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An Ideal-based Cozero-divisor Graph of a Commutative Ring

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abstract: Let R be a commutative ring and let I be an ideal of R. In this article, we introduce the cozero-divisor graph $\acute{\Gamma}_I(R)$ of R and explore some of its basic properties. This graph can be regarded as a dual notion of an ideal-based zero-divisor graph.

Key Words: Zero-divisor, Cozero-divisor, Connected, Bipartite, Secondal ideal.

Contents

1. Introduction

Throughout this paper R denotes a commutative ring with a non-zero identity. Also we denote the set of all maximal ideals and the Jacobson radical of R by *Max(R)* and $J(R)$, respectively.

Let $Z(R)$ be the set of all zero-divisors of R. Anderson and Livingston, in [5], introduced the *zero-divisor graph of R*, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z^*(R) = Z(R)\setminus\{0\}$ and for two distinct elements x and y in $Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$.

In [16], Redmond introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of R. The *zero-divisor graph of R with respect to I*, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R \setminus I \,|\, xy \in I \; for\; some\; y \in R \setminus I\}$ with distinct vertices x and y are adjacent if and only if $xy \in I$. Thus if $I = 0$, then $\Gamma_I(R) = \Gamma(R)$, and I is a non-zero prime ideal of R if and only if $\Gamma_I(R) = \emptyset$.

In [1], Afkhami and Khashayarmanesh introduced and studied the *cozero-divisor graph* $\acute{\Gamma}(R)$ of R, in which the vertices are precisely the nonzero, non-unit elements of R, denoted by $W^*(R)$, and two distinct vertices x and y are adjacent if and only if $x \notin yR$ and $y \notin xR$.

Let I be an ideal of R. In this article, we introduce and study the cozero-divisor graph $\acute{\Gamma}_I(R)$ of R with vertices $\{x \in R \setminus Ann_R(I) \,|\, xI \neq I\}$ and two distinct vertices x and y are adjacent if and only if $x \notin yI$ and $y \notin xI$. This can be regarded as a dual notion of ideal-based zero-divisor graph introduced by S.P. Redmond in [16]. Also this is a generalization of cozero-divisor graph introduced in [1] when $I = R$, i.e., we have $\acute{\Gamma}_R(R) = \acute{\Gamma}(R)$.

There is considerable researches concerning the ideal-based zero-divisor graph and this notion has attracted attention by a number of authors (for example, see [2], [3], [4], [6], [11], [14], and [15]). It is natural to ask the following question: To what extent does the dual of these results hold for ideal-based cozero-divisor graph? The main purpose of this paper is to provide some useful information in this case.

We will include some basic definitions from graph theory as needed. In a graph G, the distance between two distinct vertices a and b, denoted by $d(a, b)$ is the length of the shortest path connecting a and b. If there is not a path between a and b, $d(a, b) = \infty$. The *diameter* of a graph G is $diam(G) = sup\{d(a, b) : a\; and\; b\; are\; distinct\; vertices\; of\; G\}$. The *girth* of G, is the length of the shortest cycle in G and it is denoted by $g(G)$. If G has no cycle, we define the girth of G to be infinite. An *r-partite graph* is one whose vertex set can be partitioned into r subsets such that no edge has both ends in any one subset. A *complete r-partite graph* is one each vertex is jointed to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$.

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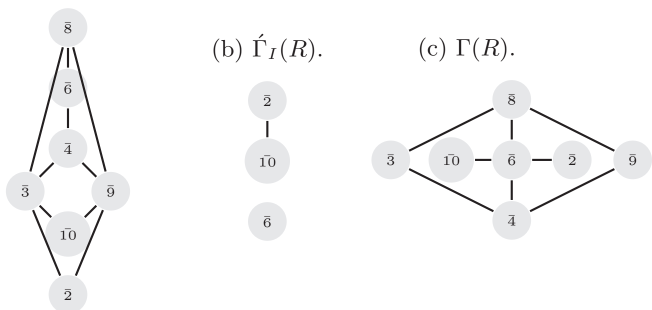
2. On the generalization of the cozero-divisor graph

Definition 2.1. *Let I be an ideal of R. We define* the ideal-based cozero-divisor graph $\acute{\Gamma}_I(R)$ *of R with vertices $\{x \in R \setminus Ann_R(I) \,|\, xI \neq I\}$. The distinct vertices x and y are adjacent if and only if $x \notin yI$ and $y \notin xI$. Clearly, when $I = R$ we have $\acute{\Gamma}_I(R) = \acute{\Gamma}(R)$.*

Example 2.2. *Let $R = \mathbb{Z}_{12}$ and $I = (\bar{3})$. Then $\Gamma_I(R) = \emptyset$. Also, in the following figures we can see the difference between the graphs $\acute{\Gamma}(R)$, $\acute{\Gamma}_I(R)$, and $\Gamma(R)$.*



(a) $\acute{\Gamma}(R)$. (b) $\acute{\Gamma}_I(R)$. (c) $\Gamma(R)$.

