An Ideal-based Cozero-divisor Graph of a Commutative Ring

H. Ansari-Toroghy, F. Farshadifard, and F. Mahboobi-Abkenar

ABSTRACT: Let $R$ be a commutative ring and let $I$ be an ideal of $R$. In this article, we introduce the cozero-divisor graph $\hat{\Gamma}_I(R)$ of $R$ and explore some of its basic properties. This graph can be regarded as a dual notion of an ideal-based zero-divisor graph.

Key Words: Zero-divisor, Cozero-divisor, Connected, Bipartite, Secondal ideal.

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1. Introduction

Throughout this paper $R$ denotes a commutative ring with a non-zero identity. Also we denote the set of all maximal ideals and the Jacobson radical of $R$ by $\text{Max}(R)$ and $J(R)$, respectively.

Let $Z(R)$ be the set of all zero-divisors of $R$. Anderson and Livingston, in [5], introduced the zero-divisor graph of $R$, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z^*(R) = Z(R) \backslash \{0\}$ and two distinct elements $x$ and $y$ in $Z^*(R)$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$.

In [16], Redmond introduced the definition of the zero-divisor graph with respect to an ideal. Let $I$ be an ideal of $R$. The zero-divisor graph of $R$ with respect to $I$, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R \backslash I | xy \in I \text{ for some } y \in R \backslash I\}$ with distinct vertices $x$ and $y$ are adjacent if and only if $xy \in I$. Thus if $I = 0$, then $\Gamma_I(R) = \Gamma(R)$, and $I$ is a non-zero prime ideal of $R$ if and only if $\Gamma_I(R) = \emptyset$.

In [1], Afkhami and Khashayarmanesh introduced and studied the cozero-divisor graph $\hat{\Gamma}(R)$ of $R$, in which the vertices are precisely the nonzero, non-unit elements of $R$, denoted by $W^*(R)$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \notin yR$ and $y \notin xR$.

Let $I$ be an ideal of $R$. In this article, we introduce and study the cozero-divisor graph $\hat{\Gamma}_I(R)$ of $R$ with vertices $\{x \in R \backslash \text{Ann}_R(I) | xI \neq I\}$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x \notin yI$ and $y \notin xI$. This can be regarded as a dual notion of ideal-based zero-divisor graph introduced by S.P. Redmond in [16]. Also this is a generalization of cozero-divisor graph introduced in [1] when $I = R$, i.e., we have $\hat{\Gamma}_R(R) = \hat{\Gamma}(R)$.

There is considerable researches concerning the ideal-based zero-divisor graph and this notion has attracted attention by a number of authors (for example, see [2], [3], [4], [6], [11], [14], and [15]). It is natural to ask the following question: To what extent does the dual of these results hold for ideal-based cozero-divisor graph? The main purpose of this paper is to provide some useful information in this case.

We will include some basic definitions from graph theory as needed. In a graph $G$, the distance between two distinct vertices $a$ and $b$, denoted by $d(a, b)$ is the length of the shortest path connecting $a$ and $b$. If there is not a path between $a$ and $b$, $d(a, b) = \infty$. The diameter of a graph $G$ is $\text{diam}(G) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The girth of $G$, is the length of the shortest cycle in $G$ and it is denoted by $g(G)$. If $G$ has no cycle, we define the girth of $G$ to be infinite. An $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets such that no edge has both ends in any one subset. A complete $r$-partite graph is one each vertex is jointed to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes $m$ and $n$ is denoted by $K_{m,n}$.
2. On the generalization of the cozero-divisor graph

**Definition 2.1.** Let $I$ be an ideal of $R$. We define the ideal-based cozero-divisor graph $\hat{\Gamma}_I(R)$ of $R$ with vertices $\{x \in R \setminus \text{Ann}_R(I) | xI \neq I\}$. The distinct vertices $x$ and $y$ are adjacent if and only if $x \not\in yI$ and $y \not\in xI$. Clearly, when $I = R$ we have $\hat{\Gamma}_I(R) = \hat{\Gamma}(R)$.

**Example 2.2.** Let $R = \mathbb{Z}_{12}$ and $I = \langle 3 \rangle$. Then $\hat{\Gamma}_I(R) = \emptyset$. Also, in the following figures we can see the difference between the graphs $\hat{\Gamma}(R)$, $\hat{\Gamma}_I(R)$, and $\Gamma(R)$.

Let $I$ be an ideal of $R$. Then $I$ is said to be a second ideal if $I \neq 0$ and for every element $r$ of $R$ we have either $rI = 0$ or $rI = I$.

