] \hspace{1cm} (3.4)

**Proof.** Putting \( n = 1 \) in the equation (3.3), we get (3.4). \( \square \)

**Remark 3.4.** By using the definition of \( \psi(q) \), \( f(q) \) and \( E_{m,n} \), it can be seen that \( E_{m,n} \) has positive real value and that the values of \( E_{m,n} \) increase as \( n \) increase when \( m > 1 \). Thus by the above corollary, \( E_{m,n} > 1 \) for all \( n > 1 \) if \( m > 1 \).

**Theorem 3.5.**

\[ \frac{E_{km,n}}{E_{nm,k}} = E_{m,\frac{n}{k}}. \] \hspace{1cm} (3.5)

**Proof.** Employing the definition of \( E_{m,n} \), we obtain

\[ \frac{E_{km,n}}{E_{nm,k}} = e^{\left(\sqrt{\frac{k}{m}} - \sqrt{\frac{n}{m}}\right) \frac{\pi}{12}} \frac{f\left(e^{-\pi \sqrt{\frac{k}{m}}}ight) \psi\left(-e^{-\pi \sqrt{\frac{k}{m}}}ight)}{f\left(e^{-\pi \sqrt{\frac{n}{m}}}ight) \psi\left(-e^{-\pi \sqrt{\frac{n}{m}}}ight)}. \] \hspace{1cm} (3.6)

Using the Lemma 2.2 in the above equation (3.6) and simplifying using the Theorems 3.1 and 3.2, we obtain (3.5). \( \square \)

**Corollary 3.6.**

\[ E_{m^2,n} = E_{nm,n} E_{m,\frac{n}{m}}. \] \hspace{1cm} (3.7)

**Proof.** Putting \( m = n \) in the above Theorem 3.5 and simplifying using the equation (3.3), we get

\[ E_{m^2,k} = E_{mk,n} E_{m,\frac{k}{m}}. \] \hspace{1cm} (3.8)

Replace \( k \) by \( n \), we obtain (3.7). \( \square \)
Theorem 3.7. If \( mn = rs \)

\[
\frac{E_{m,n}}{E_{kr,ks}} = \frac{E_{r,s}}{E_{km,kn}}.
\]  

(3.9)

Proof. Using the definition of \( E_{m,n} \) and letting \( mn = rs \) for positive real numbers \( m, n, r, s \) and \( k \), we find that

\[
\frac{E_{km,kn}}{E_{m,n}} = \frac{E_{kr,ks}}{E_{r,s}}.
\]  

(3.10)

On rearranging the above equation \((3.10)\) we obtain the required result.

\[\square\]

Corollary 3.8. If \( mn = rs \)

\[
E_{np,np} = E_{np^2,n}E_{p,p}.
\]  

(3.11)

Proof. Letting \( m = p^2 \), \( n = 1 \), \( r = s = p \) and \( k = n \) in above Theorem 3.7, we deduced the equation \((3.11)\).

\[\square\]

Theorem 3.9. For all positive real numbers \( m, n, r \) and \( s \), then

\[
E_{m/n,r/s} = E_{ms,nr}E_{mr,ns}.
\]  

(3.12)

Proof. Employing the equation \((3.3)\) in equation \((3.5)\), we find that, for all positive real numbers \( m, n \) and \( k \)

\[
E_{m/n,k} = E_{m,nk}E_{n,mk}^{-1}.
\]  

(3.13)

Letting \( k = r/s \) and again using the equation \((3.5)\) and \((3.1)\) in \((3.13)\), we get \((3.12)\).

\[\square\]

Theorem 3.10.

\[
E_{m/n,m/n} = E_{n,n}E_{m,m/n^2}.
\]  

(3.14)

Proof. Using the Theorems 3.2 and 3.9, we get \((3.14)\).

\[\square\]

Theorem 3.11.

\[
E_{m,m}E_{m,n^2/m} = E_{n,n}E_{n,m^2/n}.
\]  

(3.15)

Proof. Putting \( k = m/n \) in the equation \((3.13)\) and Employing Theorems 3.2 and 3.10, we obtain \((3.15)\).

\[\square\]

Theorem 3.12.

\[
E_{m,m}E_{n,m^2n} = E_{n,n}E_{m,mn^2}.
\]  

(3.16)

Proof. Employing the Theorems 3.1, 3.2, 3.10 and 3.11, we obtain \((3.16)\).

\[\square\]

4. Some General Theorems on \( E_{m,n} \) and their explicit evaluations

In this section we establish some general theorems and their explicit evaluations of Ramanujan’s remarkable product of theta functions involving \( E_{m,n} \).