Let I be an ideal of R. Then I is said to be a *second ideal* if $I \neq 0$ and for every element r of R we have either $rI = 0$ or $rI = I$.

Lemma 2.3. *I is a second ideal of R if and only if $\acute{\Gamma}_I(R) = \emptyset$.*

Proof. Straightforward. □

Theorem 2.4. *Let I be a proper ideal of R. Then we have the following.*

(a) The graph $\acute{\Gamma}_I(R) \setminus J(R)$ is connected.

(b) If R is a non-local ring, then diam $(\acute{\Gamma}_I(R) \setminus J(R)) \leq 2$.

Proof. (a) If R has only one maximal ideal, then $V(\acute{\Gamma}_I(R)) \setminus J(R)$ is the empty set; which is connected. So we may assume that $|Max(R)| > 1$. Let $a, b \in V(\acute{\Gamma}_I(R)) \setminus J(R)$ be two distinct elements. Without loss of generality, we may assume that $a \in bI$. Since $a \notin J(R)$, there exists a maximal ideal m such that $a \notin m$. We claim that $m \nsubseteq J(R) \cup bI$. Otherwise, $m \subseteq J(R) \cup bI$. This implies that $m \subseteq J(R)$ or $m \subseteq bI$. But $m \neq J(R)$. Hence we have $m \subseteq bI \subsetneq R$, so $m = bI$. This implies that $a \in m$, a contradiction. Choose the element $c \in m \setminus J(R) \cup bI$. It is easy to check that $a - c - b$.

(b) This follows from part (a). □

Remark 2.5. *Figure (B) in Example 2.2 shows that $J(R)$ cannot be omitted in Theorem 2.4.*

Theorem 2.6. *Let R be a non-local ring and I a proper ideal of R such that for every element $a \in J(R)$, there exists $m \in Max(R)$ and $b \in m \setminus J(R)$ with $a \notin bR$. Then $\acute{\Gamma}_I(R)$ is connected and diam$(\acute{\Gamma}_I(R)) \leqslant 3$.*

Proof. Use the technique of [1, Theorem 2.5]. □

Theorem 2.7. *Let R be a non-local ring and I be a proper ideal of R. Then $g(\acute{\Gamma}_I(R) \setminus J(R)) \leqslant 5$ or $g(\acute{\Gamma}_I(R) \setminus J(R)) = \infty$.*

Proof. Use the technique of [1, Theorem 2.8] along with Theorem 2.4. □

Theorem 2.8. *Let I be a non-zero ideal of R. If $V(\acute{\Gamma}(R)) = V(\acute{\Gamma}_I(R))$, then $Ann_R(I) = 0$ or $I = R$. The converse holds if I is finitely generated.*

Proof. Let $W^*(R) = V(\acute{\Gamma}(R)) = V(\acute{\Gamma}_I(R))$ and $Ann_R(I) \neq 0$. Then $W^*(R) = R \setminus Ann_R(I)$. Thus $W(R) \cap Ann_R(I) = \{0\}$. Now suppose contrary that $I \neq R$. Let $0 \neq x \in Ann_R(I)$ and $y \in W(R)$. Then $xy \in W(R) \cap Ann_R(I) = \{0\}$ and $x \notin W(R)$. It follows that $y = 0$ and hence $W(R) = \{0\}$. Therefore R is a field, a contradiction. Conversely, if $I = R$ the result is clear. Now suppose that $I \neq R$ is a finitely generated ideal of R such that $Ann_R(I) = 0$ and $x \in V(\acute{\Gamma}(R))$. Then $xI \neq 0$. If $xI = I$, then since I is finitely generated, there exists $t \in R$ such that $(1 + tx)I = 0$ by [13, Theoram 75]. Thus $1 + tx \in Ann_R(I) = 0$. This implies that $Rx = R$, which is a contradiction. Hence $x \in V(\acute{\Gamma}_I(R))$. Therefore $V(\acute{\Gamma}(R)) \subseteq V(\acute{\Gamma}_I(R))$. The inverse inclusion is clear. □

We will use the following lemma frequently in the sequel.

