Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–13.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.44010

Numerical Solution of Time-fractional Telegraph Equation by Using a New Class of Orthogonal Polynomials

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ABSTRACT: In this article, an efficient numerical method based on a new class of orthogonal polynomials, namely Chelyshkov polynomials, has been presented to approximate solution of time-fractional telegraph (TFT) equations. The fractional operational matrix of the Chelyshkov polynomials along with the typical collocation method is used to reduces TFT equations to a system of algebraic equations. The error analysis of the proposed collocation method is also investigated. A comparison with other published results confirms that the presented Chelyshkov collocation approach is efficient and accurate for solving TFT equations. Illustrative examples are included to demonstrate the efficiency of the Chelyshkov method.

Key Words: Time-fractional telegraph equation, Chelyshkov polynomials, Collocation method, Error analysis.

Contents

1. Introduction

In order to describe wave propagation of electric signals in a cable transmission line and also in wave phenomena, Oliver Heaviside developed the telegraph equation in 1880 [1,2]. This kind of hyperbolic partial differential equations have been frequently used to model the reaction-diffusion systems in the biological and engineering fields [3,4,5]. Recently, various numerical scheme such as splines radial basis function [6], Chebyshev Tau method [7], Legendre multiwavelet Galerkin method [8], homotopy perturbation method [9], Chebyshev spectral collocation method [10], differential quadrature method [11], B-spline collocation method [12], Haar wavelet method [13], Bessel functions [14] and dual reciprocity boundary integral equation method [15] have been applied to solve telegraph equation.

Fractional calculus, as an extension of the classical derivatives and integrals to non-integer orders, has been frequently used to model many fundamental problems in various branches of sciences and engineering [16,18,17]. More recently, it has been found that fractional operators are more suitable for modelling phenomena in sciences and engineering. By replacing the time derivative term by fractional derivative, we can obtain the time fractional telegraph (TFT) equation from the classical telegraph equation. The TFT equation of order α can be defined as:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}+u(x,t)=f(x,t),\ 1<\alpha\leq 2, \tag{1.1}$$

2010 *Mathematics Subject Classification:* 35R11, 33C47, 65M70.
Submitted August 06, 2018. Published January 03, 2019

Typeset by $\mathcal{B}^{\mathcal{S}}\mathcal{P}_{\mathcal{M}}$ style.
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subject to the following conditions:

$$u(x,0)=l_1(x),\ u(x,1)=l_2(x), \tag{1.2}$$

$$u(0,t)=g_1(t),\ u(1,t)=g_2(t), \tag{1.3}$$

where l_i and g_i are two times continuously differentiable functions on $[0,1]$, the function f is given, $1<\alpha\leq 2$ is a real number and $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the fractional derivative of order α in the Caputo sense as follow [16]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\frac{\partial^n u(x,t)}{\partial t^n}, & \alpha=n\in\{0,1,2,...\},\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\partial^n u(x,\tau)}{\partial \tau^n}(t-\tau)^{n-\alpha-1}dt,\ t>0, & 0<n-1<\alpha<n.\end{cases}$$

Numerical solution of TFT equation have been investigated by many authors. Mollahasani et. al considered the TFT equations and used hybrid Legendre functions to approximate their solutions [1]. In order to solve two-dimensional fractional telegraph equation a spectral meshless radial point interpolation method was proposed in [19]. Bhrawy et al. proposed a Chebyshev Tau method for numerical solution of the two-sided fractional-order telegraph equation [20]. A computational Tau method based on the Legendre polynomials has been proposed to solve TFT equations by Saadatmandi and Mohabbati [21]. Suleman et. al [22] have been used a new projected differential transform method for space and time fractional telegraph equations. In [23] the method of separation of variables has been applied for deriving the analytical solutions of TFT equations with different kind of boundary conditions. Sweilam et. al [24] considered the Sinc-Legendre collocation method for solving TFT equations. Fourier and Laplace transforms have been also applied to derive the analytical solutions of TFT equations [2,25,26]. Heydari et. al used an efficient Legendre wavelets method for numerical solution of TFT equations [27]. Moreover, semi-analytical methods have been imployed by the researchers in [28,30,29,31] for solving TFT equations.

In the last decade, considerable attention was paid to the application of orthogonal polynomials in the solution of fractional differential and integral equations. Numerical method based on Legendre [33,21], Chebyshev [32,7,10], second kind of Chebyshev [34] and Jacobi [35,36] polynomials were proposed. The Chelyshkov polynomials are one of the newest type of orthogonal polynomials that were introduced in 2005 by Vladimir S. Chelyshkov [37]. In spite of their different structure and formulation, the Chelyshkov polynomials have some features similar to the classical orthogonal polynomials. Indeed, they can be connected to the Jacobi polynomial, hypergeometric functions and orthogonal exponential polynomials. The main idea of this work is to present a Chelyshkov polynomials collocation method to approximate solution of TFT equations (1.1) with initial and boundary conditions (1.2)-(1.3). A comparison with other published results confirms that the presented Chelyshkov polynomials approach is efficient and accurate for solving TFT equations.

