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Fuzzy Hyers-Ulam-Rassias Stability for Generalized Additive Functional Equations

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ABSTRACT: In this paper we establish Hyers-Ulam-Rassias stability of the following

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

where m is a positive integer greater than 3, in fuzzy Banach spaces.
The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

Key Words: Hyers-Ulam-Rassias stability, Fuzzy normed space.

Contents

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [34] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [25] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. *(Th.M. Rassias): Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [33] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [4] proved the Hyers-Ulam stability of the quadratic functional equation.

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In this paper, we consider the following functional equation

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i) \tag{1.1}$$

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.1) in fuzzy Banach spaces. First, we introduce the following lemma due to A. Najati and A. Ramjbar [17] with $n=3$ in (1.1).

Lemma 1.2. *Let X and Y be linear spaces. A mapping $f:X\to Y$ satisfies the equation*

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)] \tag{1.2}$$

for all $x,y,z\in X$ if and only if f is additive.

It is noted that the following equation with $z=0$ in (1.1)

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x}{2}+y\right)+f\left(x+\frac{y}{2}\right)=2f(x)+2f(y)$$

is equivalent to $f(x+y)=f(x)+f(y)$ for all $x,y\in X$.
Second, we introduce the following lemma due to J.M. Rassias and H.M. Kim [24].

Lemma 1.3. *Let X and Y be linear spaces and let $m\geq 3$ be a fixed positive integer. A function $f:X\to Y$ satisfies the equation*

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

for all $x_1,x_2,\cdots,x_m\in X$ if and only if f is an additive function.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5]–[8], [10,12], [18]–[23], [26]–[28], [29]–[32]).

Katsaras [13] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [9], [15]– [20]).
In particular, Cheng and Mordeson [2], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [14]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [1].

2. Priliminaries

Definition 2.1. *(saadati and Vaezpour [31]) Let X be a real vector space. A function $N:X\times\mathbb{R}\to[0,1]$ is called a fuzzy norm on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,*
$(N1)$ $N(x,t)>0$ for $t>0$;
$(N2)$ $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
$(N3)$ $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$;
$(N4)$ $N(x+y,c+t)\geq min\{N(x,s),N(y,t)\}$;
$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;
$(N6)$ for $x\neq 0$, $N(x,.)$ is continuous on $\mathbb{R}^+$.

The pair (X,N) is called a fuzzy normed vector space.

Example 2.1. *Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.2. *(Saadati and Vaezpour [31]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n, x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.*

Definition 2.3. *(Saadati and Vaezpour [31]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy sequence if and only if for each $0 < \epsilon < 1$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $N(x_n, x_m, t) > 1 - \epsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.
We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X .

Definition 2.4. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:*
(1) *$d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;*
(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(3) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Theorem 2.5. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

3. Fuzzy Stability of Functional Equation (1.1): a direct method

In this section, using direct method, we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed spaces. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

Theorem 3.1. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq N'(\varphi(x_1, \cdots, x_m), t) \tag{3.1}$$

for all $x_1, \cdots, x_m \in\in X$, $t > 0$ and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{m-1}$ such that

$$N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \geq N'\left(\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \tag{3.2}$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then we can find a unique additive mapping $A : X \to Y$ satisfying (1.1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2|r|\varphi(x, x, \cdots, x)}{m(m-1)(1-|r|(m-1))}, t\right) \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.2) that

$$\begin{aligned} N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), t\right) &\geq N'\left(r^{j-1}\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \\ &= N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{|r|^j}\right). \end{aligned} \tag{3.4}$$

So

$$N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), |r|^j t\right) \geq N'\left(\varphi(x_1, x_2, \cdots, x_m), t\right)$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$.
Substituting $x_1 = x_2 = \cdots = x_m = x$ in (3.1), we obtain

$$N\left(\frac{m(m-1)}{2} f((m-1)x) - \frac{m(m-1)^2}{2} f(x), t\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{3.5}$$

So

$$\begin{aligned} &N\left(f(x) - (m-1)f\left(\frac{x}{m-1}\right), \frac{2t}{m(m-1)}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right), t\right) \end{aligned} \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{(m-1)^j}$ in (3.6), we have

$$\begin{aligned} &N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^{j-1}t}{m}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{(m-1)^{j+1}}, \frac{x}{(m-1)^{j+1}}, \cdots, \frac{x}{(m-1)^{j+1}}\right), t\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^{j+1}}\right) \end{aligned} \tag{3.7}$$