Lemma 2.9. *Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. Then $\acute{\Gamma}(R)$ is a subgraph of $\acute{\Gamma}_I(R)$.*

Proof. By Theorem 2.8, we have $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}(R))$. Now let $x, y \in V(\acute{\Gamma}(R)) = V(\acute{\Gamma}_I(R))$ and x is adjacent to y in $\acute{\Gamma}(R)$. Then clearly, they are adjacent in $\acute{\Gamma}_I(R)$. Otherwise, we may assume that $x \in yI$. This implies that $x \in yR$, which is a contradiction. Hence $\acute{\Gamma}(R)$ is a subgraph of $\acute{\Gamma}_I(R)$. □

The following example shows that the inclusion relation between $\acute{\Gamma}_I(R)$ and $\acute{\Gamma}(R)$ in Lemma 2.9 may be a restrict inclusion.

Example 2.10. *Let $R := \mathbb{Z}$ and $I := 5\mathbb{Z}$. Then $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}(R)) = \mathbb{Z} \setminus \{-1, 0, 1\}$. Now by Lemma 2.9, $\acute{\Gamma}(R)$ is subgraph of $\acute{\Gamma}_I(R)$. However, the elements 2 and 6 are adjacent in $\acute{\Gamma}_I(R)$ but they are not adjacent in $\acute{\Gamma}(R)$.*

Theorem 2.11. *Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. Suppose that $|Max(R)| \geq 3$. Then $g(\acute{\Gamma}_I(R)) = 3$.*

Proof. Use the technique of [1, Theorem 2.9]. □

As we mentioned before, $V(\Gamma_I(R)) = \{x \in R \setminus I \,|\, xy \in I \; for \; some \; y \in R \setminus I\}$. We will show this set by $Z_I(R)$. Clearly, for $I = 0$, $Z_I(R) = Z^*(R)$.

Lemma 2.12. *Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. Then $Z_I(R) \subseteq V(\acute{\Gamma}_I(R))$.*

Proof. If $I = 0$, then the claim is clear. So we assume that $I \neq 0$. Now let $x \in Z_I(R)$ then $x \neq 0$ and there exists $y \in R \setminus I$ such that $xy \in I$. Clearly, $xI \neq 0$. Further $xI \neq I$. Otherwise, $xI = I$. Since I is finitely generated, there exists $t \in R$ such that $(1 + tx)I = 0$ by [13, Theorem 75]. This implies that $1 + tx = 0$. So x is a unit element of R and hence $y \in I$, which is a contradiction. Therefore $x \in V(\acute{\Gamma}_I(R))$. □

The next example shows that the inclusion in Lemma 2.12 is not strict in general.

Example 2.13. *Let I be a finitely generated ideal of R with $Ann_R(I) = 0$. Further we assume that R is an Artinian ring with $Z(R) \cap I = 0$. Then we have $V(\acute{\Gamma}_I(R)) = Z_I(R)$. To see this, it is enough to prove that $V(\acute{\Gamma}_I(R)) \subseteq Z_I(R)$ by Lemma 2.12. Let $x \in V(\acute{\Gamma}_I(R))$. Then we have $x \neq 0$ and $xI \neq I$. This implies that $xR \neq R$ and hence x is a non-unit element of R. Since R is Artinian, the set of non-unit elements of R is the same as the set of zero-divisors of R. So $x \in Z(R)$. This shows that $x \notin I$ and there exists $0 \neq y \in R \setminus I$ such that $xy = 0 \in I$. Clearly, $x, y \in Z(R)$. Therefore, $V(\acute{\Gamma}_I(R)) \subseteq Z_I(R)$.*

Theorem 2.14. *Let I be a finitely generated ideal of R with $\sqrt{I} = I$ and $Ann_R(I) = 0$. Suppose that $Z_I(R) = V(\acute{\Gamma}_I(R))$. If $\Gamma_I(R)$ is complete, then $\acute{\Gamma}_I(R)$ is also a complete graph.*

