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Existence of Positive Solutions of Hadamard Fractional Differential Equations with Integral Boundary Conditions

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abstract: In this paper, we study the existence of positive solutions for Hadamard fractional differential equations with integral conditions. We employ Avery-Peterson fixed point theorem and properties of Green's function to show the existence of positive solutions of our problem. Furthermore, we present an example to illustrate our main result.

Key Words: Fractional differential equations, Hadamard fractional derivative, Avery-Peterson fixed point theorem.

Contents

1. Introduction

Fractional differential equations are found to be of central importance in many disciplines such as control theory, physics, neural networks, epidemiology, etc. Many researchers see [1,5,6,8,12,13,14] used different kinds of fixed point theorems (Krasnoselskii fixed-point theorem, Leray-Schauder fixed-point theorem, Leggett- Williams fixed-point theorem, Avery-Peterson fixed-point theorem) to deal with different equations. Ahmad and Nieto [2] investigated the fractional integro-differential equation with integral boundary conditions

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t, x(t), (\chi x)(t)), \quad t \in (0,1), \\ \alpha x(0) + \beta x'(0) = \int_0^1 q_1(x(s))ds, \\ \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s))ds, \end{cases}$$

where $1 < q \leq 2$, ${}^cD^q$ is the Caputo fractional derivative, and

$$(\chi x)(t) = \int_0^t \gamma(t,s)x(s)ds.$$

Wei, Pang and Ding [11] considered the nonnegative solutions for fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^cD^\alpha_{0+}x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ ax(0) - bx'(0) = 0, \ cx(1) + dx'(1) = 0, \\ x''(0) + x'''(0) = \int_0^1 x''(\tau)dp(\tau), \\ x''(1) + x'''(1) + \int_0^1 x''(\tau)dq(\tau) = 0, \end{cases}$$

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where $3 < \alpha \leq 4$ is a real number, $a, b, c, d \geq 0$, $\rho = ad + ac + bc > 0$. ${}^cD^\alpha$ is the Caputo fractional derivative and f satisfies some conditions.

Recently, Y. Wang, S.Liang and Q. Wang [9] investigated the existence of multiple positive solutions of fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^cD^\alpha_{0+}u(t) = f(t, u(t), u'(t)), \quad t \in [0, 1], \\ u''(0) = 0, \quad u^{(3)}(0) = 0, \\ u(0) + au'(0) = \int_0^1 g_1(s)u(s)ds, \\ u(1) - bu'(1) = \int_0^1 g_2(s)u(s)ds, \end{cases}$$

where $3 < \alpha < 4$, $g_1, g_2 \in C([0, 1], [0, +\infty))$, $a, b > 0$ and ${}^cD^\alpha$ denotes the Caputo fractional derivative. The problem is tackled by applying Avery-Peterson fixed point theorem and the properties of Green's function.

Motivated by these works, in this paper, we investigate the existence of multiple positive solutions of the problem

$$\begin{cases} D^\alpha u(t) = f(t, u(t), u'(t)), \quad t \in [1, T], \ T < e, \\ u''(1) = 0, \quad u^{(3)}(1) = 0, \\ u(1) + au'(1) = \int_1^T g_1(s)u(s)ds, \\ u(T) - bu'(T) = \int_1^T g_2(s)u(s)ds, \end{cases} \tag{1.1}$$

where $3 < \alpha < 4$, $g_1, \ g_2 \in C([1, T], [0, +\infty))$, $a, b > 0$ and D^α denotes the Hadamard derivative of fractional order α.

The following assumptions are needed for the sequel

(H_1) $b \geq a \geq 1 \geq \log T$,

(H_2) $f \in C([1, T] \times [0, \infty) \times (-\infty, +\infty), [0, +\infty))$,

(H_3) $g_1, g_2 \in C([1, T], [0, \infty)), 0 \leq \sigma_1 + \sigma_2 < 1, \rho = (1 - \sigma_1)(1 - \sigma_4) - \sigma_2\sigma_3 > 0$, where

$$\sigma_1 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_1(s)ds, \quad \sigma_2 = \int_1^T \frac{a - \log s}{a + \frac{b}{T} - \log T} g_1(s)ds,$$

$$\sigma_3 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_2(s)ds, \quad \sigma_4 = \int_1^T \frac{a - \log s}{a + \frac{b}{T} - \log T} g_2(s)ds.$$

The paper is organized as follows. In Section 2, we consider some definitions and lemmas. Section 3, is devoted to prove the existence of multiple positive solutions for problem(1.1). In Section 4, we provide an example to illustrate our main result.

2. Preliminaries

Definition 2.1. *(see [7]) The Hadamard fractional integral of order q for a continuous function g is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0.$$

Definition 2.2. *(see [7]) The Hadamard derivative of fractional order q for a C^{n-1} function $g : [1, \infty) \to \mathbb{R}$ is defined as*

$$D^q g(t) = \frac{1}{\Gamma(n - q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \quad n - 1 < q < n, \ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.3. *(see [7]) Let $q > 0$, then the fractional differential equation*

$$D^q u(t) = 0,$$

has the unique solution

$$u(t) = \sum_{k=0}^{[q]} \frac{u^{(k)}(1)}{k!}(\log t)^k.$$

Lemma 2.4. *(see [7]) Let $q > 0$, then*

$$I^q D^q u(t) = u(t) - \sum_{k=0}^{[q]} \frac{u^{(k)}(1)}{k!}(\log t)^k.$$

Now, we can find the exact expression of the Green's function associated to the fractional-order differential equation with nonlocal boundary value conditions:

$$\begin{cases} D^q u(t) = y(t), \qquad t \in [1, T] \quad \text{and} \quad 1 \le \mathrm{T} \le \mathrm{e}, \\ u''(1) = 0, \quad u^{(3)}(1) = 0, \\ u(1) + au'(1) = \displaystyle\int_1^T g_1(s)u(s)ds, \\ u(T) - bu'(T) = \displaystyle\int_1^T g_2(s)u(s)ds. \end{cases} \tag{2.1}$$