The rest of this paper is structured as follows: Section 2 deals with some basic definitions and properties of the Chelyshkov orthogonal polynomials. In Section 3, a collocation scheme based on the Chelyshkov orthogonal polynomials has been proposed to solve TFT equations. Section 4 is devoted to a process for estimating the error function of proposed method. In Section 5, various illustrative examples are considered to confirm accuracy of the Chelyshkov polynomials method. Finally, concluding remarks are given in Section 6.

2. Basic definition of Chelyshkov polynomials

Important classes of orthogonal polynomials are the Chelyshkov polynomials that introduced by Veladmir. S. Chelyshkov [37,38]. In this section, we are going to introduce the Chelyshkov orthogonal polynomials breifly.

2.1. Definition of Chelyshkov polynomials

The Chelyshkov polynomials are expliclty defined as:

$$\rho_{n,N}(x) = \sum_{i=n}^{N} a_{i,n} x^i, \;\; n = 0, 1, ...N, \tag{2.1}$$

where

$$a_{i,n} = (-1)^{i-n} \left(\begin{array}{c} N-n \\ i-n \end{array} \right) \left(\begin{array}{c} N+i+1 \\ N-n \end{array} \right). \tag{2.2}$$

These polynomials are orthogonal in the interval $[0, 1]$ and their orthogonality condition is

$$\int_0^1 \rho_{n,N}(x)\rho_{m,N}(x)dx = \frac{\delta_{mn}}{m+n+1},$$

where δ_{mn} is the Kronecker delta. Moreover, the Rodrigues' formula for the the Chelyshkov polynomial $\rho_{n,N}(x)$ may be expressed as follow:

$$\rho_{n,N}(x) = \frac{1}{(N-n)!} \frac{1}{x^{N-n}} \frac{d^{N-n}}{dx^{N-n}} (x^{N+n+1}(1-x)^{N-n}), \;\; n = 0, ..., N.$$

2.2. Function approximation

A function $u(x)$ defined over $[0, 1]$ may be approximated by the Chelyshkov polynomials as

$$f(x) \simeq \sum_{k=0}^{N} c_k \rho_{k,N}(x) = C^T \Theta(x), \tag{2.3}$$

where the coefficient c_k can be derived as follows:

$$c_k = \frac{\langle u, \rho_{k,N} \rangle}{\langle \rho_{k,N}, \rho_{k,N} \rangle}, \;\; k = 0, 1, ..., N, \tag{2.4}$$

and $\langle ., . \rangle$ denotes the inner product on $L^2[0, 1]$ which can be defined as follow:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx. \tag{2.5}$$

Moreover, the vectors C and $\Theta(x)$ in the relation (2.3) are given by

$$C = [c_0, c_1, ..., c_N], \tag{2.6}$$

$$\Theta(x) = [\rho_{0,N}(x), \rho_{1,N}(x), ..., \rho_{N,N}(x)]^T. \tag{2.7}$$

Any multivariate function $u(x, t)$ on $[0, 1] \times [0, 1]$ can be also expressed in terms of the Chelyshkov polynomials as:

$$u(x,t) \simeq \sum_{i=0}^{N} \sum_{j=0}^{N} u_{ij} \rho_{i,N}(x) \rho_{j,N}(t) = \Theta^T(x) U \Theta(t),$$

in which U is a $(N+1)$ square matrix and its (i, j)th element, i.e $u_{i,j}$, can be obtained as follow:

$$u_{i,j} = \frac{\langle \rho_{j,N}, \langle u, \rho_{N,i} \rangle \rangle}{\langle \rho_{i,N}, \rho_{i,N} \rangle \langle \rho_{j,N}, \rho_{j,N} \rangle}, \;\; i, j = 0, 1, ..., N. \tag{2.8}$$

2.3. Integration and fractional differentiation

The major aim of this section is to derive an explicit formula for the integration and fractional differentiation of the Chelyshkov polynomials. In the next two theorems the analytical form of these polynomials will be used to obtain the integral and Caputo fractional derivative of the Chelyshkov basis vector $\Theta(x)$.