Theorem 4.1. If \( n \) is any positive real \( P : = \{G_{n/3}G_{3n}\}^3 \) and \( Q : = E_{3,n}^3 \), then

\[
Q + \frac{1}{Q} = 2\sqrt{2}\left(P - \frac{1}{P}\right).
\]  

(4.1)
Proof. Using the Lemma 2.1 with the definition of $E_{m,n}$, we obtain

$$E_{m,n} = \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/12}. \quad (4.2)$$

Employing the above equation $(4.2)$ and definition of class invariant $(1.11), (1.12)$ in the Lemma 2.6 with $m = 3$, we obtain $(4.1) \square$

Corollary 4.2.

$$E_{3,9} = \left\{ 1 + \frac{2^{2/3} - 2^{4/3}}{2} \right\}^{1/3}. \quad (4.3)$$

Proof. Putting $n = 9$ in the above Theorem 4.1, we obtain

$$E_{3,9} + E_{−3,9} = 2\sqrt{2} \left\{ G_3 G_27 - G_3 G_27 \right\}. \quad (4.4)$$

Solving the above equation $(4.4)$ with from the table of Chapter 34 of Ramanujan’s notebooks [4, p.189,190] $G_3 = 2^{1/12}$ and $G_27 = 2^{1/12} (\sqrt{2} - 1)^{-1/3}$, we obtain $(4.3) \square$

Theorem 4.3. If $n$ is any positive real $P := \{G_{n/5}G_{5n}\}^2$ and $Q := E_{5,n}^{3/2}$, then

$$Q + \frac{1}{Q} = 2 \left\{ P - \frac{1}{P} \right\}. \quad (4.5)$$

Proof. Using the equation $(4.2)$ and definition of class invariant $(1.11), (1.12)$ in the Lemma 2.7 with $m = 5$, we obtain $(4.5) \square$

Theorem 4.4. If $n$ is any positive real $P := \{G_{n/7}G_{7n}\}^3$ and $Q := E_{7,n}^2$, then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2} \left\{ P + \frac{1}{P} \right\}. \quad (4.6)$$

Proof. Using the equation $(4.2)$ and definition of class invariant $(1.11), (1.12)$ in the Lemma 2.8 with $m = 7$, we obtain $(4.6) \square$

Theorem 4.5.

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6}f(q^3)\psi(-q)}; \quad q := e^{-\pi\sqrt{3}} \quad (4.7)$$

If

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/12}f(q^3)}, \quad \text{then} \quad (4.8)$$

$$E_{3,n}^6 = \frac{P^4 + 9}{P^4(1 + P^4)} \quad \text{if} \quad P^4 \neq -1. \quad (4.9)$$

Proof. Employing the definition of $E_{m,n}$ with $m = 3$, we get

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6}f(q^3)\psi(-q)}. \quad (4.10)$$

Raising the power by 6 in the above equation $(4.10)$ with the Lemma 2.3, we deduce that

$$E_{3,n}^6 = \frac{f^6(q)\psi^6(-q^3)}{q^{-1}f^6(q^3)\psi^6(-q)}, \quad (4.11)$$

$$E_{3,n}^6 = P^2 \left\{ \frac{P^4 + 9}{1 + P^4} \right\}. \quad (4.12)$$

On simplifying the above equation $(4.12)$, we obtain $(4.9) \square$
Corollary 4.6.

\[
E_{3,3} = \left\{ 2 - \sqrt{3} \right\}^{1/3}.
\] (4.13)

**Proof.** Putting \( n = 3 \) in the equation (4.8) and from Ramanujan’s Notebooks [4, p. 327], we have

\[
\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[3]{6\sqrt{3} - 9}.
\] (4.14)

Employing the equation (2.13) and (4.14), we obtain

\[
P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-3\pi})} = \sqrt[3]{9 + 6\sqrt{3}}.
\] (4.15)

Substituting (4.15) in (4.9), we obtain the required result. \(\square\)

Theorem 4.7.

\[
E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)};
\]

\(q := e^{-\pi\sqrt{n}}\). (4.16)

If

\[
P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/6}f(q^5)},
\]

then

\[
E_{5,n}^3 = \frac{P^2 + 5}{P^2(P^2 + 1)} \quad \text{if} \quad P^2 \neq -1.
\] (4.18)

**Proof.** Employing the definition of \( E_{m,n} \) with \( m = 5 \), we get

\[
E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}.
\] (4.19)

Raising the power by 3 in the above equation (4.19) with the Lemma 2.4, we deduce that

\[
E_{5,n}^3 = \frac{f^3(q)\psi^3(-q^5)}{q^{-1}f^3(q^5)\psi^3(-q)}.
\] (4.20)

\[
E_{5,n}^3 = \frac{P \left\{ \begin{array}{c} 5 + P^2 \\ P^2 + 1 \end{array} \right\}}{P^3}.
\] (4.21)