**Lemma 2.3.** $I$ is a second ideal of $R$ if and only if $\hat{\Gamma}_I(R) = \emptyset$.

*Proof.* Straightforward. □

**Theorem 2.4.** Let $I$ be a proper ideal of $R$. Then we have the following.

(a) The graph $\hat{\Gamma}_I(R) \setminus J(R)$ is connected.

(b) If $R$ is a non-local ring, then $\text{diam}(\hat{\Gamma}_I(R) \setminus J(R)) \leq 2$.

*Proof.* (a) If $R$ has only one maximal ideal, then $V(\hat{\Gamma}_I(R)) \setminus J(R)$ is the empty set; which is connected. So we may assume that $|\text{Max}(R)| > 1$. Let $a, b \in V(\hat{\Gamma}_I(R)) \setminus J(R)$ be two distinct elements. Without loss of generality, we may assume that $a \in bI$. Since $a \not\in J(R)$, there exists a maximal ideal $m$ such that $a \not\in m$. We claim that $m \not\subset J(R) \cup bI$. Otherwise, $m \subset J(R) \cup bI$. This implies that $m \subset J(R)$ or $m \subset bI$. But $m \neq J(R)$. Hence we have $m \subset bI \subsetneq R$, so $m = bI$. This implies that $a \in m$, a contradiction. Choose the element $c \in m \setminus J(R) \cup bI$. It is easy to check that $a - c - b$.

(b) This follows from part (a). □

**Remark 2.5.** Figure (B) in Example 2.2 shows that $J(R)$ cannot be omitted in Theorem 2.4.

**Theorem 2.6.** Let $R$ be a non-local ring and $I$ a proper ideal of $R$ such that for every element $a \in J(R)$, there exists $m \in \text{Max}(R)$ and $b \in m \setminus J(R)$ with $a \not\in bR$. Then $\hat{\Gamma}_I(R)$ is connected and $\text{diam}(\hat{\Gamma}_I(R)) \leq 3$.

*Proof.* Use the technique of [1, Theorem 2.5]. □

**Theorem 2.7.** Let $R$ be a non-local ring and $I$ be a proper ideal of $R$. Then $g(\hat{\Gamma}_I(R) \setminus J(R)) \leq 5$ or $g(\hat{\Gamma}_I(R) \setminus J(R)) = \infty$.

*Proof.* Use the technique of [1, Theorem 2.8] along with Theorem 2.4. □
Theorem 2.8. Let $I$ be a non-zero ideal of $R$. If $V(\hat{\Gamma}(R)) = V(\hat{\Gamma}_I(R))$, then $\text{Ann}_R(I) = 0$ or $I = R$. The converse holds if $I$ is finitely generated.

Proof. Let $W^*(R) = V(\hat{\Gamma}(R)) = V(\hat{\Gamma}_I(R))$ and $\text{Ann}_R(I) \neq 0$. Then $W^*(R) = R \setminus \text{Ann}_R(I)$. Thus $W(R) \cap \text{Ann}_R(I) = \{0\}$. Now suppose contrary that $I \neq R$. Let $0 \neq x \in \text{Ann}_R(I)$ and $y \in W(R)$. Then $xy \in W(R) \cap \text{Ann}_R(I) = \{0\}$ and $x \notin W(R)$. It follows that $y = 0$ and hence $W(R) = \{0\}$. Therefore $R$ is a field, a contradiction. Conversely, if $I = R$ the result is clear. Now suppose that $I \neq R$ is a finitely generated ideal of $R$ such that $\text{Ann}_R(I) = 0$ and $x \in V(\hat{\Gamma}(R))$. Then $xI \neq 0$. If $xI = I$, then since $I$ is finitely generated, there exists $t \in R$ such that $(1 + tx)I = 0$ by [13, Theorem 75]. Thus $1 + tx \in \text{Ann}_R(I) = 0$. This implies that $Rx = R$, which is a contradiction. Hence $x \in V(\hat{\Gamma}_I(R))$. Therefore $V(\hat{\Gamma}(R)) \subseteq V(\hat{\Gamma}_I(R))$. The inverse inclusion is clear. \hfill $\Box$

We will use the following lemma frequently in the sequel.

Lemma 2.9. Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Then $\hat{\Gamma}(R)$ is a subgraph of $\hat{\Gamma}_I(R)$.