Theorem 2.1. *For any Chelshkov polynomial vector $\Theta(x)$, the Caputo fractional derivative of order α for the vector can be derived as follows:*

$$D^{\alpha}\Theta(x) = \mathcal{D}^{(\alpha)}\Theta(x), \tag{2.9}$$

where $\mathcal{D}^{\alpha}$ is an $(N+1)$ square matrix and

$$\mathcal{D}^{(\alpha)}_{i,j} = \sum_{r=i}^{N}\sum_{s=j}^{N}\frac{a_{s,j}a_{r,i}\Gamma(r+1)(2j+1)}{\Gamma(r-\alpha+1)(r+s-\alpha+1)}, \quad i,j=1,2,...,N+1.$$

Proof: Consider the ith element of the vector $\Theta(x)$ i.e $\rho_{(i-1),N}(x)$. The the fractional derivative of order α for this function can be defined as

$$D^{\alpha}\rho_{(i-1),N}(x) = D^{\alpha}(\sum_{r=i-1}^{N}a_{r,i-1}x^{r}),$$

therefore

$$D^{\alpha}\rho_{(i-1),N}(x) = \sum_{r=i-1}^{N}\frac{a_{r,i-1}\Gamma(r+1)}{\Gamma(r-\alpha+1)}x^{r-\alpha}. \tag{2.10}$$

Now the term $x^{r-\alpha}$ can be approximated as

$$x^{r-\alpha} \simeq \sum_{j=0}^{N}u_{r,j}\rho_{j,N}(x), \tag{2.11}$$

where

$$u_{r,j} \simeq (2j+1)\int_0^1 x^{r-\alpha}\rho_{j,N}(x)dx = (2j+1)\int_0^1 x^{r-\alpha}(\sum_{s=j}^{N}a_{s,j}x^{s})dx \tag{2.12}$$

$$= (2j+1)\sum_{s=j}^{N}a_{s,j}\int_0^1 x^{r+s-\alpha}dx = \sum_{s=j}^{N}\frac{(2j+1)a_{s,j}}{r+s-\alpha+1}.$$

By putting Eqs. (2.11) and (2.12) in (2.10), we have:

$$D^{\alpha}\rho_{(i-1),N}(x)(x) = \sum_{j=0}^{N}\left(\sum_{r=i-1}^{N}\sum_{s=j}^{N}\frac{a_{s,j}a_{r,i-1}(2j+1)\Gamma(r+1)}{(r+s-\alpha+1)\Gamma(r-\alpha+1)}\right)\rho_{j,N}(x),$$

and this yields the desired result in Eq. (2.9).
Now, by a similar process, we can derive integration of the Chelyshkov polynomials vector.

Theorem 2.2. *If $\Theta(x)$ is the Chelshkov polynomial vectors. Then the integration of this vector is given by*

$$\int_0^x \Theta(t)dt = \mathcal{P}\Theta(x), \tag{2.13}$$

where $\mathcal{P}$ is an $(N+1)$ square matrix and

$$\mathcal{P}_{i,j} = \sum_{r=i}^{N}\sum_{s=j}^{N}\frac{a^{N}_{s,j}a^{N}_{r,i}(2j+1)}{(r+s+2)(r+1)}, \quad i,j=1,2,...,N.$$

Proof Consider $\rho_{(i-1),N}(x)$, the ith element of the vector $\Theta(x)$. The integration of this function can be derived as follows

$$\int_0^x \rho_{(i-1),N}(t)dt = \sum_{r=i-1}^{N} a_{r,i-1}\int_0^x t^r dt = \sum_{r=i-1}^{N} \frac{a_{r,i-1}}{r+1}x^{r+1} = \sum_{r=i-1}^{N} \frac{a_{r,i-1}}{(r+1)}x^{r+1}, \tag{2.14}$$

by expanding x^{r+1} by the Chelyshkov polynomials, we get

$$x^{r+1} \simeq \sum_{j=0}^{N} u_{r,j}\rho_{j,N}(x), \tag{2.15}$$

where $u_{r,j}$ can be derived as

$$u_{r,j} \simeq (2j+1)\int_0^1 x^{r+1}\rho_{j,N}(x)dx = (2j+1)\int_0^1 x^{r+1}(\sum_{s=j}^{N} a_{s,j}x^s)dx \tag{2.16}$$

$$= (2j+1)\sum_{s=j}^{N} a_{s,j}\int_0^1 x^{r+s+1}dx = \sum_{s=j}^{N}\frac{(2j+1)a_{s,j}}{r+s+2}.$$

Now, by substituting Eqs. (2.15) and (2.16) in (2.14) we obtain:

$$\int_0^x \rho_{(i-1),N}(t)dt = \sum_{j=0}^{N}\left(\sum_{r=i-1}^{N}\sum_{s=j}^{N}\frac{a_{s,j}a_{r,i-1}(2j+1)}{(r+s+2)(r+1)}\right)\rho_{j,N}(x),$$

and this proves the desired result.