Proof. Assume on the contrary that $\acute{\Gamma}_I(R)$ is not complete. So there exist $a, b \in V(\acute{\Gamma}_I(R))$ such that $a \in bI$ or $b \in aI$. Without loss of generality, we may assume that $a \in bI$. So, there exists $i \in I$ such that $a = bi$. We claim that i is a unit element. Otherwise, $i \in V(\acute{\Gamma}(R)$. Thus we have $i \in V(\acute{\Gamma}_I(R))$ by Lemma 2.9. Hence $i \in Z_I(R)$ by assumption, which is a contradiction. Now $ab = b^2 i \in I$. So there exist $i_1 \in I$ such that $b^2 i = i_1$. Then $b^2 = i^{-1} i_1 \in I$. Therefore, $b \in \sqrt{I} = I$, a contradiction. □

Proposition 2.15. *Let I be a proper ideal of R and $\acute{\Gamma}_I(R)$ a complete bipartite graph with parts V_i, $i = 1, 2$. Then every cyclic ideal $\boldsymbol{a}$, $\boldsymbol{b} \subseteq V_i$, for some $i = 1, 2$, are totally ordered.*

Proof. Assume on the contrary that there exist ideals aR and bR in V_1 such that $aR \nsubseteq bR$ and $bR \nsubseteq aR$. It follows that $b \notin aR$ and $a \notin bR$. Hence $b \notin aI$ and $a \notin bI$. This means a is adjacent to b, a contradiction. □

Proposition 2.16. *Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. If the graph $\acute{\Gamma}_I(R) \setminus J(R)$ is n-partite for some positive integer n, then $|Max(R)| \leq n$.*

Proof. Assume contrary that $|Max(R)| > n$. Since $\acute{\Gamma}_I(R) \setminus J(R)$ is a n-partite graph and $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}(R))$ by Lemma 2.9, there exist $m, \acute{m} \in Max(R)$ and $a \in m \setminus \acute{m}, b \in \acute{m} \setminus m$ such that a, b belong to a same part. Clearly, $a \notin bI$ and $b \notin aI$, which is a contradiction. □

For a graph G, let $\chi(G)$ denote the *chromatic number* of the graph G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* of a graph G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $clique(G)$, is called the *clique number* of G.

Theorem 2.17.

(1) Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. Then if R has infinite member of maximal ideal, then clique $\acute{\Gamma}_I(R)$ is also infinite; otherwise clique $(\acute{\Gamma}_I(R)) \geqslant |Max(R)|$.

(2) If $\chi(\acute{\Gamma}_I(R)) < \infty$, then $|Max(R)| < \infty$.

Proof. (1) This follows from Lemma 2.9 and [1, Theorem 2.14].
(2) Use part (1) along with [1, Theorem 2.14]. □

Theorem 2.18. *Let $R = S_1 + S_2$, where S_1 and S_2 are second ideals of R. If $P_1 = Ann_R(S_1)$ and $P_2 = Ann_R(S_2)$, then $V(\acute{\Gamma}(R)) = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$ and $\acute{\Gamma}(R)$ is a complete bipartite graph.*

Proof. Let $x \in V(\acute{\Gamma}(R))$, so we have $xR \neq 0$ and $xR \neq R$. Since $xR \neq 0$, $xS_1 \neq 0$ or $xS_2 \neq 0$. First we show that $V(\acute{\Gamma}(R)) = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$. If $xS_1 \neq 0$, then $x \notin P_1$. So $xS_1 = S_1$. We claim that $xS_2 = 0$. Otherwise, $xS_2 \neq 0$ so that $x \notin P_2$. It means that $xS_2 = S_2$. Thus $xR = R$, a contradiction. So we have $x \in P_2$ hence $x \in (P_2 \setminus P_1) \cup (P_2 \setminus P_1)$. We have similar arguments for reverse inclusion. Now let $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. We show that $x \notin yR$ and $y \notin xR$. Otherwise, $x \in yR$ or $y \in xR$. Without loss of generality, $x \in yR$. Then there exists $t \in R$ such that $x = ty$. But $x \notin P_2$ implies that $ty \notin Ann_R(S_2)$ so that $tyS_2 \neq 0$, a contradiction. Thus, x is adjacent to y. Now we show that x and y can not lie in $P_1 \setminus P_2$ or $P_1 \setminus P_2$. To see this let $x, y \in P_1 \setminus P_2$ and assume that they are adjacent. Then we have $x \notin yR$ and $y \notin xR$. Now by using our assumptions, we conclude that $x \notin xR$, a contradiction. □

Theorem 2.19. *Let $I \neq R$ be a finitely generated ideal of R with $Ann_R(I) = 0$. Assume that $|Max(R)| \geq 5$. Then $\acute{\Gamma}_I(R)$ is not planar.*

Proof. This follows from Lemma 2.9 and [1, Theorem 3.9]. □

Proposition 2.20. *Let I be a proper ideal. Then the following hold.*

(a) $V(\Gamma_{Ann(I)}(R)) \subseteq V(\acute{\Gamma}_I(R))$.