Proof. By Theorem 2.8, we have $V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}(R))$. Now let $x, y \in V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}_I(R))$ and $x$ is adjacent to $y$ in $\hat{\Gamma}(R)$. Then clearly, they are adjacent in $\hat{\Gamma}_I(R)$. Otherwise, we may assume that $x \in yI$. This implies that $x \in yR$, which is a contradiction. Hence $\hat{\Gamma}(R)$ is a subgraph of $\hat{\Gamma}_I(R)$. \hfill $\Box$

The following example shows that the inclusion relation between $\hat{\Gamma}_I(R)$ and $\hat{\Gamma}(R)$ in Lemma 2.9 may be a strict inclusion.

Example 2.10. Let $R := \mathbb{Z}$ and $I := 5\mathbb{Z}$. Then $V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}(R)) = \mathbb{Z} \setminus \{-1, 0, 1\}$. Now by Lemma 2.9, $\hat{\Gamma}(R)$ is subgraph of $\hat{\Gamma}_I(R)$. However, the elements 2 and 6 are adjacent in $\hat{\Gamma}_I(R)$ but they are not adjacent in $\hat{\Gamma}(R)$.

Theorem 2.11. Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Suppose that $|\text{Max}(R)| \geq 3$. Then $g(\hat{\Gamma}_I(R)) = 3$.

Proof. Use the technique of [1, Theorem 2.9]. \hfill $\Box$

As we mentioned before, $V(\Gamma_I(R)) = \{x \in R \setminus I | xy \in I \text{ for some } y \in R \setminus I\}$. We will show this set by $Z_I(R)$. Clearly, for $I = 0$, $Z_I(R) = Z^*(R)$.

Lemma 2.12. Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Then $Z_I(R) \subseteq V(\hat{\Gamma}_I(R))$.

Proof. If $I = 0$, then the claim is clear. So we assume that $I \neq 0$. Now let $x \in Z_I(R)$ then $x \neq 0$ and there exists $y \in R \setminus I$ such that $xy \in I$. Clearly, $xI \neq 0$. Further $xI \neq I$. Otherwise, $xI = I$. Since $I$ is finitely generated, there exists $t \in R$ such that $(1 + tx)I = 0$ by [13, Theorem 75]. This implies that $1 + tx = 0$. So $x$ is a unit element of $R$ and hence $y \in I$, which is a contradiction. Therefore $x \in V(\hat{\Gamma}_I(R))$. \hfill $\Box$

The next example shows that the inclusion in Lemma 2.12 is not strict in general.

Example 2.13. Let $I$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Further we assume that $R$ is an Artinian ring with $Z(R) \cap I = 0$. Then we have $V(\hat{\Gamma}_I(R)) = Z_I(R)$. To see this, it is enough to prove that $V(\hat{\Gamma}_I(R)) \subseteq Z_I(R)$ by Lemma 2.12. Let $x \in V(\hat{\Gamma}_I(R))$. Then we have $x \neq 0$ and $xI \neq I$. This implies that $xR \neq R$ and hence $x$ is a non-unit element of $R$. Since $R$ is Artinian, the set of non-unit elements of $R$ is the same as the set of zero-divisors of $R$. So $x \in Z(R)$. This shows that $x \notin I$ and there exists $0 \neq y \in R \setminus I$ such that $xy = 0 \in I$. Clearly, $x, y \in Z(R)$. Therefore, $V(\Gamma_I(R)) \subseteq Z_I(R)$.

Theorem 2.14. Let $I$ be a finitely generated ideal of $R$ with $\sqrt{I} = I$ and $\text{Ann}_R(I) = 0$. Suppose that $Z_I(R) = V(\hat{\Gamma}_I(R))$. If $\Gamma_I(R)$ is complete, then $\hat{\Gamma}_I(R)$ is also a complete graph.
Proof. Assume on the contrary that $\hat{\Gamma}_I(R)$ is not complete. So there exist $a, b \in V(\hat{\Gamma}_I(R))$ such that $a \in bI$ or $b \in aI$. Without loss of generality, we may assume that $a \in bI$. So, there exists $i \in I$ such that $a = bi$. We claim that $i$ is a unit element. Otherwise, $i \in V(\hat{\Gamma}_I(R))$. Thus we have $i \in V(\hat{\Gamma}_I(R))$ by Lemma 2.9. Hence $i \in Z_I(R)$ by assumption, which is a contradiction. Now $ab = b^2i \in I$. So there exist $i_1 \in I$ such that $b^2i = i_1$. Then $b^2 = i_1^{-1}i_1 \in I$. Therefore, $b \in \sqrt{I}$, a contradiction. □

Proposition 2.15. Let $I$ be a proper ideal of $R$ and $\hat{\Gamma}_I(R)$ a complete bipartite graph with parts $V_i$, $i = 1, 2$. Then every cyclic ideal $a, b \subseteq V_i$, for some $i = 1, 2$, are totally ordered.