2.4. Convergence analysis

In this section, by approximating the function $u(x)$, we state and prove a convergence theorem and find the error bound of Chelyshkov polynomials expansion. Consider the truncated Chelyshkov polynomials series for the function $u(x)$ as

$$u_N(x) \simeq \sum_{i=0}^{N} c_i\rho_{i,N}(x).$$

The error function of this truncated series is defined as follow

$$E_N(x) = |u(x) - u_N(x)|.$$

Moreover, we define the following norm

$$\|u\|^2 = \langle u(x), u(x)\rangle = \int_0^1 |u(x)|^2 dx, \tag{2.17}$$

on $L^2[0,1]$, the space of all functions $u(x)$ defined on $[0,1]$ such that

$$\int_0^1 |u(x)|^2 dx < \infty.$$

Theorem 2.3. *Suppose that $u(x)$ is a real-valued function, defined and continuous on the interval $[0,1]$, and such that the derivative of $u(x)$ of order $N+1$ is continuous on $[0,1]$. Let $u_N(x)$ denote the truncated*

Chelyshkov polynomials series $u(x)$. Then, the mean error bound for this truncated series can be derived as follows

$$\|E_N\| \leq \frac{L_{N+1}}{2^{(2N+1)}\,(N+1)!},$$

in which

$$L_{N+1} = \max_{x\in[0,1]} \left|u^{(N+1)}(x)\right|.$$

Proof: The truncated Chelyshkov polynomials series $u_N(x)$ is the best approximation of $u(x)$ by a polynomial of degree less than N with respect to the norm $\|.\|$ defined in (2.17). Therefore, if $Q_N(x)$ be the well-known polynomial which interpolates $u(x)$ at shifted zeros of Chebyshev polynomials $T_{N+1}(x)$ in the interval $[0,1]$, it is convenient to write

$$\|E_N\|^2 = \|u-u_N\|^2 \leq \|u-Q_N\|^2 = \int_0^1 |u(x)-Q_N(x)|^2 dx,$$

Now, by employing the error bound for the interpolation polynomial $Q_n(t)$ (Theorem 8.7 in [39]) in the above relation, we have

$$\|E_N\|^2 \leq \|u-Q_N\|^2 = \int_0^1 |u(x)-Q_N(x)|^2 dx \leq \frac{L_{N+1}^2}{2^{2(2N+1)}\,(N+1)!^2}$$

which completes the proof.

3. Description of the numerical method

The purpose of this section is to solve the TFT equations using the fractional operational matrices of the Chelyshkov polynomials and the collocation method. To this end, consider the TFT equation (1.1) subject to the initial and boundry conditions (1.2) and (1.3). First, we expand the functions $\frac{\partial^2 u(x,t)}{\partial x^2}$, $f(x,t)$, $g_1(t)$ and $g_2(t)$ via the Chelyshkov basis as:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \Psi^T(x)U\Psi(t), \tag{3.1}$$

and

$$f(x,t) = \Psi^T(x)F\Psi(t),\ g_1(t) = G_1\Psi(t),\ g_2(t) = G_2\Psi(t), \tag{3.2}$$

where U is an $(N+1)\times(N+1)$ unknown coefficient matrix as defined in (2.8), F is the coefficient matrix of the function $f(x,t)$ and $G_i,\ i=1,2$ are the $(N+1)$ Chelyshkov coefficient vectors for the functions $g_i(t), i=1,2$. Then, by integrating with respect to x in relation (3.1) and using the integration matrix defined in (2.13), we have:

$$\frac{\partial u(x,t)}{\partial x} = \Psi^T(x)\mathcal{P}^T U\Psi(t) + \frac{\partial u(0,t)}{\partial x}, \tag{3.3}$$

$$u(x,t) = \Psi^T(x)(\mathcal{P}^T)^2 U\Psi(t) + x\frac{\partial u(0,t)}{\partial x} + G_1\Psi(t), \tag{3.4}$$

Now, by putting $x=1$ in Eq. (3.4), we have:

$$\frac{\partial u(0,t)}{\partial x} = G_2\Psi(t) - \Psi^T(1)(\mathcal{P}^T)^2 U\Psi(t) - G_1\Psi(t), \tag{3.5}$$

Substituting Eq. (3.5) into (3.4), we obtain:

$$\begin{aligned} u(x,t) &= \Psi^T(x)(\mathcal{P}^T)^2 U\Psi(t) - x\Psi^T(1)(\mathcal{P}^T)^2 U\Psi(t) \\ &\quad +(1-x)G_1\Psi(t) + xG_2\Psi(t). \end{aligned} \tag{3.6}$$