(b) *If* R *be a reduced ring, then* $\Gamma_{Ann(I)}(R)$ *is a subgraph of* $\acute{\Gamma}_I(R)$.

Proof. (a) Let $x \in V(\Gamma_{Ann(I)}(R))$. Then there exists $y \in R \setminus Ann_R(I)$ such that $xy \in Ann_R(I)$. We claim that $xI \neq I$. Otherwise, $xI = I$. Then $xyI = yI$ so that $yI = 0$. This implies that $y \in Ann_R(I)$, a contradiction. Therefore, $V(\Gamma_{Ann(I)}(R)) \subseteq V(\acute{\Gamma}_I(R))$.

(b) By part (a), $V(\Gamma_{Ann(I)}(R)) \subseteq V(\acute{\Gamma}_I(R))$. Now we suppose that x is adjacent to y in $\Gamma_{Ann(I)}(R)$. We show that x is adjacent to y in $\acute{\Gamma}_I(R)$. Otherwise, without loss of generality, we assume that $x \in yI$. So that $x^2 \in xyI$. Thus $x^2 = 0$. This implies that $x \in Ann_R(I)$, a contradiction. □

Proposition 2.21. *Let* I *be a finitely generated non-zero ideal of* R. *Suppose that* $x, y \in R \setminus Ann_R(I)$.

(a) $x \in V((\acute{\Gamma}_I(R))$ *if and only if* $x + Ann_R(I) \in V(\acute{\Gamma}(R/Ann_R(I)))$.

(b) If $x + Ann_R(I)$ *is adjacent to* $y + Ann_R(I)$ *in* $\acute{\Gamma}(R/Ann_R(I))$, *then* x *is adjacent to* y *in* $\acute{\Gamma}_I(R)$.

Proof. a) Let $x \in V(\acute{\Gamma}_I(R))$ and $x \in V((\acute{\Gamma}(R/Ann_R(I)))$. Then there exists $y + Ann_R(I)$ such that $xy + Ann_R(I) = 1 + Ann_R(I)$. Thus $(xy - 1) \in Ann_R(I)$. Since I is a finite generated ideal, there exists $r \in R$ such that $(r(xy-1)+1)I = 0$ and so $r(xy-1)+1 \in Ann_R(I)$. Thus $1 \in Ann_R(I)$ which implies that $I = 0$, a contradiction.

b) This is straightforward. □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = Ann_M(I)$, equivalently, for each submodule N of M, we have $N = Ann_M(Ann_R(N))$ [7]. R is said to be a *comultiplication ring* if R is a comultiplication R-module.

Theorem 2.22. *Let* I *be a proper ideal of* R. *Then* $V(\acute{\Gamma}_I(R)) = V(\Gamma_{Ann(I)}(R))$ *if one of the following conditions hold.*

(a) R *is a comultiplication ring.*

(b) $R/Ann_R(I) = Z(R/Ann_R(I)) \cup U(R/Ann_R(I))$.

Proof. Clearly $V(\Gamma_{Ann(I)}(R)) \subseteq V(\acute{\Gamma}_I(R))$.

(a) Let $x \in V(\acute{\Gamma}_I(R))$. Then $xI \neq 0$ and $xI \neq I$. Since R is a comultiplication ring, this implies that $Ann_R(xI) \neq Ann_R(I)$. Thus there exists $y \in Ann_R(xI) \setminus Ann_R(I)$. Therefore, $x \in V(\Gamma_{Ann(I)}(R))$.

(b) Let $x \in V(\acute{\Gamma}_I(R))$. Then $xI \neq 0$ and $xI \neq I$. By assumption, $x + Ann_R(I) \in Z(R/Ann_R(I))$ or $x + Ann_R(I) \in U(R/Ann_R(I))$. If $x + Ann_R(I) \in Z(R/Ann_R(I))$, then there exists $y \in R \setminus Ann_R(I)$ such that $xy \in Ann_R(I)$. Therefore, $x \in V(\Gamma_{Ann(I)}(R))$. If $x + Ann_R(I) \in U(R/Ann_R(I))$, then there exists $z + Ann_R(I) \in R/Ann_R(I)$ such that $xz + Ann_R(I) = 1 + Ann_R(I)$. Thus $1 = xz + a$ for some $a \in Ann_R(I)$. Now we have $I = 1I = (xz + a)I = xzI \subseteq xI$, a contradiction. □

Theorem 2.23. *Let* $I \subseteq J$ *be non-zero ideals of* R. *Then we have the following.*

(a) If $R/Ann_R(J) = Z(R/Ann_R(J)) \bigcup U(R/Ann_R(J))$, *then* $V(\acute{\Gamma}_I(R)) \subseteq V(\acute{\Gamma}_J(R))$.