for all $x \in X$, all $t > 0$ and any integer $j \geq 0$. So

$$\begin{aligned} &N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &= N\left(\sum_{j=0}^{n-1}\left[(m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right)\right], \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &\geq \min_{0 \leq j \leq n-1}\left\{N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right)\right\} \\ &\geq N'(\varphi(x, x, \cdots, x), t) \end{aligned}$$

which yields

$$N\left((m-1)^{n+p}f\left(\frac{x}{(m-1)^{n+p}}\right)-(m-1)^p f\left(\frac{x}{(m-1)^p}\right),\sum_{j=0}^{n-1}\frac{2(m-1)^{j+p}|r|^{j+1}t}{m(m-1)}\right)$$
$$\geq N'\left(\varphi\left(\frac{x}{(m-1)^p},\frac{x}{(m-1)^p},\cdots,\frac{x}{(m-1)^p}\right),t\right)$$
$$\geq N'\left(\varphi(x,x,\cdots,x),\frac{t}{|r|^p}\right)$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. So

$$N\left((m-1)^{n+p}f\left(\frac{x}{(m-1)^{n+p}}\right)-(m-1)^p f\left(\frac{x}{(m-1)^p}\right),\sum_{j=0}^{n-1}\frac{2(m-1)^{j+p}|r|^{j++p+1}t}{m(m-1)}\right)$$
$$\geq N'(\varphi(x,x,\cdots,x),t)$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Hence one obtains

$$N\left((m-1)^{n+p}f\left(\frac{x}{(m-1)^{n+p}}\right)-(m-1)^p f\left(\frac{x}{(m-1)^p}\right),t\right) \tag{3.8}$$
$$\geq N'\left(\varphi(x,x,\cdots,x),\frac{t}{\frac{2(m-1)^{p-1}|r|^{p+1}}{m}\sum_{j=0}^{n-1}(m-1)^j|r|^j}\right)$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Since, the series

$$\sum_{j=0}^{\infty}(m-1)^j|r|^j$$

is convergent series, we see by taking the limit $p\to\infty$ in the last inequality that a sequence $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y,N) and so it converges in Y. Therefore a mapping $A:X\to Y$ defined by $A(x):=N-\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ is well defined for all $x\in X$. It means that

$$\lim_{n\to\infty}N\left(A(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right),t\right)=1 \tag{3.9}$$

for all $x\in X$ and all $t>0$. In addition, it follows from (3.8) that

$$N\left(f(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right),t\right)\geq N'\left(\varphi(x,x,\cdots,x),\frac{t}{\frac{2|r|}{m(m-1)}\sum_{j=0}^{n-1}(m-1)^j|r|^j}\right)$$

for all $x\in X$ and all $t>0$. So

$$N(f(x)-A(x),t)$$
$$\geq\min\left\{N\left(f(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right),(1-\epsilon)t\right),N\left(A(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right),\epsilon t\right)\right\}$$
$$\geq N'\left(\varphi(x,x,\cdots,x),\frac{t}{\frac{2|r|}{m(m-1)}\sum_{j=0}^{n-1}(m-1)^j|r|^j}\right)$$
$$\geq N'\left(\varphi(x,x,\cdots,x),\frac{m(m-1)(1-|r|(m-1))\epsilon t}{2|r|}\right)$$

for sufficiently large n and for all $x \in X$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{2|r|}\right)$$

for all $x \in X$ and $t > 0$. It follows from (3.1) that

$$\begin{aligned} &N\bigg((m-1)^n\bigg[\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2(m-1)^n} + \sum_{l=1, k_l\neq i,j}^{m-2} \frac{x_{k_l}}{(m-1)^n}\right) \\ &-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\bigg], t\bigg) \\ &\geq N'\left(\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right), \frac{t}{(m-1)^n}\right) \\ &\geq N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{(m-1)^n|r|^n}\right) \end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{(m-1)^n|r|^n}\right) = 1$$

and so

$$\begin{aligned} &N\bigg((m-1)^n\bigg[\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2(m-1)^n} + \sum_{l=1, k_l\neq i,j}^{m-2} \frac{x_{k_l}}{(m-1)^n}\right) \\ &-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\bigg], t\bigg) \to 1 \end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Therefore, we obtain in view of (3.9)

$$\begin{aligned} &N\left(\sum_{1\leq i<j\leq m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i), t\right) \\ &\geq \min\bigg\{N\bigg(\sum_{1\leq i<j\leq m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i) \\ &-(m-1)^n\bigg[\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2(m-1)^n} + \sum_{l=1, k_l\neq i,j}^{m-2} \frac{x_{k_l}}{(m-1)^n}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\bigg], \frac{t}{2}\bigg) \\ &, N\bigg((m-1)^n\bigg[\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2(m-1)^n} + \sum_{l=1, k_l\neq i,j}^{m-2} \frac{x_{k_l}}{(m-1)^n}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\bigg], \frac{t}{2}\bigg)\bigg\} \\ &= N\bigg((m-1)^n\bigg[\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2(m-1)^n} + \sum_{l=1, k_l\neq i,j}^{m-2} \frac{x_{k_l}}{(m-1)^n}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\bigg], \frac{t}{2}\bigg) \\ &\geq N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{2(m-1)^n|r|^n}\right) \to 1 \text{ as } n \to \infty \end{aligned}$$