Proof. Assume on the contrary that there exist ideals $aR$ and $bR$ in $V_i$ such that $aR \not\subseteq bR$ and $bR \not\subseteq aR$. It follows that $b \notin aR$ and $a \notin bR$. Hence $b \notin aI$ and $a \notin bI$. This means $a$ is adjacent to $b$, a contradiction. □

Proposition 2.16. Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. If the graph $\hat{\Gamma}_I(R) \setminus J(R)$ is $n$-partite for some positive integer $n$, then $|\text{Max}(R)| \leq n$.

Proof. Assume contrary that $|\text{Max}(R)| > n$. Since $\hat{\Gamma}_I(R) \setminus J(R)$ is a $n$-partite graph and $V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}_I(R))$ by Lemma 2.9, there exist $m, \hat{m} \in \text{Max}(R)$ and $a \in m \setminus \hat{m}, b \in \hat{m} \setminus m$ such that $a, b$ belong to a same part. Clearly, $a \notin bI$ and $b \notin aI$, which is a contradiction. □

For a graph $G$, let $\chi(G)$ denote the chromatic number of the graph $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. A clique of a graph $G$ is a complete subgraph of $G$ and the number of vertices in the largest clique of $G$, denoted by clique$(G)$, is called the clique number of $G$.

Theorem 2.17.

(1) Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Then if $R$ has infinite member of maximal ideal, then clique $\hat{\Gamma}_I(R)$ is also infinite; otherwise clique $(\hat{\Gamma}_I(R)) \geq |\text{Max}(R)|$.

(2) If $\chi(\hat{\Gamma}_I(R)) < \infty$, then $|\text{Max}(R)| < \infty$.

Proof. (1) This follows from Lemma 2.9 and [1, Theorem 2.14].

(2) Use part (1) along with [1, Theorem 2.14]. □

Theorem 2.18. Let $R = S_1 + S_2$, where $S_1$ and $S_2$ are second ideals of $R$. If $P_1 = \text{Ann}_R(S_1)$ and $P_2 = \text{Ann}_R(S_2)$, then $V(\hat{\Gamma}(R)) = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$ and $\hat{\Gamma}(R)$ is a complete bipartite graph.

Proof. Let $x \in V(\hat{\Gamma}(R))$, so we have $xR \neq 0$ and $xR \neq R$. Since $xR \neq 0$, $xS_1 \neq 0$ or $xS_2 \neq 0$. First we show that $V(\hat{\Gamma}(R)) = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$. If $xS_1 \neq 0$, then $x \notin P_1$. So $xS_1 = S_1$. We claim that $xS_2 = 0$. Otherwise, $xS_2 \neq 0$ so that $x \notin P_2$. It means that $xS_2 = S_2$. Thus $xR = R$, a contradiction. So we have $x \in P_2$ hence $x \in (P_2 \setminus P_1) \cup (P_2 \setminus P_1)$. We have similar arguments for reverse inclusion. Now let $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. We show that $x \notin yR$ and $y \notin xR$. Otherwise, $x \in yR$ or $y \in xR$. Without loss of generality, $x \in yR$. Then there exists $t \in R$ such that $x = ty$. But $x \notin P_2$ implies that $ty \notin \text{Ann}_R(S_2)$ so that $tyS_2 \neq 0$, a contradiction. Thus, $x$ is adjacent to $y$. Now we show that $x$ and $y$ can not lie in $P_1 \setminus P_2$ or $P_2 \setminus P_1$. To see this let $x, y \in P_1 \setminus P_2$ and assume that they are adjacent. Then we have $x \notin yR$ and $y \notin xR$. Now by using our assumptions, we conclude that $x \notin xR$, a contradiction. □

Theorem 2.19. Let $I \neq R$ be a finitely generated ideal of $R$ with $\text{Ann}_R(I) = 0$. Assume that $|\text{Max}(R)| \geq 5$. Then $\hat{\Gamma}_I(R)$ is not planar.