(b) If $dim(R) = 0$, *then* $V(\acute{\Gamma}_I(R)) \subseteq V(\acute{\Gamma}_J(R))$. *In particular, this holds if* R *is a finite ring.*

Proof. (a) This follows from Theorem 2.22 (b) and [6, Theorem 2.8].

(b) $dim(R) = 0$ implies that $dim(R/J) = 0$. It follows that

$$R/Ann_R(J) = Z(R/Ann_R(J)) \bigcup U(R/Ann_R(J)).$$

Now the result follows from part (a). □

Proposition 2.24. *Let* I *be a non-zero ideal* R *with* $R = Z(R) \cup U(R)$ *and* $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}(R))$. *Then* $Ann_R(I) = 0$.

Proof. Suppose that $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}(R))$. Since $V(\acute{\Gamma}_I(R)) \subseteq R \setminus Ann_R(I)$, we have $V(\acute{\Gamma}(R)) \subseteq R \setminus Ann_R(I)$. Thus $Ann_R(I) \subseteq R \setminus V(\acute{\Gamma}(R)) = \{0\} \cup U(R)$ by hypothesis. Therefore, $Ann_R(I) = 0$. □

3. Secondal ideals

In this section, we will study the ideal-based cozero-divisor graph with respect to secondal ideals.

The element $a \in R$ is called *prime to an ideal* I of R if $ra \in I$ (where $r \in R$) implies that $r \in I$. The set of elements of R which are not prime to I is denoted by $S(I)$. A proper ideal I of R is said to be *primal* if $S(I)$ is an ideal of R [12].

A non-zero submodule N of an R-module M is said to be *secondal* if $W_R(N) = \{a \in R : aN \neq N\}$ is an ideal of R [8]. A *secondal ideal* is defined similarly when $N = I$ is an ideal of R. In this case, we say I is P-*secondal*, where $P = W(I)$ is a prime ideal of R.

Lemma 3.1. *Let I be a non-zero ideal of R. Then the following hold.*

(a) $Ann_R(I) \subseteq W(I)$.

(b) $Z_R(R/Ann_R(I)) \subseteq W(I)$.

(c) $V(\acute{\Gamma}_I(R)) = W(I) \setminus Ann_R(I)$. *In particular,* $V(\acute{\Gamma}_I(R)) \cup Ann_R(I) = W(I)$.

(d) If $Ann_R(I)$ is a radical ideal of R, then $\bigcup_{P \in Min(Ann_R(I))} P \subseteq W(I)$.

Proof. (a) Let $r \in Ann_R(I)$. Then $rI = 0 \neq I$. Thus $r \in W(I)$.

(b) Let $x \in Z_R(R/Ann_R(I))$ and $x \notin W(I)$. Then there exists $y \in R \setminus Ann_R(I)$ such that $xyI = 0$. Hence $xI = I$ implies that $yI = 0$, a contradiction.

(c) Let $r \in V(\acute{\Gamma}_I(R))$. Then $r \in R \setminus Ann_R(I)$ and $rI \neq I$; hence $r \in W(I) \setminus Ann_R(I)$. Thus $V(\acute{\Gamma}_I(R)) \subseteq W(I) \setminus Ann_R(I)$. Conversely, we assume that $x \in W(I) \setminus Ann_R(I)$. So $xI \neq I$ and $xI \neq 0$. Then $x \in V(\acute{\Gamma}_I(R))$, so we have equality.