Also, by fractional differentiation of order α and $\alpha-1$ with respect to variable t in Eq. (3.6), considering fractional operational matrix $\mathcal{D}$, we derive

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \Psi^T(x)(\mathcal{P}^T)^2 U\mathcal{D}^\alpha\Psi(t) - x\Psi^T(1)(\mathcal{P}^T)^2 U\mathcal{D}^\alpha\Psi(t) \\ &\quad +(1-x)G_1\mathcal{D}^\alpha\Psi(t) + xG_2\mathcal{D}^\alpha\Psi(t), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} &= \Psi^T(x)(\mathcal{P}^T)^2 U\mathcal{D}^{\alpha-1}\Psi(t) - x\Psi^T(1)(\mathcal{P}^T)^2 U\mathcal{D}^{\alpha-1}\Psi(t) \\ &\quad +(1-x)G_1\mathcal{D}^{\alpha-1}\Psi(t) + xG_2\mathcal{D}^{\alpha-1}\Psi(t), \end{aligned} \tag{3.8}$$

Substituing Eqs. (3.1), (3.2) and (3.6)-(3.8) in Eq. (1.1), we have the following residual function as:

$$\begin{aligned} R(x,t) &= \Psi^T(x)\left[(\mathcal{P}^T)^2 U\mathcal{D}^\alpha + (\mathcal{P}^T)^2 U\mathcal{D}^{\alpha-1} + (\mathcal{P}^T)^2 U - U - F\right]\Psi(t) \\ &\quad -x\Psi^T(1)\left[(\mathcal{P}^T)^2 U\mathcal{D}^\alpha + (\mathcal{P}^T)^2 U\mathcal{D}^{\alpha-1}\right]\Psi(t) + (1-x)G_1\left(\mathcal{D}^\alpha + \mathcal{D}^{\alpha-1}\right)\Psi(t) \\ &\quad +xG_2\left(\mathcal{D}^\alpha + \mathcal{D}^{\alpha-1}\right)\Psi(t). \end{aligned} \tag{3.9}$$

To obtain the $(N+1)\times(N+1)$ unknown matrix U, we collocate the residual $R(x,t)$ at the $N+1$ zeros of shifted Chebyshev polynomials as follow:

$$R(x_i,t_j) = 0, i = 1,2,...N+1, j = 1,2,...,N-1. \tag{3.10}$$

This gives N^2-1 algebraic equations. Moreover, by taking collocation points x_i, for the initial and boundary conditions (1.2) we have $2(N+1)$ algebraic equations as:

$$u(x_i,0) - l_1(x_i) = 0,\ i = 1,2,...,N+1, \tag{3.11}$$

$$u(x_i,1) - l_2(x_i) = 0,\ i = 1,2,...,N+1, \tag{3.12}$$

Equations (3.11) and (3.12) together with (3.10) result in a system of $(N+1)^2$ for the unknown matrix U. By solving this system and determining U, we get the numerical solution of the main problem by substituting U into (3.6).

4. Error analysis

Let us assume that the exact solution of (1.1) is $u(x,t)$ and the approximate solution derived by the Chelyshkov collocation method is $u_N(x,t)$. It is our aim to introduce a new procedure to approximate the error function of the proposed method, i.e. $e_N(x,t) = u_N(x,t) - u(x,t)$. Due to the fact that $u_N(x,t)$ is supposed to be an approximate solution of TFT equation (1.1), it does satisfy that the following problem:

$$\frac{\partial^\alpha}{\partial t^\alpha}u_N(x,t) + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}u_N(x,t) + u_N(x,t) - \frac{\partial^2}{\partial x^2}u_N(x,t) = f(x,t) + R_N(x,t), \tag{4.1}$$

where the perturbation term $R_N(x,t)$ can be obtained by substituting the estimated solution $u_N(x,t)$ into TFT equation (1.1) as follow:

$$R_N(x,t) = \frac{\partial^\alpha}{\partial t^\alpha}u_N(x,t) + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}u_N(x,t) + u_N(x,t) - \frac{\partial^2}{\partial x^2}u_N(x,t) - f(x,t). \tag{4.2}$$

Subtracting Eq.(4.1) from (1.1), we get the following equation:

$$\frac{\partial^\alpha}{\partial t^\alpha}e_N(x,t) + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}e_N(x,t) + e_N(x,t) - \frac{\partial^2}{\partial x^2}e_N(x,t) = -R_N(x,t).$$

Obviously the above equation is a TFT equation in which the error function $e_N(x,t)$, is the unknown function. We can easily apply our proposed method to solve this equation to find an approximation of the error function $e_N(x,t)$.