Proof. This follows from Lemma 2.9 and [1, Theorem 3.9]. □

Proposition 2.20. Let $I$ be a proper ideal. Then the following hold.
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(a) \( V(\Gamma_{\text{Ann}(I)}(R)) \subseteq V(\hat{\Gamma}_I(R)) \).

(b) If \( R \) be a reduced ring, then \( \Gamma_{\text{Ann}(I)}(R) \) is a subgraph of \( \hat{\Gamma}_I(R) \).

Proof. (a) Let \( x \in V(\Gamma_{\text{Ann}(I)}(R)) \). Then there exists \( y \in R \setminus \text{Ann}_R(I) \) such that \( xy \in \text{Ann}_R(I) \). We claim that \( xI \neq I \). Otherwise, \( xI = I \). Then \( xyI = yI \) so that \( yI = 0 \). This implies that \( y \in \text{Ann}_R(I) \), a contradiction. Therefore, \( V(\Gamma_{\text{Ann}(I)}(R)) \subseteq V(\hat{\Gamma}_I(R)) \).

(b) By part (a), \( V(\Gamma_{\text{Ann}(I)}(R)) \subseteq V(\hat{\Gamma}_I(R)) \). Now we suppose that \( x \) is adjacent to \( y \) in \( \Gamma_{\text{Ann}(I)}(R) \). We show that \( x \) is adjacent to \( y \) in \( \hat{\Gamma}_I(R) \). Otherwise, without loss of generality, we assume that \( x \in yI \). So that \( x^2 = xyI \). Thus \( x^2 = 0 \). This implies that \( x \in \text{Ann}_R(I) \), a contradiction.

Proposition 2.21. Let \( I \) be a finitely generated non-zero ideal of \( R \). Suppose that \( x, y \in R \setminus \text{Ann}_R(I) \).

(a) \( x \in V(\hat{\Gamma}_I(R)) \) if and only if \( x + \text{Ann}_R(I) \in V(\hat{\Gamma}(R/\text{Ann}_R(I))) \).

(b) If \( x + \text{Ann}_R(I) \) is adjacent to \( y + \text{Ann}_R(I) \) in \( \hat{\Gamma}(R/\text{Ann}_R(I)) \), then \( x \) is adjacent to \( y \) in \( \hat{\Gamma}_I(R) \).

Proof. a) Let \( x \in V(\hat{\Gamma}_I(R)) \) and \( x \in V(\hat{\Gamma}(R/\text{Ann}_R(I))) \). Then there exists \( y + \text{Ann}_R(I) \) such that \( xy + \text{Ann}_R(I) = 1 + \text{Ann}_R(I) \). Thus \( (xy - 1) + 1 \in \text{Ann}_R(I) \). Since \( I \) is a finite generated ideal, there exists \( r \in R \) such that \( (r(xy - 1) + 1)I = 0 \) and so \( r(xy - 1) + 1 \in \text{Ann}_R(I) \). Thus \( 1 \in \text{Ann}_R(I) \) which implies that \( I = 0 \), a contradiction.

b) This is straightforward.

An \( R \)-module \( M \) is said to be a \textit{comultiplication module} if for every submodule \( N \) of \( M \) there exists an ideal \( I \) of \( R \) such that \( N = \text{Ann}_M(I) \), equivalently, for each submodule \( N \) of \( M \), we have \( N = \text{Ann}_M(\text{Ann}_M(N)) \) [7]. \( R \) is said to be a \textit{comultiplication ring} if \( R \) is a comultiplication \( R \)-module.

Theorem 2.22. Let \( I \) be a proper ideal of \( R \). Then \( V(\hat{\Gamma}_I(R)) = V(\Gamma_{\text{Ann}(I)}(R)) \) if one of the following conditions hold.

(a) \( R \) is a comultiplication ring.

(b) \( R/\text{Ann}_R(I) = Z(R/\text{Ann}_R(I)) \cup U(R/\text{Ann}_R(I)) \).

Proof. Clearly \( V(\Gamma_{\text{Ann}(I)}(R)) \subseteq V(\hat{\Gamma}_I(R)) \).

(a) Let \( x \in V(\hat{\Gamma}_I(R)) \). Then \( xI \neq 0 \) and \( xI \neq I \). Since \( R \) is a comultiplication ring, this implies that \( \text{Ann}_R(xI) \neq \text{Ann}_R(I) \). Thus there exists \( y \in \text{Ann}_R(xI) \setminus \text{Ann}_R(I) \). Therefore, \( x \in V(\Gamma_{\text{Ann}(I)}(R)) \).