On simplifying the above equation (4.21), we obtain (4.18). \(\square\)

Corollary 4.8.

\[
E_{5,5} = \left\{ 9 - 4\sqrt{5} \right\}^{2/3}.
\] (4.22)

**Proof.** Putting \( n = 5 \) in the equation (4.17) and from Ramanujan’s Notebooks [4, p. 327], we have

\[
\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5} - 10}.
\] (4.23)

Employing the equation (2.14) and (4.23), we obtain

\[
P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-5\pi})} = \sqrt{5\sqrt{5} + 10}.
\] (4.24)

Substituting (4.24) in (4.18), we obtain the required result. \(\square\)
Theorem 4.9.

\[ E_{9,n} = \frac{f(q)\psi(-q^9)}{q^{2/3}f(q^9)\psi(q)}, \quad q := e^{-\pi \sqrt{\frac{3}{2}}}. \]  \hspace{1cm} (4.25)

If

\[ P := \frac{\psi(-q)}{q \psi(-q^9)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/3}f(q^9)}, \]  \hspace{1cm} (4.26)

\[ E_{9,n}^3 = \left\{ \frac{P + 3}{P(P + 1)} \right\}^2, \quad \text{if} \ P \neq -1. \]  \hspace{1cm} (4.27)

Proof. Employing the definition of \( E_{m,n} \) with \( m = 9 \), we get

\[ E_{9,n} = \frac{f(q)\psi(-q^9)}{q^{2/3}f(q^9)\psi(q)}. \]  \hspace{1cm} (4.28)

Raising the power by 3 in the above equation (4.28) with the Lemma 2.5, we deduce that

\[ E_{9,n}^3 = \frac{f^3(q)\psi^3(-q^9)}{q^{-2/3}f^3(q^9)\psi^3(-q)}, \]  \hspace{1cm} (4.29)

\[ E_{9,n}^3 = \frac{P + 3}{P(P + 1)} \cdot \frac{P^3}{P^3}. \]  \hspace{1cm} (4.30)

On simplifying the above equation (4.30), we obtain (4.27). \( \square \)

Corollary 4.10.

\[ E_{9,9} = \left\{ \frac{[33s^2 - (39 + \sqrt{3})s - 21\sqrt{3} + 6][54 - 31\sqrt{3}]}{33} \right\}^{1/3}. \]  \hspace{1cm} (4.31)

where \( s = (2\sqrt{3} + 2)^{1/3} \)

Proof. Putting \( n = 9 \) in the equation (4.26) and from Ramanujan’s Notebooks [4, p. 327] we have,

\[ P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1 + \sqrt{2(\sqrt{3} + 1)}}. \]  \hspace{1cm} (4.32)

Employing the equation (2.15) and (4.32), we obtain

\[ P := \frac{\psi(e^{-\pi})}{\psi(e^{-9\pi})} = \frac{(s^2 + 2s + \sqrt{3} + 1)(3 + \sqrt{3})}{2}. \]  \hspace{1cm} (4.33)

Substituting (4.33) in (4.27), we obtain the required result. \( \square \)

Theorem 4.11.

\[ E_{m,n} = \left\{ \frac{G_{n/m}}{G_{mn}} \right\}^2. \]  \hspace{1cm} (4.34)

Proof. Employing the Lemma 2.1 in the definition of \( E_{m,n} \), we obtain

\[ E_{m,n} = \left\{ \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right\}^{1/12}. \]  \hspace{1cm} (4.35)

Using the equation (1.11) and (1.12), we get

\[ G_{nm} = \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\}^{1/24}. \]  \hspace{1cm} (4.36)

By observing the equations (4.35) and (4.36), we obtain (4.34). \( \square \)
Corollary 4.12.

\[ E_{n,n} = G_n^{-2}. \]  

(4.37)

Proof. Setting \( m = n \) in the above Theorem 4.7 with the value \( G_1 = 1 \), we obtain required result. \( \Box \)

Corollary 4.13.

(i) \( E_{2,2} = 2^{3/8}(1 + \sqrt{2})^{-1/2} \),

(ii) \( E_{4,3} = \left\{ 2 - \sqrt{3} \right\}^{1/3} \),

(iii) \( E_{5,5} = \frac{3 - \sqrt{5}}{2} \),

(iv) \( E_{9,9} = \left\{ \frac{[2(\sqrt{3} + 1)]^{1/3} + 1}{[2(\sqrt{3} - 1)]^{1/3} - 1} \right\}^{-2/3} \).

(4.41)

Proof. For (i), we use the values of \( G_4 \) from [3, p.114, Theorem 6.2.2(ii)]. For (ii) – (iv), we use corresponding values of \( G_n \) from [2, p.189-193]. \( \Box \)

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