5. Numerical Results and Discussion

In this section, we demonstrate the performance of the proposed Chelyshkov collocation method for solving TFT equation. Several numerical examples are given to illustrate the properties of the present method. All results are computed by using MAPLE 17 and MATLAB R2010a. To show the efficiency of the present method, we report the root mean square error as:

$$\|e_N\|_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}|e_N(x_i,t_i)|^2}.$$

Example 5.1. *Consider the following TFT equation:*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} + u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = x^2 + t - 1,$$

with the initial and boundary conditions:

$$u(x,0) = x^2,\ u(x,1) = x^2 + 1,\ u(0,t) = t,\ u(1,t) = t + 1.$$

We see that, for $\alpha = 2$, $u(x,t) = x^2 + t$ is the exact solution of the problem. The numerical solutions obtained by the proposed Chelyshkov collocation method and its absolute error for $\alpha = 2$ and $N = 10$ are shown in Fig. 1. The approximate solutions for different values of α and t with $N = 10$ are listed in Table 1. From the results in this Table it is possible to see that the Chelyshkov collocation method is efficient for solving this TFT equation and the numerical solution converges to exact solution as α tends to 2. In order to demonstrate the efficiency of our proposed method, we compare the results to other existing methods. Table 2 presents the mean square error of the achieved results for some values of t and N with a comparison to presented methods in Refs. [1,40]. Moreover, the mean square of $R_N(x,t)$ for different values of α are presented in Table 3. From these Tables we can observe that the Chelyshkov collocation method is more efficient and accurate for solving this TFT equation.

Table 1: The numerical results for different choices of t and α (Example 5.1).

(x,t)	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$	$\alpha = 2.0$
$(0.2, 0.2)$	0.28064689	0.26411951	0.25678350	0.24939991	0.24581595
$(0.4, 0.4)$	0.60444716	0.58653283	0.58035115	0.57528088	0.57329439
$(0.6, 0.6)$	0.98897368	0.97885865	0.97665831	0.97595363	0.97627808
$(0.8, 0.8)$	1.54941319	1.54907370	1.54948758	1.55032077	1.55092799
$(1.0, 1.0)$	2.00000004	1.00000002	2.00000001	2.00000000	2.00000000

Table 2: The mean square error for $\alpha = 2$ and different choices of t (Example 5.1).

	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$N = 10$	1.4960×10^{-16}	2.2819×10^{-14}	4.6867×10^{-13}	4.3187×10^{-12}	2.4030×10^{-11}
Ref [1] $m = 10$	1.4312×10^{-4}	2.8533×10^{-4}	–	1.1572×10^{-4}	–
Ref [40] $m = 12$	8.6400×10^{-4}	8.0600×10^{-4}	–	7.5000×10^{-4}	–

Table 3: The mean square of $R_N(x,t)$ for different choices of α (Example 5.1).

α	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$
$\|R_N\|_2$	6.4×10^{-9}	1.2×10^{-9}	3.7×10^{-10}	7.9×10^{-11}

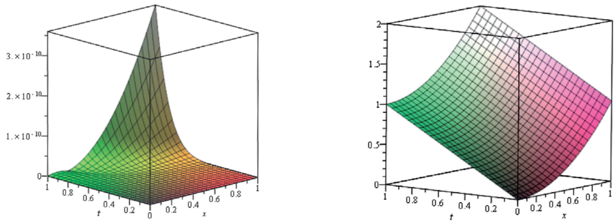

Figure 1: The absolute error (Left) and approximate solution (Right) for $\alpha = 2$ and $N = 10$ (Example 5.1).

Example 5.2. *Consider the following TFT equation:*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} + u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,$$

with the initial and boundary conditions:

$$u(x,0) = e^x, \; u(x,1) = e^{x-1}, \; u(0,t) = e^{-t}, \; u(1,t) = e^{1-t}.$$

We see that, for $\alpha = 2$, $u(x,t) = e^{x-t}$ is the exact solution of the problem. The approximate solution and its absolute error for this TFT equation with $\alpha = 2$ and $N = 10$ are plotted in Fig. 2. Moreover, the numerical solutions derived by the proposed Chelyshkov collocation method for different values of α and t with $N = 10$ are listed in Table 4. The mean square error for $\alpha = 2$ and different values of t and N are presented in Table 5. Furthermore, the mean square of $R_N(x,t)$ for different values of α are presented in Table 6. From these Tables we can see that the Chelyshkov collocation method is efficient for solving the TFT equation and the numerical solution converges to the exact solution as the fractional order α approaches 2.