(b) Let \( x \in V(\hat{\Gamma}_I(R)) \). Then \( xI \neq 0 \) and \( xI \neq I \). By assumption, \( x + \text{Ann}_R(I) \in Z(R/\text{Ann}_R(I)) \) or \( x + \text{Ann}_R(I) \in U(R/\text{Ann}_R(I)) \). If \( x + \text{Ann}_R(I) \in Z(R/\text{Ann}_R(I)) \), then there exists \( y \in R \setminus \text{Ann}_R(I) \) such that \( xy \in \text{Ann}_R(I) \). Therefore, \( x \in V(\Gamma_{\text{Ann}(I)}(R)) \). If \( x + \text{Ann}_R(I) \in U(R/\text{Ann}_R(I)) \), then there exists \( z + \text{Ann}_R(I) \in R/\text{Ann}_R(I) \) such that \( xz + \text{Ann}_R(I) = 1 + \text{Ann}_R(I) \). Thus \( 1 = xz + a \) for some \( a \in \text{Ann}_R(I) \). Now we have \( I = 1I = (xz + a)I = xzI \subseteq xI \), a contradiction.

Theorem 2.23. Let \( I \subseteq J \) be non-zero ideals of \( R \). Then we have the following.

(a) If \( R/\text{Ann}_R(J) = Z(R/\text{Ann}_R(J)) \cup U(R/\text{Ann}_R(J)) \), then \( V(\hat{\Gamma}_I(R)) \subseteq V(\hat{\Gamma}_J(R)) \).

(b) If \( \dim(R) = 0 \), then \( V(\hat{\Gamma}_I(R)) \subseteq V(\hat{\Gamma}_J(R)) \). In particular, this holds if \( R \) is a finite ring.

Proof. (a) This follows from Theorem 2.22 (b) and [6, Theorem 2.8].

(b) \( \dim(R) = 0 \) implies that \( \dim(R/J) = 0 \). It follows that

\[
R/\text{Ann}_R(J) = Z(R/\text{Ann}_R(J)) \cup U(R/\text{Ann}_R(J)).
\]

Now the result follows from part (a).

Proposition 2.24. Let \( I \) be a non-zero ideal \( R \) with \( R = Z(R) \cup U(R) \) and \( V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}(R)) \). Then \( \text{Ann}_R(I) = 0 \).

Proof. Suppose that \( V(\hat{\Gamma}_I(R)) = V(\hat{\Gamma}(R)) \). Since \( V(\hat{\Gamma}_I(R)) \subseteq R \setminus \text{Ann}_R(I) \), we have \( V(\hat{\Gamma}(R)) \subseteq R \setminus \text{Ann}_R(I) \). Thus \( \text{Ann}_R(I) \subseteq R \setminus V(\hat{\Gamma}(R)) = \{0\} \cup U(R) \) by hypothesis. Therefore, \( \text{Ann}_R(I) = 0 \).
3. Secondal ideals

In this section, we will study the ideal-based cozero-divisor graph with respect to secondal ideals. The element $a \in R$ is called prime to an ideal $I$ of $R$ if $ra \in I$ (where $r \in R$) implies that $r \in I$. The set of elements of $R$ which are not prime to $I$ is denoted by $S(I)$. A proper ideal $I$ of $R$ is said to be primal if $S(I)$ is an ideal of $R$ [12].

A non-zero submodule $N$ of an $R$-module $M$ is said to be secondal if $W_R(N) = \{a \in R : aN \neq N\}$ is an ideal of $R$ [8]. A secondal ideal is defined similarly when $N = I$ is an ideal of $R$. In this case, we say $I$ is $P$-secondal, where $P = W(I)$ is a prime ideal of $R.

Lemma 3.1. Let $I$ be a non-zero ideal of $R$. Then the following hold.

(a) $\text{Ann}_R(I) \subseteq W(I)$.

(b) $Z_R(R/\text{Ann}_R(I)) \subseteq W(I)$.

(c) $V(\Gamma_I(R)) = W(I) \setminus \text{Ann}_R(I)$. In particular, $V(\Gamma_I(R)) \cup \text{Ann}_R(I) = W(I)$.

(d) If $\text{Ann}_R(I)$ is a radical ideal of $R$, then $\bigcup_{P \in \text{Min}(\text{Ann}_R(I))} P \subseteq W(I)$.

Proof. (a) Let $r \in \text{Ann}_R(I)$. Then $rI = 0 \neq I$. Thus $r \in W(I)$.