(d) By [13, Exer 13, page 63], $Z_R(R/I) = \bigcup_{P \in Min(I)} P$, where I is a radical ideal of R. Thus $Z_R(R/Ann_R(I)) = \bigcup_{P \in Min(Ann_R(I))} P$. Hence $\bigcup_{P \in Min(Ann_R(I))} P \subseteq W(I)$ by part (b). □

Remark 3.2. *Let $R = \mathbb{Z}$, $I = 2\mathbb{Z}$. Then $Z_R(R/Ann_R(I)) = Z_R(R) = 0$ and $W(I) = \mathbb{Z} \setminus \{-1, 1\}$. Therefore the converse of part (b) of the above lemma is not true in general.*

Proposition 3.3. *Let I and P be ideals of R with $Ann_R(I) \subseteq P$. Then I is a P-secondal ideal of R if only if $V(\acute{\Gamma}_I(R)) = P \setminus Ann_R(I)$.*

Proof. Straightforward. □

Theorem 3.4. *Let I be an ideal of R. Then I is a secondal ideal of R if and only if $V(\acute{\Gamma}_I(R)) \cup Ann_R(I)$ is an (prime) ideal of R.*

Proof. Let I be a secondal ideal. Then $W(I)$ is a prime ideal and by Lemma 3.1(c), $V(\acute{\Gamma}_I(R)) \cup Ann_R(I) = W(I)$. Thus $V(\acute{\Gamma}_I(R)) \cup Ann_R(I)$ is an ideal of R. Conversely, suppose that $V(\acute{\Gamma}_I(R)) \cup Ann_R(I)$ is a (prime) ideal. Then by Lemma 3.1(c) , $V(\acute{\Gamma}_I(R)) \cup Ann_R(I) = W(I)$ is a prime ideal. Hence I is a secondal ideal. □

Theorem 3.5. *Let I and J be P-secondal ideals of R. Then $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}_J(R))$ if and only if $Ann_R(I) = Ann_R(J)$.*

Proof. By Lemma 3.1 (a), $Ann_R(I) \subseteq P$ and $Ann_R(J) \subseteq P$. It then follows from Proposition 3.3 that $V(\acute{\Gamma}_I(R)) = V(\acute{\Gamma}_J(R))$ if and only if $P \setminus Ann_R(I) = P \setminus Ann_R(J)$; and this holds if and only if $Ann_R(I) = Ann_R(J)$. □

Lemma 3.6. *Let N be a secondary submodule of an R-module M. Then $\sqrt{Ann_R(N)} = W(N)$.*

Proof. Let $x \in W(N)$. Then $xN \neq N$. Since N is a secondary R-module, there exists a positive integer n such that $x^n N = 0$. Thus $x \in \sqrt{Ann_R(N)}$. Hence $W(N) \subseteq \sqrt{Ann_R(N)}$. To see the reverse inclusion, let $x \in \sqrt{Ann_R(N)}$ and $x \notin W(N)$. Then $x^n N = 0$ for some positive integer n and $xN = N$. Therefore $N = 0$, a contradiction. □

Theorem 3.7. *Let I be an ideal of R. Then I is secondary ideal if and only if $V(\acute{\Gamma}_I(R)) = \sqrt{Ann_R(I)} \setminus Ann_R(I)$.*

Proof. If I is secondary, then $\sqrt{Ann_R(I)} = W(I)$ by Lemma 3.6. Hence I is a $\sqrt{Ann_R(I)}$-secondal ideal of R. Then Proposition 3.3 implies that $V(\acute{\Gamma}_I(R)) = \sqrt{Ann_R(I)} \setminus Ann_R(I)$. Conversely, suppose that $x \in R, xI \neq I$, and $x \notin \sqrt{Ann_R(I)}$. Then $x \in W(I)$ and $x \notin Ann_R(I)$. Thus $x \in V(\acute{\Gamma}_I(R))$ and so $x \in \sqrt{Ann_R(I)} \setminus Ann_R(I)$ by assumption, a contradiction. □

Definition 3.8. *Let I be an ideal of R. We say that an ideal J of R is* second *to I if $IJ = I$.*

Proposition 3.9. *Let I be an ideal of R. If I is not secondal, then there exist $x, y \in V(\acute{\Gamma}_I(R))$ such that $< x, y >$ is second to I.*

Proof. Suppose that I is an ideal of R such that it is not secondal. Then by Lemma 3.1 (c), $V(\acute{\Gamma}_I(R)) \cup Ann_R(I) = W(I)$ is not an ideal of R, so there exist $x, y \in W(I)$ with $x - y \notin W(I)$ and so $(x - y)I = I$. Hence $< x, y > I = I$. Now we claim that $x, y \notin Ann_R(I)$. Otherwise, we have $x \in Ann_R(I)$ or $y \in Ann_R(I)$. If $x, y \in Ann_R(I)$, then $x - y \in Ann_R(I) \subseteq W(I)$, a contradiction. If $x \in Ann_R(I)$ and $y \notin Ann_R(I)$, then $I = (x - y)I \subseteq xI + yI = 0 + yI$, a contradiction. Similarly, we get a contradiction when $x \notin Ann_R(I)$ and $y \in Ann_R(I)$. Thus we have $x, y \notin Ann_R(I)$. □