Table 4: The numerical results for different choices of t and α (Example 5.2).

	(x,t)	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$	$\alpha = 2.0$
	$(0.2, 0.2)$	0.99044221	0.97960466	0.97281827	0.96403646	0.95877914
	$(0.4, 0.4)$	1.01055328	1.02363321	1.03286365	1.04499639	1.05222872
v	$(0.6, 0.6)$	1.03414801	1.03152311	1.03329187	1.03903915	1.04419474
	$(0.8, 0.8)$	1.07909885	1.07314580	1.07071861	1.06857003	1.06762792
	$(1.0, 1.0)$	1.00000004	0.99999999	0.99999999	1.00000000	1.00000000

Table 5: The mean square error for different choices of t and N. (Example 5.2)

t	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$N = 8$	4.2964×10^{-9}	2.8407×10^{-9}	8.3189×10^{-10}	6.4244×10^{-9}	8.3837×10^{-9}
$N = 10$	2.2449×10^{-11}	2.1644×10^{-11}	1.0071×10^{-12}	1.5993×10^{-11}	1.4707×10^{-11}

Table 6: The mean square of $R_N(x,t)$ for different choices of α (Example 5.2).

α	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$
$\|R_N\|_2$	6.3×10^{-10}	7.3×10^{-9}	9.1×10^{-10}	8.2×10^{-10}

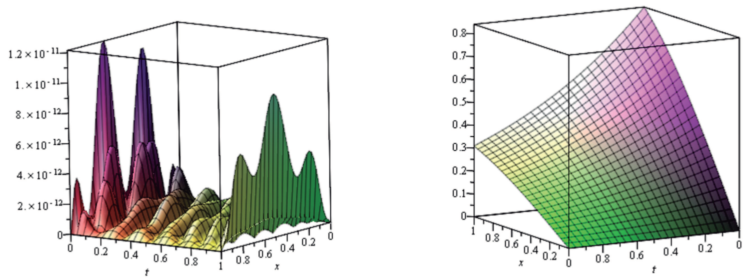

Figure 2: The absolute error (Left) and approximate solution (Right) for $\alpha = 2$ and $N = 10$ (Example 5.2).

Example 5.3. *Consider the following TFT equation:*

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} + u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,$$

with the initial and boundary conditions:

$$u(x,0) = \sin(x),\ \ u(x,1) = \sin(x)e^{-1},\ \ u(0,t) = 0,\ \ u(1,t) = \sin(1)e^{-t}.$$

We see that, for $\alpha = 2$, $u(x,t) = sin(x)e^{-t}$ is the exact solution of the problem. The numerical solution and its absolute error for $\alpha = 2$ and $N = 10$ are plotted in Fig. 3. Moreover, the numerical solution derived by the proposed Chelyshkov collocation method for different values of α and t with $N = 10$ are presented in Table 7. Table 8 presents the mean square error of the obtained numerical solution for $\alpha = 2$ and different values of t and N. The mean square of $R_N(x,t)$ for different values of α are also presented in Table 6. From these results we see the Chelyshkov collocation method is efficient for solving this TFT equation and the numerical solution converges to the exact solution as α gets close to 2.

Table 7: The numerical results for different values of t and α (Example 5.3).

(x,t)	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$	$\alpha = 2.0$
$(0.2, 0.2)$	0.16096692	0.15987705	0.15908018	0.15791606	0.15729048
$(0.4, 0.4)$	0.26141821	0.26083154	0.26062439	0.26003810	0.25550755
$(0.6, 0.6)$	0.30151890	0.30265641	0.30395617	0.30495458	0.31274605
$(0.8, 0.8)$	0.29781447	0.29664391	0.29569637	0.29464543	0.29586048
$(1.0, 1.0)$	0.30955983	0.30955985	0.30955985	0.30955987	0.30955987

Table 8: The mean square error for different choices of t and N (Example 5.3).

t	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$N = 8$	1.7874×10^{-8}	3.4375×10^{-9}	1.8324×10^{-9}	4.2809×10^{-9}	6.9250×10^{-8}
$N = 10$	3.0209×10^{-9}	1.0438×10^{-10}	2.6151×10^{-11}	1.1739×10^{-10}	7.7683×10^{-9}

Table 9: The mean square of $R_N(x,t)$ for different choices of α (Example 5.3).