Lemma 2.5. *Let $3 < q < 4$. Assume $y \in C([1, T])$ and (H_1) holds, then the problem(2.1) has a solution $u(t)$ given by*

$$u(t) = \int_1^T G(t, s)y(s)\ ds + \int_1^T R(t, s) \int_1^T G(s, \tau)y(\tau)\ d\tau ds, \tag{2.2}$$

where

$$G(t, s) = \begin{cases} \dfrac{(a + \frac{b}{T} - \log T)(\log\frac{t}{s})^{q-1} + (\log t - a)(\log\frac{T}{s})^{q-1}}{(a + \frac{b}{T} - \log T)\Gamma(q)} \\ + \dfrac{b(q-1)(a - \log t)\log(\frac{T}{s})^{q-2}}{(a + \frac{b}{T} - \log T)\Gamma(q)} & 1 \le s \le t \le T, \\ \dfrac{(\log t - a)(\log\frac{T}{s})^{q-1} + b(q-1)(a - \log t)\log(\frac{T}{s})^{q-2}}{(a + \frac{b}{T} - \log T)\Gamma(q)} & 1 \le t \le s \le T, \end{cases} \tag{2.3}$$

and

$$\begin{aligned} R(t, s) \quad &= \quad \frac{[(a - \log t)\sigma_3 + (\log t + \frac{b}{T} - \log T)(1 - \sigma_4)]g_1(s)}{\rho(a + \frac{b}{T} - \log T)} \\ &\quad + \frac{[(a - \log t)(1 - \sigma_1) + (\log t + \frac{b}{T} - \log T)\sigma_2]g_2(s)}{\rho(a + \frac{b}{T} - \log T)}. \end{aligned} \tag{2.4}$$

Proof. Applying the result of lemma (2.4), we get the general solution of problem (2.1)

$$u(t) = \alpha_0 + \alpha_1 \log t + \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{T}{s}\right)^{(q-1)} \frac{y(s)}{s}\ ds, \tag{2.5}$$

where $\alpha_0, \alpha_1 \in \mathbb{R}$, are arbitrary constants. Using boundary conditions yields the system of equations:

$$\begin{cases} \alpha_0 + a\alpha_1 = & \displaystyle\int_1^T g_1(s)u(s)ds, \\ \alpha_0 + (\log T - \frac{b}{T})\alpha_1 = & \displaystyle\frac{b}{\Gamma(q-1)} \int_1^T \left(\log\frac{T}{s}\right)^{(q-2)} \frac{y(s)}{s}ds \\ & \displaystyle -\frac{1}{\Gamma(q)} \int_1^T \left(\log\frac{T}{s}\right)^{(q-1)} \frac{y(s)}{s}\ ds + \int_1^T g_2(s)u(s)ds. \end{cases} \tag{2.6}$$

Solving the system (2.6), we obtain

$$\begin{aligned}
\alpha_0 &= \frac{ab}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q-1)}\Big(\log\frac{T}{s}\Big)^{(q-2)}\frac{y(s)}{s}ds \\
&\quad -\frac{a}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q)}\Big(\log\frac{T}{s}\Big)^{(q-1)}\frac{y(s)}{s}\;ds \\
&\quad +\frac{\frac{b}{T}-\log T}{a+\frac{b}{T}-\log T}\int_1^T g_1(s)\;u(s)\;ds \\
&\quad +\frac{a}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)\;u(s)\;ds,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 &= \frac{1}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q)}\Big(\log\frac{T}{s}\Big)^{(q-1)}\frac{y(s)}{s}ds \\
&\quad -\frac{b}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q-1)}\Big(\log\frac{T}{s}\Big)^{(q-2)}\frac{y(s)}{s}\;ds \\
&\quad +\frac{1}{a+\frac{b}{T}-\log T}\int_1^T g_1(s)\;u(s)\;ds \\
&\quad -\frac{1}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)\;u(s)\;ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
u(t) &= \alpha_0+\alpha_1\log t+\frac{1}{\Gamma(q)}\int_1^T\Big(\log\frac{T}{s}\Big)^{(q-1)}\;\frac{y(s)}{s}\;ds \\
&= \int_1^T G(t,s)\;y(s)\;ds+\frac{\frac{b}{T}+\log t-\log T}{a+\frac{b}{T}-\log T}\int_1^T g_1(s)\;u(s)\;ds \\
&\quad +\frac{a-\log t}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)\;u(s)\;ds.
\end{aligned}$$