Proposition 3.10. *Let I be an ideal of R. Then the following hold.*

(a) *Let x, y be distinct elements of $\sqrt{Ann_R(I)} \setminus Ann_R(I)$ with $xy \notin Ann_R(I)$. Then the ideal $< x, y >$ is not second to I.*

(b) *If I is a secondary ideal, then the $diam(\Gamma_{Ann(I)}(R)) \leq 2$.*

Proof. (a) Let ideal $< x, y >$ be second to I. Since $x, y \in \sqrt{Ann_R(I)} \setminus Ann_R(I)$, there exists the least positive integer n such that $x^n y \in Ann(I)$. As $xy \notin Ann_R(I)$, we have $n \geqslant 2$. Let m be the least positive such that $x^{n-1}y^m \in Ann_R(I)$. Now clearly $m \geqslant 2$ because $x^{n-1}y \notin Ann_R(I)$. This yields that the contradiction

$$0 = x^{n-1}y^{m-1}(x, y)I = x^{n-1}y^{m-1}I \neq 0.$$

(b) If I is secondary, then $W(I) = \sqrt{Ann_R(I)}$ by Lemma 3.6. Choose two distinct vertices x, y in $\Gamma_{Ann(I)}(R)$. If $xy \in Ann_R(I)$, then $d(x, y) = 1$. So we assume that $xy \notin Ann_R(I)$. Then by Proposition 2.20 (a) and Lemma 3.1, $x, y \in W(I) \setminus Ann_R(I)$. Also we have $x, y \in \sqrt{Ann_R(I)} \setminus Ann_R(I)$ by Theorem 3.7. As in the proof of (a), we have the path $x - x^{n-1}y^{m-1} - y$ from x to y in $\Gamma_{Ann(I)}(R)$. Hence $d(x, y) = 2$. Therefore, $diam(\Gamma_{Ann(I)}(R)) \leq 2$. □

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References

1. M. Afkhami and K. Khashyarmanesh, *The cozero-divisor graph of a commutative ring*, Southeast Asian Bull. Math. 35, 753–762, (2011).
2. M. Afkhami and K. Khashyarmanesh, *On the cozero-divisor graphs of commutative rings and their complements*, Bull. Malays. Math. Sci. Soc. 35, 935-944, (2012).
3. M. Afkhami and K. Khashyarmanesh, *On the cozero-divisor graphs of commutative rings*, Appl. Math. 4, 979-985, (2013).
4. D. F. Anderson, Sh. Ghalandarzadeh, S. Shirinkam, and P. Malakooti Rad, *On the diameter of the graph $\Gamma_{Ann(M)}(R)$*, Filomat 26 (3), 623-629, (2012).
5. D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217, 434–447, (1999).
6. D.F. Anderson and S. Shirinkam, *Some remarks on the graph $\Gamma_I(R)$*, Comm. Algebra 42, 545—562, (2014).

7. H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math. 11 (4), 1189–1201, (2007).
8. H. Ansari-Toroghy and F. Farshadifar, *The dual notion of some generalizations of prime submodules*, Comm. Algebra 39, 2396-2416, (2011).
9. J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, American Elsevier, New York, 1976.
10. S. Ebrahimi Atani and A. Yousefian Darani, *Zero-divisor graphs with respect to primal and weakly primal ideals*, J. Korean Math. Soc. 46, 313–325, (2009).
11. P. Dheena and B. Elavarasan, *An ideal-based zero-divisor graph of 2-primal near-rings*, Bull. Korean Math. Soc. 46, 1051-1060, (2009).
12. L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc. 1, 1–6, (1950).
13. I. Kaplansky, *Commutative rings*, University of Chicago, (1978).
14. M.J. Nikmehr and S. Khojasteh, *A generalized ideal-based zero-divisor graph*, J. Alegbra Appl. 14 (6), 1550079, (2015).
15. H. R. Maimani, M. R. Pournaki, and S. Yassemi, *Zero-divisor graph with respect to an ideal*, Comm. Algebra 34 (3), 923-929, (2006).
16. S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra 31, 4425–4443, (2003).
17. A. Yousefian Darani, *Notes on the ideal-based zero-divisor graph*, J. Math. Appl. 32, 103–107, (2010).

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