which implies

$$\sum_{1\leq i<j\leq m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) = \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$. Thus $A : X \to Y$ is a mapping satisfying the equation (1.1) and the inequality (3.3).
To prove the uniqueness, let there is another mapping $L : X \to Y$ which satisfies the inequality (3.3). Since $L((m-1)^n x) = (m-1)^n L(x)$ for all $x \in X$, we have

$$\begin{aligned}
&N(A(x) - L(x), t) \\
&= N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), t\right) \\
&\geq \min\left\{N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right. \\
&\left., N\left((m-1)^n f\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right\} \\
&\geq N'\left(\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right), \frac{m(m-1)(1-|r|(m-1))t}{4|r|(m-1)^n}\right) \\
&\geq N\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{4|r|^{n+1}(m-1)^n}\right) \to 1 \text{ as } n \to \infty \text{ by (N5)}
\end{aligned}$$

for all $t > 0$. Therefore $A(x) = L(x)$ for all $x \in X$, this completes the proof. □

Corollary 3.2. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exists real number $\theta \geq 0$ and $0 < p < 2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfying the following inequality*

$$N\left(\sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq N'\left(\theta\left(\sum_{j=1}^{m} \|x_j\|^p\right), t\right)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1.1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m^3 - 4m^2 + 5m - 2}, t\right)$$

Proof. Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{j=1}^{m} \|x_j\|^p\right)$ and $|r| = \frac{1}{(m-1)^2}$. Apply Theorem 3.1, we get desired results. □

Theorem 3.3. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.1) and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < m - 1$ such that*

$$N'(\varphi(x_1, \cdots, x_m), |r|t) \geq N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \tag{3.10}$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then we can find a unique additive mapping $A : X \to Y$ that satisfying (1.1) and the following inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\varphi(x, x, \cdots, x)}{m(m-1)(m-1-|r|)}, t\right). \tag{3.11}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Replacing x by $(m-1)^n x$ in (3.12), we obtain

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2t}{m(m-1)^{n+2}}\right) \tag{3.13}$$
$$\geq N'(\varphi((m-1)^n x, (m-1)^n x, \cdots, (m-1)^n x), t)$$
$$\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^n}\right).$$

So

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2|r|^n t}{m(m-1)^{n+2}}\right) \tag{3.14}$$
$$\geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, \sum_{j=0}^{n-1} \frac{2|r|^j t}{m(m-1)^{j+2}}\right) \geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, all $t > 0$ and any integer $n > 0$. So

$$\begin{aligned} N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, t\right) &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{j=0}^{n-1} \frac{2|r|^j}{m(m-1)^{j+2}}}\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(m-1-|r|)t}{2}\right). \end{aligned} \tag{3.15}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exists real number $\theta \geq 0$ and $0 < p = \sum_{j=1}^m p_j < 2$ such that a mapping $f : X \to Y$ satisfying the following inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2} \sum_{i=1}^m f(x_i), t\right) \geq N'\left(\theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right), t\right)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1.1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m(m-1)}, t\right)$$

Proof. Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right)$ and $r = m - 2$. Applying Theorem 3.3, we get the desired results. □

4. Fuzzy Stability of Functional Equation (1.1): a fixed point method

In this section, using the fixed point alternative approach we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 4.1. *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right) \leq \frac{L\varphi(x_1, x_2, \cdots, x_m)}{m-1}$$

for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ with $f(0) = 0$ is a mapping satisfying

$$N\left(\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq \frac{t}{t+\varphi(x_1,x_2,\cdots,x_m)} \tag{4.1}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then, the limit

$$A(x) := N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \geq \frac{(m(m-1)^2-m(m-1)^2L)t}{(m(m-1)^2-m(m-1)^2L)t+2L\varphi(x,x,\cdots,x)}. \tag{4.2}$$