We obtain the exact form of $u(t)$ by solving the following equations

$$\begin{aligned}
(1-\sigma_1)\int_1^T g_1(s)u(s)\;ds-\sigma_2\int_1^T g_2(s)u(s)\;ds &= \int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds, \\
(1-\sigma_4)\int_1^T g_2(s)u(s)\;ds-\sigma_3\int_1^T g_1(s)u(s)\;ds &= \int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds,
\end{aligned}$$

and

$$\begin{aligned}
\int_1^T g_1(s)u(s)\;ds &= \frac{(1-\sigma_4)\int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3} \\
&\quad +\frac{\sigma_2\int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3}, \\
\int_1^T g_2(s)u(s)\;ds &= \frac{\sigma_3\int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3} \\
&\quad +\frac{(1-\sigma_1)\int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)\;d\tau\;ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3}.
\end{aligned}$$

We find

$$u(t)=\int_1^T G(t,s)y(s)\ ds+\int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)\ d\tau ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}+(\log t-a)(\log\frac{T}{s})^{q-1}}{(a+\frac{b}{T}-\log T)\Gamma(q)} & \\ +\dfrac{b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{(a+\frac{b}{T}-\log T)\Gamma(q)} & 1\le s\le t\le T,\\ \dfrac{(\log t-a)(\log\frac{T}{s})^{q-1}+b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{(a+\frac{b}{T}-\log T)\Gamma(q)} & 1\le t\le s\le T,\end{cases}$$

and

$$\begin{aligned}R(t,s) &= \frac{[(a-\log t)\sigma_3+(\log t+\frac{b}{T}-\log T)(1-\sigma_4)]g_1(s)}{\rho(a+\frac{b}{T}-\log T)}\\ &\quad+\frac{[(a-\log t)(1-\sigma_1)+(\log t+\frac{b}{T}-\log T)\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)}.\end{aligned}$$

□

Lemma 2.6. *Let $3<q<4$, and assume (H_1) holds. Let $G(t,s)$ be the Green function related to problem (2.1) given by the expression (2.3), then we have*

$$(1-\log t)\ G(1,s)\le G(t,s)\le G(1,s) \tag{2.7}$$

Proof. Denote $K(t,s)=\frac{G(t,s)}{G(1,s)}$.

Case 1: $1\le t\le s\le T\le e$,

$$\begin{aligned}K(t,s) &= \frac{(\log t-a)(\log\frac{T}{s})^{q-1}+b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{-a(\log\frac{T}{s})^{q-1}+b(q-1)a\log(\frac{T}{s})^{q-2}}\\ &= \frac{a-\log t}{a}=1-\frac{\log t}{a};\end{aligned}$$

we have

$$(1-\log t)\le(1-\frac{\log t}{a})\le 1,$$

then,

$$(1-\log t)\le K(t,s)\le 1.$$

Case 2: $1\le s\le t\le T\le e$,

$$\begin{aligned}K(t,s) &= \frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}+(\log t-a)(\log\frac{T}{s})^{q-1}}{-a(\log\frac{T}{s})^{q-1}+b(q-1)a(\log\frac{T}{s})^{q-2}}\\ &\quad+\frac{b(q-1)(a-\log t)(\log\frac{T}{s})^{q-2}}{-a(\log\frac{T}{s})^{q-1}+b(q-1)a(\log\frac{T}{s})^{q-2}},\end{aligned}$$

we can write

$$K(t,s)=\frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]}+1-\frac{\log t}{a}.$$

As $1-\frac{\log t}{a}\ge(1-\log t)$, taking account of assumptions $b\ge a\ge 1$ and $3<q<4$, it comes that

$$\frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]}>\frac{\log T(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}}>0;$$

$$K(t,s)=\frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]}+1-\frac{\log t}{a}\geq(1-\log t).$$

Now, we differentiate twice $K(t,s)$ with respect to t,

$$\begin{aligned}\frac{\partial^2K(t,s)}{\partial t^2} &= \frac{(a+\frac{b}{T}-\log T)}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]}\\ &\quad\times[(q-1)(q-2)\frac{1}{t^2}(\log\frac{t}{s})^{q-3}-\frac{1}{t^2}(q-1)(\log\frac{t}{s})^{q-2}]+\frac{1}{at^2}\\ &= \frac{1}{t^2}\frac{(a+\frac{b}{T}-\log T)}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]}\\ &\quad\times(q-1)(\log\frac{t}{s})^{q-3}[(q-2)-(\log\frac{t}{s})]+\frac{1}{at^2}\geq 0,\end{aligned}$$

if $1\leq s\leq t\leq T\leq e$. Whereupon, the maximum values of $K(t,s)$ are at either $t=s$ or $t=T$;

$$\begin{aligned}K(s,s) &= 1-\frac{s}{a}\leq 1,\\ K(T,s) &= \frac{(a+\frac{b}{T}-\log T)}{a[b(q-1)(\log\frac{T}{s})^{-1}-1]}+1-\frac{\log T}{a}\\ &= \frac{ab(q-1)-b(q-1)\log T+\frac{b}{T}\log\frac{T}{s}}{ab(q-1)-a\log\frac{T}{s}}\\ &\leq \frac{ab(q-1)-a[(q-1)\log T-\frac{1}{T}\log\frac{T}{s}]}{ab(q-1)-a\log\frac{T}{s}}\\ &\leq \frac{ab(q-1)-a[(q-1)-\log\frac{T}{s}]}{ab(q-1)-a\log\frac{T}{s}}\\ &\leq \frac{ab(q-1)-a\log\frac{T}{s}}{ab(q-1)-a\log\frac{T}{s}}=1,\end{aligned}$$

hence, $(1-\log t)\leq K(t,s)\leq 1$ for $1\leq s\leq t\leq T\leq e$. This completes the proof. □

Now, we present some definitions and recall Avery-Peterson fixed point theorem.

