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Construction Of Inverse Curves Of General Helices In The Sol Space $\mathfrak{Sol}^3$

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ABSTRACT: In this paper, we study inverse curves of general helices in the $\mathfrak{Sol}^3$. Finally, we find out explicit parametric equations of inverse curves in the $\mathfrak{Sol}^3$.

Key Words: General helix, Sol space, Curvature, Inverse curves.

Contents

1. Introduction

An inversion with respect to the sphere $S_C(r)$ with the center $C \in \mathfrak{Sol}^3$ is given by

$$C + \frac{r^2}{\|P - C\|^2}(P - C),$$

where r is radious, $P \in \mathfrak{Sol}^3$. The inversion is a conformal mapping and also is differentiable and a transformation defining between open subsets of $\mathfrak{Sol}^3$, [1,2,7].

A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802, [4].

In this paper, we study inverse curves of general helices in the $\mathfrak{Sol}^3$. Finally, we find out explicit parametric equations of inverse curves in the $\mathfrak{Sol}^3$.

2. Preliminaries

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{2.1}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i, \tag{2.2}$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z}dy, \quad \boldsymbol{\omega}^3 = dz, \tag{2.3}$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.4}$$

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Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

3. Inverse Curves of General Helices in Sol Space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ, [3,5,6,8]. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{N}) = 1,\ g_{\mathfrak{Sol}^3}(\mathbf{B},\mathbf{B}) = 1, \\ g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{B}) = g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4, \\ y(s) &= \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s+\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5, \\ z(s) &= \cos\mathfrak{P}s+\mathfrak{C}_3, \end{aligned} \tag{3.4}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration, [8].

We can use Mathematica in Theorem 3.1, yields

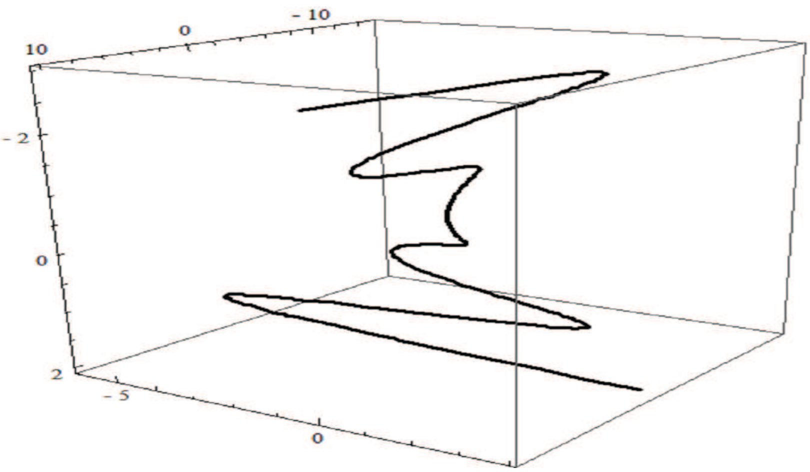

Figure 1: Parametric equation for $\gamma : I \longrightarrow \mathfrak{Sol}^3$

An inversion with respect to the sphere $S_C(r)$ with the center $C \in \mathfrak{Sol}^3$ is given by

$$C + \frac{r^2}{\|P - C\|^2}(P - C).$$

Let $C \in \mathfrak{Sol}^3$ and $r \in \mathbb{R}^+$. We denote that $\left(\mathfrak{Sol}^3\right)^* = \mathfrak{Sol}^3 - \{C\}$. Then, an inversion of $\mathfrak{Sol}^3$ with the center $C \in \mathfrak{Sol}^3$ and the radius r is the map

$$\Phi[C, r] : \left(\mathfrak{Sol}^3\right)^* \longrightarrow \left(\mathfrak{Sol}^3\right)^*$$

given by

$$\Phi[C, r](P) = C + \frac{r^2}{\|P - C\|^2}(P - C). \tag{3.5}$$

Definition 3.2. Let $\Phi[C, r]$ be an inversion with the center C and the radius r. Then, the tangent map of Φ at $P \in \left(\mathfrak{Sol}^3\right)^*$ is the map

$$\Phi_{*p} = T_p\left(\left(\mathfrak{Sol}^3\right)^*\right) \longrightarrow T_{\Phi(p)}\left(\left(\mathfrak{Sol}^3\right)^*\right)$$

given by

$$\Phi_{*p}(v_p) = \frac{r^2 v_p}{\|P - C\|^2} - \frac{2r^2 \langle (P - C), v_P \rangle}{\|P - C\|^4}(P - C),$$

where $v_p \in T_{\ p}\left(\left(\mathfrak{Sol}^3\right)^*\right)$, [1].

