Construction Of Inverse Curves Of General Helices In The Sol Space $\mathfrak{Sol}^3$

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ABSTRACT: In this paper, we study inverse curves of general helices in the $\mathfrak{Sol}^3$. Finally, we find out explicit parametric equations of inverse curves in the $\mathfrak{Sol}^3$.

Key Words: General helix, Sol space, Curvature, Inverse curves.

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1. Introduction

An inversion with respect to the sphere $S_C (r)$ with the center $C \in \mathfrak{Sol}^3$ is given by

$$C + \frac{r^2}{\|P - C\|^2} (P - C),$$

where $r$ is radius, $P \in \mathfrak{Sol}^3$. The inversion is a conformal mapping and also is differentiable and a transformation defining between open subsets of $\mathfrak{Sol}^3$, [1,2,7].

A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802, [4].

In this paper, we study inverse curves of general helices in the $\mathfrak{Sol}^3$. Finally, we find out explicit parametric equations of inverse curves in the $\mathfrak{Sol}^3$.

2. Preliminaries

Sol space, one of Thurston’s eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = \epsilon^{2z}dx^2 + \epsilon^{-2z}dy^2 + dz^2,$$

where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = \epsilon^z dx, \quad \omega^2 = \epsilon^{-z} dy, \quad \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$e_1 = \epsilon^{-z} \frac{\partial}{\partial x}, \quad e_2 = \epsilon^z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$
Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric \( g_{\mathfrak{sol}^3} \), defined above the following is true:

\[
\nabla = \begin{pmatrix}
-\mathbf{e}_3 & 0 & \mathbf{e}_1 \\
0 & \mathbf{e}_3 & -\mathbf{e}_2 \\
0 & 0 & 0
\end{pmatrix},
\]

(2.5)

where the \((i, j)\)-element in the table above equals \( \nabla_{\mathbf{e}_i} \mathbf{e}_j \) for our basis \( \{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \).

3. Inverse Curves of General Helices in Sol Space \( \mathfrak{sol}^3 \)

Assume that \( \{\mathbf{T}, \mathbf{N}, \mathbf{B}\} \) be the Frenet frame field along \( \gamma \), \([3, 5, 6, 8]\). Then, the Frenet frame satisfies the following Frenet–Serret equations:

\[
\begin{align*}
\nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\
\nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\
\nabla_{\mathbf{T}} \mathbf{B} &= -\tau \mathbf{N},
\end{align*}
\]

(3.1)

where \( \kappa \) is the curvature of \( \gamma \) and \( \tau \) its torsion and

\[
\begin{align*}
g_{\mathfrak{sol}^3}(\mathbf{T}, \mathbf{T}) &= 1, \\
g_{\mathfrak{sol}^3}(\mathbf{N}, \mathbf{N}) &= 1, \\
g_{\mathfrak{sol}^3}(\mathbf{B}, \mathbf{B}) &= 1, \\
g_{\mathfrak{sol}^3}(\mathbf{T}, \mathbf{N}) &= 0, \\
g_{\mathfrak{sol}^3}(\mathbf{T}, \mathbf{B}) &= 0, \\
g_{\mathfrak{sol}^3}(\mathbf{N}, \mathbf{B}) &= 0.
\end{align*}
\]

(3.2)

With respect to the orthonormal basis \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \), we can write

\[
\begin{align*}
\mathbf{T} &= T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2 + T_3 \mathbf{e}_3, \\
\mathbf{N} &= N_1 \mathbf{e}_1 + N_2 \mathbf{e}_2 + N_3 \mathbf{e}_3, \\
\mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1 \mathbf{e}_1 + B_2 \mathbf{e}_2 + B_3 \mathbf{e}_3.
\end{align*}
\]

(3.3)

Theorem 3.1. Let \( \gamma : I \rightarrow \mathfrak{sol}^3 \) be a unit speed non-geodesic general helix. Then, the parametric equations of \( \gamma \) are

\[
\begin{align*}
x(s) &= \sin \mathfrak{P} e^{-\cos \mathfrak{P} s - \mathfrak{C}_3} \left[ \frac{\mathfrak{C}_1 \cos \mathfrak{C}_1 s + \mathfrak{C}_2}{\mathfrak{C}_1^2 + \cos^2 \mathfrak{P}} \right] + \mathfrak{C}_4, \\
y(s) &= \sin \mathfrak{P} e^{\cos \mathfrak{P} s + \mathfrak{C}_3} \left[ -\mathfrak{C}_1 \cos \mathfrak{C}_1 s + \mathfrak{C}_2 \right] + \mathfrak{C}_5, \\
z(s) &= \cos \mathfrak{P} s + \mathfrak{C}_3,
\end{align*}
\]

(3.4)

where \( \mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5 \) are constants of integration, \([8]\).

We can use Mathematica in Theorem 3.1, yields
Construction Of Inverse Curves Of General Helices In The Sol Space \( \mathfrak{Sol}^3 \)

Figure 1: Parametric equation for \( \gamma : I \rightarrow \mathfrak{Sol}^3 \)

An inversion with respect to the sphere \( S_C(r) \) with the center \( C \in \mathfrak{Sol}^3 \) is given by

\[
C + \frac{r^2}{\|P - C\|^2} (P - C).
\]

Let \( C \in \mathfrak{Sol}^3 \) and \( r \in \mathbb{R}^+ \). We denote that \( (\mathfrak{Sol}^3)^* = \mathfrak{Sol}^3 - \{C\} \). Then, an inversion of \( \mathfrak{Sol}^3 \) with the center \( C \in \mathfrak{Sol}^3 \) and the radius \( r \) is the map

\[
\Phi \left[ C, r \right] : (\mathfrak{Sol}^3)^* \rightarrow (\mathfrak{Sol}^3)^*
\]
given by

\[
\Phi \left[ C, r \right] (P) = C + \frac{r^2}{\|P - C\|^2} (P - C).
\]