Definition 2.7. *([9,10]) Let E be a real Banach space. A nonempty convex closed set $P\subset E$ is said to be a cone provided that*
(i) $au\in P$ for all $u\in P$ and all $a\geq 0$,
(ii) $u,-u\in P$ implies $u=0$.

Note that every cone $P\subset E$ includes an ordering in E given by $x\leq y$ if $y-x\in P$.

Definition 2.8. *([9,10]) The map ϕ is defined as a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\phi:P\to[0,+\infty)$ satisfies*

$$\phi(\theta x+(1-\theta)y)\geq\theta\phi(x)+(1-\theta)\phi(y)\quad\text{for all } x,y\in P \text{ and } 0\leq\theta\leq 1.$$

Similarly, we say the map η is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\eta:P\to[0,+\infty)$ satisfies

$$\eta(\theta x+(1-\theta)y)\leq\theta\eta(x)+(1-\theta)\eta(y)\quad\text{for all } x,y\in P \text{ and } 0\leq\theta\leq 1.$$

Let γ and θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then for positive real numbers a', b', c' and d', we define the following convex sets

$$\begin{aligned}&P(\gamma, d') = \{x \in P|\ \gamma(x) < d'\},\\ &P(\gamma, \phi, b', d') = \{x \in P|\ b' \leq \phi(x), \gamma(x) \leq d'\},\\ &P(\gamma, \theta, \phi, b', c', d') = \{x \in P|\ b' \leq \phi(x), \theta(x) \leq c', \gamma(x) \leq d'\},\end{aligned}$$

and the closed set

$$R(\gamma, \psi, a', d') = \{x \in P|\ a' \leq \psi(x), \gamma(x) \leq d'\}.$$

Theorem 2.9. *([4,9]) Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d',*

$$\phi(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, d')}$. Suppose $\mathcal{A} : \overline{P(\gamma, d')} \to \overline{P(\gamma, d')}$ is completely continuous and there exist positive numbers a', b' and c' with $a' < b'$ such that
(S_1)

$$\{x \in P(\gamma, \theta, \phi, b', c', d')|\ \phi(x) > b'\} \neq \varnothing$$

and

$$\phi(\mathcal{A}x) > b' \text{ for } x \in P(\gamma, \theta, \phi, b', c', d'),$$

(S_2) $\phi(\mathcal{A}x) > b'$ for $x \in P(\gamma, \phi, b', d')$ with $\theta(\mathcal{A}x) > c'$,
(S_3) $0 \notin R(\gamma, \psi, a', d')$ and $\psi(\mathcal{A}x) < a'$ for $x \in R(\gamma, \psi, a', d')$ with $\psi(x) = a'$.
Then $\mathcal{A}$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d')}$ such that

$$\begin{aligned}&\gamma(x_i) \leq d', \ for\ i = 1, 2, 3;\\ &b' < \phi(x_1);\\ &a' < \psi(x_2) \ with\ \phi(x_2) < b';\\ &\psi(x_3) < a'.\end{aligned}$$

3. Main Results

Let us define the operator $\mathcal{A} : C^1[1, T] \to C^1[1, T]$ as:

$$\begin{aligned}\mathcal{A}u(t) \quad &= \quad \int_1^T G(t, s)f(s, u(s), u'(s))\ ds\\ &\quad + \int_1^T R(t, s) \int_1^T G(s, \tau)f(\tau, u(\tau), u'(\tau))\ d\tau ds.\end{aligned} \tag{3.1}$$

Let the Banach space $E = (C^1[1, T], \|\cdot\|)$ equipped with the norm

$$\|u\| = \max\{\max_{t\in[1,T]} |u(t)|, \max_{t\in[1,T]} |u'(t)|\}.$$

We define the cone $P \subset E$ by

$$P = \Big\{u \in E \mid u(t) \geq 0 \text{ and } u(t) \text{ is convex on } [1, T]\Big\}.$$

And denote a nonnegative continuous concave functional ϕ, the nonnegative continuous convex functional θ, the positive continuous convex functional γ and the nonnegative continuous functional ψ on the cone P as follows:

$$\gamma(u) = \max_{t\in[1,T]} |u'(t)|\ ,\ \psi(u) = \theta(u) = \max_{t\in[1,T]} |u(t)|$$

and

$$\phi(u) = \min_{t\in[\delta,T-\delta]} |u(t)| \text{ for } \delta \in \Big[1, \frac{1 + T}{2}\Big].$$

Lemma 3.1. *If $u \in P$ and satisfies the boundary conditions $u(1) = -au'(1) + \int_1^T g_1(s)u(s)ds$, then*

$$\max_{t\in[1,T]} |u(t)| \leq \frac{(T-1)+a}{1-\sigma_1-\sigma_2} \max_{t\in[1,T]} |u'(t)|.$$