(b) Let $x \in Z_R(R/\text{Ann}_R(I))$ and $x \notin W(I)$. Then there exists $y \in R \setminus \text{Ann}_R(I)$ such that $xyI = 0$. Hence $xI = I$ implies that $yI = 0$, a contradiction.

(c) Let $r \in V(\Gamma_I(R))$. Then $r \in R \setminus \text{Ann}_R(I)$ and $rI \neq I$; hence $r \in W(I) \setminus \text{Ann}_R(I)$. Thus $V(\Gamma_I(R)) \subseteq W(I) \setminus \text{Ann}_R(I)$. Conversely, assume that $x \in W(I) \setminus \text{Ann}_R(I)$. Then $x \notin V(\Gamma_I(R))$, so we have equality.

(d) By [13, Exer 13, page 63], $Z_R(R/\text{Ann}_R(I)) = \bigcup_{P \in \text{Min}(\text{Ann}_R(I))} P$, where $P$ is a radical ideal of $R$. Thus $Z_R(R/\text{Ann}_R(I)) = \bigcup_{P \in \text{Min}(\text{Ann}_R(I))} P$. Hence $\bigcup_{P \in \text{Min}(\text{Ann}_R(I))} P \subseteq W(I)$ by part (b).

Remark 3.2. Let $R = \mathbb{Z}$, $I = 2\mathbb{Z}$. Then $Z_R(R/\text{Ann}_R(I)) = Z_R(R) = 0$ and $W(I) = \mathbb{Z} \setminus \{1, \{-1\}$. Therefore the converse of part (b) of the above lemma is not true in general.

Proposition 3.3. Let $I$ and $P$ be ideals of $R$ with $\text{Ann}_R(I) \subseteq P$. Then $I$ is a $P$-secondal ideal of $R$ if only if $V(\Gamma_I(R)) = P \setminus \text{Ann}_R(I)$.

Proof. Straightforward.

Theorem 3.4. Let $I$ be an ideal of $R$. Then $I$ is a secondal ideal of $R$ if and only if $V(\Gamma_I(R)) \cup \text{Ann}_R(I)$ is an (prime) ideal of $R$.

Proof. Let $I$ be a secondal ideal. Then $W(I)$ is a prime ideal and by Lemma 3.1(c), $V(\Gamma_I(R)) \cup \text{Ann}_R(I) = W(I)$. Thus $V(\Gamma_I(R)) \cup \text{Ann}_R(I)$ is an ideal of $R$. Conversely, suppose that $V(\Gamma_I(R)) \cup \text{Ann}_R(I)$ is a (prime) ideal. Then by Lemma 3.1(c), $V(\Gamma_I(R)) \cup \text{Ann}_R(I) = W(I)$ is a prime ideal. Hence $I$ is a secondal ideal.

Theorem 3.5. Let $I$ and $J$ be $P$-secondal ideals of $R$. Then $V(\Gamma_I(R)) = V(\Gamma_J(R))$ if and only if $\text{Ann}_R(I) = \text{Ann}_R(J)$.

Proof. By Lemma 3.1(a), $\text{Ann}_R(I) \subseteq P$ and $\text{Ann}_R(J) \subseteq P$. It then follows from Proposition 3.3 that $V(\Gamma_I(R)) = V(\Gamma_J(R))$ if and only if $P \setminus \text{Ann}_R(I) = P \setminus \text{Ann}_R(J)$; and this holds if and only if $\text{Ann}_R(I) = \text{Ann}_R(J)$.

Lemma 3.6. Let $N$ be a secondary submodule of an $R$-module $M$. Then $\sqrt{\text{Ann}_R(N)} = W(N)$.

Proof. Let $x \in W(N)$. Then $xN \neq N$. Since $N$ is a secondary $R$-module, there exists a positive integer $n$ such that $x^nN = 0$. Thus $x \in \sqrt{\text{Ann}_R(N)}$. Hence $W(N) \subseteq \sqrt{\text{Ann}_R(N)}$. To see the reverse inclusion, let $x \in \sqrt{\text{Ann}_R(N)}$ and $x \notin W(N)$. Then $x^nN = 0$ for some positive integer $n$ and $xN = N$. Therefore $N = 0$, a contradiction.
Theorem 3.7. Let $I$ be an ideal of $R$. Then $I$ is secondary ideal if and only if $V(\hat{\Gamma}_I(R)) = \sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$.