Theorem 3.3. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the equation inverse curve of γ is*

$$\begin{aligned} \tilde{\gamma}(s) \quad = \quad & [ae^c + \frac{r^2}{\Pi(s)}[\frac{\sin \mathfrak{P}}{\mathfrak{C}_1^2 + \cos^2 \mathfrak{P}}[-\cos \mathfrak{P} \cos[\mathfrak{C}_1 s + \mathfrak{C}_2] \\ & + \mathfrak{C}_1 \sin[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_4 e^{\cos \mathfrak{P} s + \mathfrak{C}_3} - ae^c]]\mathbf{e}_1 \\ & + [be^{-c} + \frac{r^2}{\Pi(s)}[\frac{\sin \mathfrak{P}}{\mathfrak{C}_1^2 + \cos^2 \mathfrak{P}}[-\mathfrak{C}_1 \cos[\mathfrak{C}_1 s + \mathfrak{C}_2] \\ & + \cos \mathfrak{P} \sin[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_5 e^{-\cos \mathfrak{P} s - \mathfrak{C}_3}] - be^{-c}]\mathbf{e}_2 \\ & + [c + \frac{r^2}{\Pi(s)}[[\cos \mathfrak{P} s + \mathfrak{C}_3] - c]]\mathbf{e}_3, \end{aligned} \tag{3.6}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ *are constants of integration and*

$$\begin{aligned}\Pi(s) &= [\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}-ae^c]^2\\ &+[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}-be^{-c}]^2\\ &+[\cos\mathfrak{P}s+\mathfrak{C}_3-c]^2.\end{aligned}$$

Proof. Suppose that γ be a unit speed non-geodesic general helix.

Setting

$$C=(a,b,c),$$

where $a,b,c\in\mathbb{R}$.

Using above equation and (2.1), we have (3.6) as desired. This completes the proof.

Theorem 3.4. *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *be a unit speed non-geodesic general helix. Then, the parametric equations of* γ *are*

$$\begin{aligned}x &= e^{-[c+\frac{r^2}{\Pi(s)}[[\cos\mathfrak{P}s+\mathfrak{C}_3]-c]]}[ae^c+\frac{r^2}{\Pi(s)}[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\\ &+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}-ae^c]], \qquad (3.7)\\ y &= e^{[c+\frac{r^2}{\Pi(s)}[[\cos\mathfrak{P}s+\mathfrak{C}_3]-c]]}[be^{-c}+\frac{r^2}{\Pi(s)}[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\\ &+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}]-be^{-c}],\\ z &= [c+\frac{r^2}{\Pi(s)}[[\cos\mathfrak{P}s+\mathfrak{C}_3]-c]],\end{aligned}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ *are constants of integration and*

$$\begin{aligned}\Pi(s) &= [\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}-ae^c]^2\\ &+[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}-be^{-c}]^2\\ &+[\cos\mathfrak{P}s+\mathfrak{C}_3-c]^2.\end{aligned}$$

Proof. Using orthonormal basis in Theorem 3.3, we easily have above system.

Finally, the obtained parametric equations for Eqs. (3.4) and (3.7) is illustrated in Fig.2:

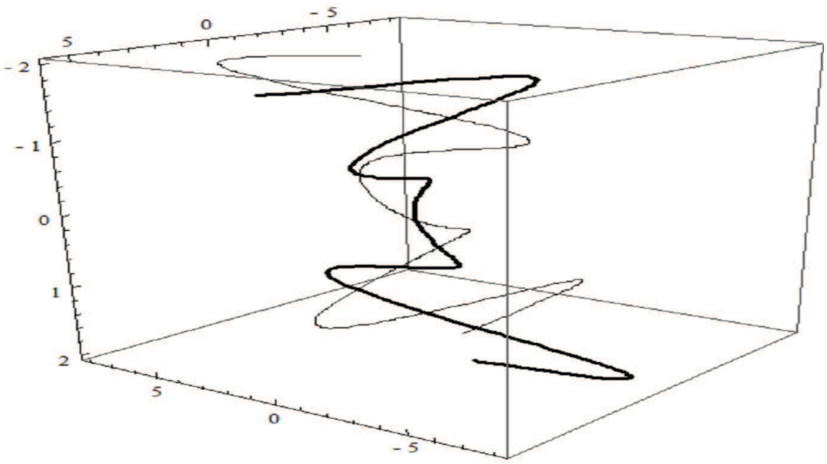

Figure 2: Parametric equations for Eqs. (3.4) and (3.7)

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