Proof. Since $u(t) - u(1) = \displaystyle\int_1^t u'(s)ds$, we get

$$\begin{aligned}
\max_{t\in[1,T]} |u(t)| &\leq |u(1)| + (T-1) \max_{t\in[1,T]} |u'(t)| \\
&\leq \left| -au'(1) + \int_1^T g_1(s)u(s)ds \right| + (T-1) \max_{t\in[1,T]} |u'(t)| \\
&\leq |\, au'(1) \,| + \max_{t\in[1,T]} |\, u(t) \,| \int_1^T g_1(s)ds + (T-1) \max_{t\in[1,T]} |u'(t)|
\end{aligned}$$

then,

$$\begin{aligned}
\max_{t\in[1,T]} |u(t)| \left[1 - \int_1^T g_1(s)ds\right] &\leq (T-1) \max_{t\in[1,T]} |u'(t)| + a \max_{t\in[1,T]} |u'(t)| \\
&= [(T-1)+a] \max_{t\in[1,T]} |u'(t)|.
\end{aligned}$$

Therefore,

$$\max_{t\in[1,T]} |u(t)| \leq \frac{(T-1)+a}{1-\sigma_1-\sigma_2} \max_{t\in[1,T]} |u'(t)|.$$

□

Lemma 3.2. *For $\delta \in \left[1, \frac{1+T}{2}\right]$, we have*

$$\min_{t\in[\delta,(T+1)-\delta]} R(t,s) > (\log\delta) \max_{t\in[1,T]} R(t,s),$$

where $R(t,s)$ defined by (2.4).

Proof. From lemma (2.5), we have

$$\begin{aligned}
R(t,s) &= \frac{[(a-\log t)\sigma_3 + (\log t + \frac{b}{T} - \log T)(1-\sigma_4)]g_1(s)}{\rho(a + \frac{b}{T} - \log T)} \\
&\quad + \frac{[(a-\log t)(1-\sigma_1) + (a + \frac{b}{T} - \log T)\sigma_2]g_2(s)}{\rho(a + \frac{b}{T} - \log T)}
\end{aligned}$$

we can write

$$\begin{aligned}
R(t,s) &= \frac{(1-\sigma_3-\sigma_4)g_1(s) + (\sigma_1+\sigma_2-1)g_2(s)}{\rho(a + \frac{b}{T} - \log T)} \log t \\
&+ \frac{[a\sigma_3 + (\frac{b}{T} - \log T)(1-\sigma_4)]g_1(s) + [a(1-\sigma_1) + (\frac{b}{T} - \log T)\sigma_2]g_2(s)}{\rho(a + \frac{b}{T} - \log T)}.
\end{aligned}$$

Denote

$$j(s) = \frac{(1-\sigma_3-\sigma_4)g_1(s) + (\sigma_1+\sigma_2-1)g_2(s)}{\rho(a + \frac{b}{T} - \log T)},$$

and

$$l(s) = \frac{[a\sigma_3 + (\frac{b}{T} - \log T)(1-\sigma_4)]g_1(s) + [a(1-\sigma_1) + (\frac{b}{T} - \log T)\sigma_2]g_2(s)}{\rho(a + \frac{b}{T} - \log T)}.$$

When $j(s) < 0$, we have tow cases:

Case 1: $0 < (1-\sigma_3-\sigma_4)g_1(s) < (1-\sigma_1-\sigma_2)g_2(s)$, so due to monotonicity of $R(t,s)$, $\max_{t\in[1,T]} R(t,s) = R(1,s) = l(s) > 0$, the minimum value is

$$R(T,s) = j(s)\log T + l(s) > 0.$$

Then, it holds

$$\begin{aligned}
\frac{\min_{t\in[\delta,(T+1)-\delta]} R(t,s)}{\max_{t\in[1,T]} R(t,s)} &= \frac{R((T+1)-\delta,s)}{R(1,s)} \\
&= \frac{\log((T+1)-\delta)j(s)+l(s)}{l(s)} \\
&= \frac{\log((T+1)-\delta)j(s)}{l(s)} + 1 > \log\delta.
\end{aligned} \tag{3.2}$$

Case 2: $(1-\sigma_3-\sigma_4)g_1(s) < 0 < (1-\sigma_1-\sigma_2)g_2(s)$. The minimum value of $R(t,s)$ is $R(T,s)$ and

$$\begin{aligned}
R(T,s) &= \frac{[(a-\log T)\sigma_3 + (\log T + \frac{b}{T} - \log T)(1-\sigma_4)]g_1(s)}{\rho(a+\frac{b}{T}-\log T)} \\
&\quad + \frac{[(a-\log T)(1-\sigma_1) + (\log T + \frac{b}{T} - \log T)\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)} \\
&= \frac{[(a-\log T)\sigma_3 + \frac{b}{T}(1-\sigma_4)]g_1(s) + [(a-\log T)(1-\sigma_1) + \frac{b}{T}\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)}.
\end{aligned}$$

Since, $b \geq a \geq 1 \geq \log T$ and $1 \leq T \leq e$, we have $\frac{b}{T} \geq \frac{1}{e}$. Then

$$\begin{aligned}
R(T,s) &\geq \frac{\frac{1}{e}[(1-\sigma_4)]g_1(s) + \sigma_2 g_2(s)]}{\rho(a+\frac{b}{T}-\log T)} \\
&\geq \frac{\frac{1}{e}[(1-\sigma_4)]g_1(s) + \sigma_2 \frac{1-\sigma_3-\sigma_4}{1-\sigma_1-\sigma_2} g_1(s)]}{\rho(a+\frac{b}{T}-\log T)} \\
&\geq \frac{\frac{1}{e}[(1-\sigma_4)(1-\sigma_1-\sigma_2) + \sigma_2(1-\sigma_3-\sigma_4)]g_1(s)}{(1-\sigma_1-\sigma_2)\rho(a+\frac{b}{T}-\log T)} \\
&= \frac{g_1(s)}{e(1-\sigma_1-\sigma_2)(a+\frac{b}{T}-\log T)} > 0,
\end{aligned}$$

the expression (3.2) also holds.
Therefore, for $\delta \in [1, \frac{T+1}{2}]$, we get

$$\min_{t\in[\delta,(T+1)-\delta]} R(t,s) > (\log\delta) \max_{t\in[1,T]} R(t,s).$$