Proof. If $I$ is secondary, then $\sqrt{\text{Ann}_R(I)} = W(I)$ by Lemma 3.6. Hence $I$ is a $\sqrt{\text{Ann}_R(I)}$-secondal ideal of $R$. Then Proposition 3.3 implies that $V(\hat{\Gamma}_I(R)) = \sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$. Conversely, suppose that $x \in R, xI \neq I$, and $x \notin \sqrt{\text{Ann}_R(I)}$. Then $x \in W(I)$ and $x \notin \text{Ann}_R(I)$. Thus $x \in V(\hat{\Gamma}_I(R))$ and so $x \notin \sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$ by assumption, a contradiction. □

Definition 3.8. Let $I$ be an ideal of $R$. We say that an ideal $J$ of $R$ is second to $I$ if $IJ = I$.

Proposition 3.9. Let $I$ be an ideal of $R$. If $I$ is not secondal, then there exist $x, y \in V(\hat{\Gamma}_I(R))$ such that $<x,y>$ is second to $I$.

Proof. Suppose that $I$ is an ideal of $R$ such that $I$ is not secondal. Then by Lemma 3.1 (c), $V(\hat{\Gamma}_I(R)) \cup \text{Ann}_R(I) = W(I)$ is not an ideal of $R$, so there exist $x, y \in W(I)$ with $x - y \notin W(I)$ and so $(x-y)I = I$. Hence $<x,y> = I = I$. Now we claim that $x, y \notin \text{Ann}_R(I)$. Otherwise, we have $x \in \text{Ann}_R(I)$ or $y \in \text{Ann}_R(I)$. If $x, y \in \text{Ann}_R(I)$, then $x - y \in \text{Ann}_R(I) \subseteq W(I)$, a contradiction. If $x \in \text{Ann}_R(I)$ and $y \notin \text{Ann}_R(I)$, then $I = (x-y)I \subseteq xI + yI = 0 + yI$, a contradiction. Similarly, we get a contradiction when $x \notin \text{Ann}_R(I)$ and $y \in \text{Ann}_R(I)$. Thus we have $x, y \notin \text{Ann}_R(I)$. □

Proposition 3.10. Let $I$ be an ideal of $R$. Then the following hold.

(a) Let $x, y$ be distinct elements of $\sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$ with $xy \notin \text{Ann}_R(I)$. Then the ideal $<x,y>$ is not second to $I$.

(b) If $I$ is a secondary ideal, then the $\text{diam} (\Gamma_{\text{Ann}_R(I)}(R)) \leq 2$.

Proof. (a) Let ideal $<x,y>$ be second to $I$. Since $x, y \in \sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$, there exists the least positive integer $n$ such that $x^n y \in \text{Ann}(I)$. As $xy \notin \text{Ann}_R(I)$, we have $n \geq 2$. Let $m$ be the least positive such that $x^{n-1} y^m \in \text{Ann}_R(I)$. Now clearly $m \geq 2$ because $x^{n-1} y \notin \text{Ann}_R(I)$. This yields that the contradiction $0 \neq x^{n-1} y^{m-1} (x,y)I = x^{n-1} y^{m-1} I$.

(b) If $I$ is secondary, then $W(I) = \sqrt{\text{Ann}_R(I)}$ by Lemma 3.6. Choose two distinct vertices $x, y$ in $\Gamma_{\text{Ann}_R(I)}(R)$. If $xy \notin \text{Ann}_R(I)$, then $d(x, y) = 1$. So we assume that $xy \notin \text{Ann}_R(I)$. Then by Proposition 2.20 (a) and Lemma 3.1, $x, y \in W(I) \setminus \text{Ann}_R(I)$. Also we have $x, y \in \sqrt{\text{Ann}_R(I)} \setminus \text{Ann}_R(I)$ by Theorem 3.7. As in the proof of (a), we have the path $x - x^{n-1} y^{m-1} - y$ from $x$ to $y$ in $\Gamma_{\text{Ann}_R(I)}(R)$. Hence $d(x, y) = 2$. Therefore, $\text{diam} (\Gamma_{\text{Ann}_R(I)}(R)) \leq 2$. □

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References


H. Ansari-Toroghy,
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan,
Rasht,
Iran.
E-mail address: ansari@guilan.ac.ir

and

F. Farshadifar,
Assistant Professor,
Department of Mathematics,
Farhangian University,
Tehran,
Iran.
E-mail address: f.farshadifar@cfu.ac.ir

and

F. Mahboobi-Abkenar,
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan,
Rasht,
Iran.
E-mail address: mahboobi@phd.guilan.ac.ir