α	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$
$\Vert R_N \Vert_2$	1.1×10^{-9}	1.3×10^{-8}	6.7×10^{-9}	2.0×10^{-9}

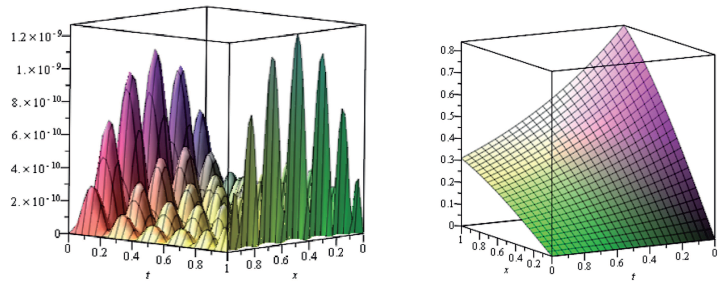

Figure 3: The absolute error (Left) and approximate solution (Right) for $\alpha = 2$ and $N = 10$ (Example 5.3).

Example 5.4. *Consider the following TFT equation:*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} + u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = (t^2 - 2t + 2)(x - x^2)exp(-t) + 2t^2 e^{-t},$$

with the initial and boundary conditions:

$$u(x,0) = 0, \; u(x,1) = (x - x^2)e^{-1}, \; u(0,t) = 0, \; u(1,t) = 0.$$

We see that, for $\alpha = 2$, $u(x,t) = (x - x^2)t^2 e^{-t}$ is the exact solution of the problem. Fig. 4 displays the approximate solution of this TFT equation and its absolute error for $\alpha = 2$ and $N = 8$. The numerical solution derived by the proposed Chelyshkov collocation method for different values of α and t with $N = 10$ are listed in Table 10. The mean square error for different values of t and N are also presented in Table 11. Moreover, the mean square of $R_N(x,t)$ for different values of α are also presented in Table 12. From these results we see the Chelyshkov collocation method is efficient for solving the TFT equation and the numerical solution converges to the exact solution as the fractional order α gets close to 2.

Table 10: The numerical results for different values of t and α (Example 5.4).

(x,t)	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$	$\alpha = 2.0$
$(0.2, 0.2)$	0.01085248	0.00779192	0.00657095	0.00546551	0.00499218
$(0.4, 0.4)$	0.02538437	0.02395859	0.02386314	0.02414798	0.02444074
$(0.6, 0.6)$	0.03768723	0.03836135	0.03910208	0.04015172	0.04080823
$(0.8, 0.8)$	0.00998294	0.01044287	0.01068900	0.01097418	0.01113382
$(1.0, 1.0)$	3.4×10^{-19}	1.8×10^{-18}	1.9×10^{-19}	8.7×10^{-19}	7.3×10^{-18}

Table 11: The mean square error for different choices of t and N. (Example 5.4)

t	$t = 0.1$	$t = 0.3$	$t = 0.6$	$t = 0.7$	$t = 0.8$
$N = 8$	1.3859×10^{-8}	1.3659×10^{-9}	6.6085×10^{-10}	8.7466×10^{-10}	1.3149×10^{-8}
$N = 10$	3.4432×10^{-11}	6.9154×10^{-11}	5.1166×10^{-13}	8.4484×10^{-11}	1.1516×10^{-11}
Ref [1] $m = 10$	–	–	2.1875×10^{-3}	1.4545×10^{-3}	1.6187×10^{-3}

Table 12: The mean square of $R_N(x,t)$ for different choices of α (Example 5.4).

α	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	$\alpha = 1.95$
$\|R_N\|_2$	9.3×10^{-10}	1.1×10^{-9}	8.6×10^{-8}	9.7×10^{-10}

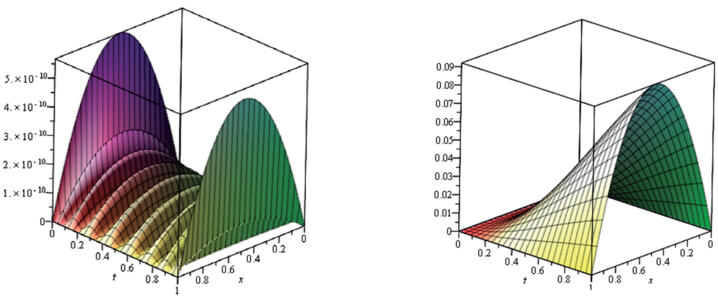

Figure 4: The absolute error (Left) and approximate solution (Right) for $\alpha = 2$ and $N = 10$ (Example 5.4).

6. Conclusion

An efficient numerical approach has been presented to solve time-fractional telegraph equations. The presented method is mainly based on the Chelyshkov polynomials, their fractional differentiation and integration and the typical collocation method. The main advantage of the presented method is that it reduces the time-fractional telegraph equations into a system of algebraic equations. A comparison between the achieved results and those available in the literatures confirms the good accuracy and superiority of the proposed method.

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