□

Lemma 3.3. *If $u \in P, \delta \in [1, \frac{T+1}{2}]$, then*

$$\min_{t\in[\delta,(T+1)-\delta]} u(t) \geq (\log\delta) \max_{t\in[1,T]} u(t).$$

Proof.

$$\begin{aligned}\min_{t\in[\delta,T+1-\delta]} u(t) =& \min_{t\in[\delta,T+1-\delta]}\left\{\int_1^T G(t,s)y(s)ds+\int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\right\}.\\ \geq& \min_{t\in[\delta,T+1-\delta]}\{(1-\log t)\int_1^T G(1,s)y(s)ds+\int_1^T \min_{t\in[\delta,T+1-\delta]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\\ \geq&(1-\log(T+1-\delta))\int_1^T \max_{t\in[1,T]} G(t,s)y(s)ds\\ &+(\log\delta)\int_1^T \max_{t\in[1,T]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\\ \geq&(\log\delta)\int_1^T \max_{t\in[1,T]} G(t,s)y(s)ds+(\log\delta)\int_1^T \max_{t\in[1,T]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\\ \geq&(\log\delta)\max_{t\in[1,T]}\left\{\int_1^T G(t,s)y(s)ds+\int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\right\}\\ =&(\log\delta)\max_{t\in[1,T]} u(t).\end{aligned}$$

Hence,

$$\min_{t\in[\delta,T+1-\delta]} u(t)\geq(\log\delta)\max_{t\in[1,T]} u(t).$$

□

Lemma 3.4. *$\mathcal{A}: P\to P$ is completely continuous.*

Proof. From the continuity and the non-negativeness of functions G and f on their domains of definition.

We know that if $u\in P$, then $\mathcal{A}u\in E$ and $\mathcal{A}u(t)\geq 0$ for all $t\in[1,T]$. Take $u\in P$, then

$$(\mathcal{A}u)''(t) = \int_1^T \frac{\partial^2 G(t,s)}{\partial t^2} f(s,u(s),u'(s))ds \tag{3.3}$$

$$+\int_1^T \frac{\partial^2 R(t,s)}{\partial t^2}\int_1^T G(s,\tau)f(\tau,u(\tau),u'(\tau))d\tau ds \ \geq 0 \tag{3.4}$$

□

Let us set

$$\begin{aligned} M_1 &= \max\left\{\left|\frac{\partial R(t,s)}{\partial t}\right| t,s\in[1,T]\right\}, M_2=\max\{|R(t,s)|\ t,s\in[1,T]\},\\ K &= \{A+M_1B\},\end{aligned}$$

where

$$\begin{cases} A=\dfrac{(a+b/T-\log T)\left(T^{\alpha-1}-1+\frac{\alpha-1}{3-\alpha}\left(1-T^{\alpha-3}\right)\right)}{a+b/T-\log T}\\ \dfrac{+\frac{1}{2-\alpha}\left(T-T^{\alpha-1}\right)-b(\alpha-1)\left[\left(T-T^{\alpha-2}\right)+\left(1-T^{\alpha-3}\right)\right]}{a+b/T-\log T}\\ B=\dfrac{\frac{1}{\alpha}(a+b/T-\log T)\left(T^{\alpha+1}-T\right)-\frac{a}{2-\alpha}\left(T^2-T^{\alpha}\right)+\frac{ab(\alpha-1)}{3-\alpha}\left(T-T^{\alpha-2}\right)}{a+b/T-\log T}\end{cases}$$

$$\begin{aligned} L &= \{(1-\log(T+1-\delta))+(\log\delta)M\left[2(T-1)-T\log T\right]\}(T-1)\\ &\quad\times\frac{-a(\log T)^{\alpha-1}+ab/T.(\alpha-1)(\log T)^{\alpha-2}}{[a-\log T+b/T].\Gamma(\alpha)},\\ N &= \frac{\frac{1}{\alpha}.(a+b\backslash T-\log T)(T^{\alpha}-1)-\frac{aT^{\alpha-1}}{2-\alpha}(T^{2-\alpha}-1)+\frac{ab(\alpha-1)}{3-\alpha}(T^{\alpha-5}-T^{2-\alpha})}{a+b/T-\log T}.\end{aligned}$$

Theorem 3.5. *Assume there exist constants $0 < a' < b' < c' < d'$, where $c' = \delta b'$, and suppose the function $f(t, u(t), u'(t))$ satisfies*

(A1) $f(t,u,v) \leq d'/K,$ *for* $(t,u,v) \in [1,T] \times \left[0, \frac{a+T-1}{1-\sigma_1-\sigma_2} d'\right] \times [-d', d']$,

(A2) $f(t,u,v) \geq b'/L,$ *for* $(t,u,v) \in [\delta, 1+T-\delta] \times [b', \delta b'] \times [-d', d']$,

(A3) $f(t,u,v) < a'/N,$ *for* $(t,u,v) \in [1,T] \times [0,a'] \times [-d', d']$.