Proof. Putting $x_1 = x_2 = \cdots = x_m = x$ in (4.1), we have

$$N\left(\frac{m(m-1)f((m-1)x)}{2}-\frac{m(m-1)^2 f(x)}{2}, t\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)} \tag{4.3}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\ ;\ g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu\in\mathbb{R}^+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}, \forall x\in X,\, t>0\right\},$$

where $\inf\emptyset = +\infty$. It is easy to show that (S, d) is complete (see [16, Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := (m-1)g\left(\frac{x}{m-1}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x)-h(x),\epsilon t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
N(Jg(x)-Jh(x),L\epsilon t) &= N\left((m-1)g\left(\frac{x}{m-1}\right)-(m-1)h\left(\frac{x}{m-1}\right),L\epsilon t\right)\\
&= N\left(g\left(\frac{x}{m-1}\right)-h\left(\frac{x}{m-1}\right),\frac{L\epsilon t}{m-1}\right)\\
&\geq \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1}+\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)}\\
&\geq \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1}+\frac{L\varphi(x_1,x_2,\cdots,x_m)}{m-1}}\\
&= \frac{t}{t+\varphi(x,x,\cdots,x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (4.3) that

$$N\left(\frac{m(m-1)\left[f((m-1)x)-(m-1)f(x)\right]}{2}, t\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}.$$

So

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2t}{m(m-1)}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)} \geq \frac{t}{t+\frac{L\varphi(x,x,\cdots,x)}{m-1}} = \frac{\frac{(m-1)t}{L}}{\frac{(m-1)t}{L}+\varphi(x,x,\cdots,x)}. \tag{4.4}$$

Therefore

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2Lt}{m(m-1)^2}\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}. \tag{4.5}$$

This means that

$$d(f,Jf) \leq \frac{2L}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{m-1}\right)=\frac{A(x)}{m-1} \tag{4.6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (4.6) such that there exists $\mu \in (0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)=A(x)$$

for all $x \in X$.
(3) $d(f,A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f,A) \leq \frac{2L}{m(m-1)^2-m(m-1)^2L}.$$

This implies that the inequality (4.2) holds. Furthermore, since

$$\begin{aligned}
&N\left(\sum_{1\leq i<j\leq m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i),t\right)\\
&= N-\lim_{n\to\infty}\left((m-1)^n\left[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)\right.\right.\\
&\left.\left.-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\right],t\right)\\
&\geq \lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\varphi\left(\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}\right)}\\
&\geq \lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\frac{L^n\varphi(x_1,x_2,\cdots,x_m)}{(m-1)^n}}
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n} + \frac{L^n \varphi(x_1,x_2,\cdots,x_m)}{(m-1)^n}} = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$, we deduce that

$$N\left(\sum_{1\le i<j\le m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i), t\right) = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Thus the mapping $A : X \to Y$ is additive, as desired. □

Corollary 4.2. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality*

$$N\left(\sum_{1\le i<j\le m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \ge \frac{t}{t + \theta\left(\sum_{i=1}^{m} \|x_i\|^p\right)}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then, the limit

$$A(x) := N\text{-}\lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{((m-1)^p - 1)t}{((m-1)^p - 1)t + 2(m-1)^{-2}\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{i=1}^{m} \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

Theorem 4.3. *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_m) \le (m-1)L\varphi\left(\frac{x}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right)$$

for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (4.1). Then

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{m(m-1)^2(1-L)t}{m(m-1)^2(1-L)t + 2\varphi(x, x, \cdots, x)} \tag{4.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 4.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g((m-1)x)}{m-1}$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \ge \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{g((m-1)x)}{m-1} - \frac{h((m-1)x)}{m-1}, L\epsilon t\right) \\ &= N\Big(g((m-1)x) - h((m-1)x), (m-1)L\epsilon t\Big) \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + \varphi((m-1)x, , (m-1)x, \cdots, (m-1)x)} \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + (m-1)L\varphi(x, x, \cdots, x)} \\ &= \frac{t}{t + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (4.3) that

$$N\left(\frac{m(m-1)^2}{2}\left[\frac{f((m-1)x)}{m-1} - f(x)\right], t\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)} \tag{4.8}$$

for all $x \in X$ and $t > 0$. So

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

Therefore

$$d(f, Jf) \leq \frac{2}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$(m-1)A(x) = A((m-1)x) \tag{4.9}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that A is a unique mapping satisfying (4.9) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{2}{m(m-1)^2(1-L)}.$$

This implies that the inequality (4.7) holds. The rest of the proof is similar to that of the proof of Theorem 4.1. □

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < \frac{1}{m}$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq \frac{t}{t+\theta\left(\prod_{i=1}^{m}\|x_i\|^p\right)}$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{m((m-1)^{p+2} - (m-1)^2)t}{m((m-1)^{p+2} - (m-1)^2)t + 2(m-1)^p\theta\|x\|^{mp}}.$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{i=1}^{m}\|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

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