**Definition 3.2.** Let \( \Phi \left[ C, r \right] \) be an inversion with the center \( C \) and the radius \( r \). Then, the tangent map of \( \Phi \) at \( P \in (\mathfrak{Sol}^3)^* \) is the map

\[
\Phi_{*p} = T_p \left( (\mathfrak{Sol}^3)^* \right) \rightarrow T_{\Phi(p)} \left( (\mathfrak{Sol}^3)^* \right)
\]
given by

\[
\Phi_{*p} (v_p) = \frac{r^2 v_p}{\|P - C\|^2} - \frac{2r^2 \langle (P - C), v_P \rangle}{\|P - C\|^4} (P - C),
\]

where \( v_p \in T_p \left( (\mathfrak{Sol}^3)^* \right) \), [1].

**Theorem 3.3.** Let \( \gamma : I \rightarrow \mathfrak{Sol}^3 \) be a unit speed non-geodesic general helix. Then, the equation inverse curve of \( \gamma \) is

\[
\tilde{\gamma} (s) = \left[ ae^c + \frac{r^2}{\Pi(s)} \left( \frac{\sin \mathcal{P}}{\mathcal{C}_1^2 + \cos^2 \mathcal{P}} \left[ -\cos \mathcal{P} \cos \mathcal{C}_1 s + \mathcal{C}_2 \right] + \mathcal{C}_1 \sin \left[ \mathcal{C}_1 s + \mathcal{C}_2 \right] + \mathcal{C}_1 e^{\cos \mathcal{P} s + \varepsilon_3 - ae^c} \right] e_1 \\
+ \left[ be^c + \frac{r^2}{\Pi(s)} \left( \frac{\sin \mathcal{P}}{\mathcal{C}_1^2 + \cos^2 \mathcal{P}} \left[ \mathcal{C}_1 \cos \mathcal{C}_1 s + \mathcal{C}_2 \right] + \cos \mathcal{P} \sin \left[ \mathcal{C}_1 s + \mathcal{C}_2 \right] + \mathcal{C}_1 e^{-\cos \mathcal{P} s - \varepsilon_1} - be^c \right] e_2 \\
+ \left[ c + \frac{r^2}{\Pi(s)} \left[ \cos \mathcal{P} s + \mathcal{C}_3 - c \right] \right] e_3, \tag{3.6}
\]
where \( e_1, e_2, e_3, e_4, e_5 \) are constants of integration and

\[
\Pi(s) = \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -\cos \mathcal{P} \cos [e_1 s + e_2] + e_1 \sin [e_1 s + e_2] + e_4 e^{-\cos \mathcal{P} s + e_5} - ae^c \right]^2 \\
+ \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -e_1 \cos [e_1 s + e_2] + \cos \mathcal{P} \sin [e_1 s + e_2] + e_5 e^{-\cos \mathcal{P} s + e_5} - be^{-c} \right]^2 \\
+ [\cos \mathcal{P} s + e_3 - c]^2 \right].
\]

**Proof.** Suppose that \( \gamma \) be a unit speed non-geodesic general helix.

Setting

\[ C = (a,b,c), \]

where \( a, b, c \in \mathbb{R} \).

Using above equation and (2.1), we have (3.6) as desired. This completes the proof.

**Theorem 3.4.** Let \( \gamma : I \rightarrow \mathfrak{so}(3) \) be a unit speed non-geodesic general helix. Then, the parametric equations of \( \gamma \) are

\[
x = e^{-[c + \frac{r^2}{\Pi(s)}][\cos \mathcal{P} s + e_3 - c]} \left[ ae^c + \frac{r^2}{\Pi(s)} \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -\cos \mathcal{P} \cos [e_1 s + e_2] + e_1 \sin [e_1 s + e_2] + e_4 e^{-\cos \mathcal{P} s + e_5} - ae^c \right]^2 \\
+ [\cos \mathcal{P} s + e_3 - c]^2 \right] \right],
\]

\[
y = e^{[c + \frac{r^2}{\Pi(s)}][\cos \mathcal{P} s + e_3 - c]} \left[ be^{-c} + \frac{r^2}{\Pi(s)} \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -e_1 \cos [e_1 s + e_2] + \cos \mathcal{P} \sin [e_1 s + e_2] + e_5 e^{-\cos \mathcal{P} s + e_5} - be^{-c} \right]^2 \\
+ [\cos \mathcal{P} s + e_3 - c]^2 \right] \right],
\]

\[
z = [c + \frac{r^2}{\Pi(s)}][\cos \mathcal{P} s + e_3 - c],
\]

where \( e_1, e_2, e_3, e_4, e_5 \) are constants of integration and

\[
\Pi(s) = \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -\cos \mathcal{P} \cos [e_1 s + e_2] + e_1 \sin [e_1 s + e_2] + e_4 e^{-\cos \mathcal{P} s + e_5} - ae^c \right]^2 \\
+ \left[ \frac{\sin \mathcal{P}}{\mathcal{P}^2 + \cos^2 \mathcal{P}} \left[ -e_1 \cos [e_1 s + e_2] + \cos \mathcal{P} \sin [e_1 s + e_2] + e_5 e^{-\cos \mathcal{P} s + e_5} - be^{-c} \right]^2 \\
+ [\cos \mathcal{P} s + e_3 - c]^2 \right].
\]

**Proof.** Using orthonormal basis in Theorem 3.3, we easily have above system.

Finally, the obtained parametric equations for Eqs. (3.4) and (3.7) is illustrated in Fig.2:
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Figure 2: Parametric equations for Eqs. (3.4) and (3.7)

References


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