Then the boundary value problem (1.1) has at least three solutions u_1, u_2 and u_3 satisfying

$$\begin{aligned}
\max_{t\in[1,T]} |u_i'(t)| &\leq d', \ \textit{for} \quad i = 1,2,3;\\
b' &< \min_{t\in[\delta,T+1-\delta]} |u_1(t)|;\\
a' &< \max_{t\in[1,T]} |u_2(t)| \qquad \textit{with} \quad \min_{t\in[\delta;T+1-\delta]} |u_2(t)| < b';\\
\max_{t\in[1,T]} |u_3(t)| &< a'.
\end{aligned}$$

Proof. The BVP (1.1) has a solution $u = u(t)$ if and only if u is the solution of the operator equation

$$\begin{aligned}
u \quad &= \quad \mathcal{A}u(t) = \int_1^T G(t,s) f(s,u(s),u'(s))ds\\
&\quad + \int_1^T R(t,s) \int_1^T G(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau ds.
\end{aligned}$$

Let us that operator T satisfies the conditions in the Avery-Peterson fixed point theorem.

If $u \in \overline{P(\gamma, d')}$, then $\gamma(u) = \max_{t\in[1,T]} |u'(t)| \leq d'$, with lemma(3.1), we get

$$\max_{t\in[1,T]} |u(t)| \leq \frac{a+T-1}{1-\sigma_1-\sigma_2} \max_{t\in[1,T]} |u'(t)| \leq \frac{a+T-1}{1-\sigma_1-\sigma_2} d'.$$

With the condition $u \in P$ and its non-negativeness on its domain and because of $\mathcal{A}(u) \in P$ with $\mathcal{A}u \geq 0$ and $\mathcal{A}u$ is convex on $[1,T]$, the maximum value of $|\mathcal{A}u'(t)|$ is either $|\mathcal{A}u'(1)|$ or $\left|\mathcal{A}u^(T)\right|$.
Combining with the assumption (A1), we have

$$\begin{aligned}
\gamma(\mathcal{A}u(t)) &= \max_{t\in[1,T]} |(\mathcal{A}u)'(t)|\\
&= \max\left\{|(\mathcal{A}u)'(1)|, |(\mathcal{A}u)'(T)|\right\}\\
&\leq \int_1^T \left| \frac{(a+b\backslash T - \log T)(\alpha-1)(-\log s)^{\alpha-2} + (\log T/s)^{\alpha-2}\left[\log\frac{T}{s} - b(\alpha-1)\right]}{(a+b\backslash T - \log T)\Gamma(\alpha)}\right.\\
&\times f(s,u(s),u'(s))ds|\\
&+ \int_1^T \frac{(a+b\backslash T - \log T)(\alpha-1) + (\log\frac{T}{s}) - b(\alpha-1)]1/T.(\log T/s)^{\alpha-2}}{(a+b\backslash T - \log T)\Gamma(\alpha)}\\
&\times f(s,u(s),u'(s))ds\\
&+ \left|\int_1^T \frac{\partial R(t,s)}{\partial t}\right| \left|\int_1^T G(s,\tau) f(\tau,u(\tau),u'(\tau)) d\tau ds\right|
\end{aligned}$$

$$\leq \frac{d'}{K}\left[\frac{(a+b/T-\log T)\left(T^{\alpha-1}-1\right)+\frac{1}{2-\alpha}\left(T-T^{\alpha-1}\right)-b(\alpha-1)\left(T-T^{\alpha-2}\right)}{a+b/T-\log T}\right]$$
$$+\frac{d'}{K}\left[\frac{\left[(a+b/T-\log T)\frac{\alpha-1}{3-\alpha}1-b(\alpha-1)\right]\left(1-T^{\alpha-3}\right)}{a+b/T-\log T}\right]$$
$$+\frac{d'M_1}{K}\left[\frac{\frac{1}{\alpha}(a+b/T-\log T)\left(T^{\alpha+1}-T\right)-\frac{a}{2-\alpha}\left(T^2-T^{\alpha}\right)+\frac{ab(\alpha-1)}{3-\alpha}\left(T-T^{\alpha-2}\right)}{a+b/T-\log T}\right]$$
$$\leq \frac{d'}{K}\{A+M_1B\}=d'.$$

Therefore, $\mathcal{A}:\overline{P(\gamma,d')}\to\overline{P(\gamma,d')}$. To confirm the condition $(S1)$ of Theorem(2.9), we choose $u(t)=\frac{b'+c'}{2}$, $1\leq t\leq T$. So $\varphi(u)=\frac{b'+c'}{2}>b'$,
$\theta(u)=\frac{b'+c'}{2}<c'$ and $\gamma(u)=0<d'$.
Consequently, $\{u\in P(\gamma,\theta,\varphi,b',c',d'),\varphi(u)>b'\}\neq\phi$. Moreover, if

$$u\in P(\gamma,\theta,\varphi,b',c',d')$$

then $b'\leq u(t)\leq c'$ and $|u'(t)|\leq d'$ hold for $t\in[\delta,T+1-\delta]$. By using the assumption $(A2)$, we will check the condition $(S1)$ of Theorem(2.9).

$$\varphi(\mathcal{A}u(t))=\min_{t\in[\delta,T+1-\delta]}|(\mathcal{A}u)(t)|$$
$$=\min_{t\in[\delta,T+1-\delta]}\{\int_1^T G(t,s)f(s,u(s),u'(s))ds$$
$$+\int_1^T R(t,s)\int_1^T G(s,\tau)f(\tau,u(\tau),u'(\tau))d\tau ds\}$$
$$\geq \min_{t\in[\delta,T+1-\delta]}\{(1-\log\delta)\int_1^T G(1,s)f(s,u(s),u'(s))ds$$
$$+\int_1^T(1-\log s)R(t,s)\int_1^T G(1,\tau)f(\tau,u(\tau),u'(\tau))d\tau ds.\}$$
$$\geq \frac{b'}{L}.\min_{t\in[\delta,T+1-\delta]}\left\{(1-\log t)+\int_1^T(1-\log s)R(t,s)ds\right\}\int_1^T G(1,\tau)ds$$
$$\geq \frac{b'}{L}.\left\{(1-\log(T+1-\delta))+(\log\delta)\max_{t\in[1,T]}R(t,s)\int_1^T(1-\log s)ds\right\}\int_1^T G(1,s)ds$$
$$\geq \frac{b'}{L}.\{(1-\log(T+1-\delta))+(\log\delta)M_2[2(T-1)-T\log T]\}\int_1^T G(1,s)ds$$
$$\geq \frac{b'}{L}.\{(1-\log(T+1-\delta))+(\log\delta)M_2[2(T-1)-T\log T]\}(T-1)$$
$$\times\frac{-aT(\log T)^{\alpha-1}+ab(\alpha-1)(\log T)^{\alpha-2}}{[T(a-\log T)+b]\Gamma(\alpha)}=b'.$$

So the condition $(S1)$ is satisfied. If $u\in P(\gamma;\varphi,b',d')$ and $\theta(\mathcal{A}u)>\delta b'$, then

$$\begin{aligned}\varphi(\mathcal{A}u) &= \min_{t\in[\delta,T+1-\delta]}(\mathcal{A}u)(t)\geq(\log\delta)\max_{t\in[1,T]}(\mathcal{A}u)(t)\\ &= (\log\delta).\theta(\mathcal{A}u)>(\log\delta).\delta b'>b'.\end{aligned}$$

So condition $(S2)$ of theorem(2.9) follows.
Finally, we show that the condition $(S3)$ of Theorem(2.9) holds.

As $\psi(0) = 0 < a'$, so $0 \notin R(\gamma, \psi, a', d')$. For $u \in R(\gamma, \psi, a', d')$ with $\psi(u) = a'$ we have $0 \le u(t) \le a', t \in [1, T]$. Using assumption $(A3)$, we can estimate

$$\begin{aligned}
\psi(\mathcal{A}u) &= \max_{t\in[1,T]} |(\mathcal{A}u)(t)| \\
&= \max_{t\in[1,T]} \left| \int_1^T G(t,s) f(s,u(s),u'(s))ds + \int_1^T R(t,s) \int_1^T G(s,\tau) f(\tau,u(\tau),u'(\tau))d\tau ds \right|. \\
&\le \max_{t\in[1,T]} \left| \int_1^T G(1,s) f(s,u(s),u'(s))ds \right. \\
&\quad \left. + \max_{t\in[1,T]} \int_1^T R(t,s) \int_1^T G(1,\tau) f(\tau,u(\tau),u'(\tau))d\tau ds \right| \\
&\le \max_{t\in[1,T]} \left\{ 1 + \int_1^T R(t,s)ds \right\} \int_1^T G(1,s) f(s,u(s),u'(s))ds \\
&< (1+M_2) \frac{a'}{N} \int_1^T G(1,s)ds \\
&< (1+M_2) \frac{a'}{N} \left[\frac{\frac{1}{\alpha}.(a + b\backslash T - \log T)(T^\alpha - 1) - \frac{aT^{\alpha-1}}{2-\alpha}(T^{2-\alpha} - 1)}{a + b/T - \log T} \right] \\
&\quad + (1+M_2) \left[\frac{\frac{ab(\alpha-1)}{3-\alpha}(T^{\alpha-5} - T^{2-\alpha})}{a + b/T - \log T} \right] \\
&= a'.
\end{aligned}$$

Therefore, problem (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\begin{aligned}
\max_{t\in[1,T]} |u_i'(t)| &\le d', \text{ for } \quad i = 1,2,3; \\
b' &< \min_{t\in[\delta,T+1-\delta]} |u_1(t)|; \\
a' &< \max_{t\in[1,T]} |u_2(t)| \quad \text{with} \quad \min_{t\in[\delta,T+1-\delta]} |u_2(t)| < b'; \\
\max_{t\in[1,T]} |u_3(t)| &< a'.
\end{aligned}$$

The proof of the theorem is completed. □

4. Example

We consider the problem

$$\begin{cases} D^{\frac{7}{2}} u(t) = f(t,u(t),u'(t)), \quad t \in [1,2], \\ u''(1) = 0, \quad u^{(3)}(1) = 0, \\ u(1) + u'(1) = \frac{1}{10} \displaystyle\int_1^2 u(s)ds, \\ u(2) - 2u'(2) = \displaystyle\int_1^2 su(s)ds, \end{cases} \tag{4.1}$$

where

$$f(t,u,v) = \begin{cases} \log(2+t) + \frac{u^8}{8} + (\frac{v}{10^4})^3 & u \in [0,4], \\ \log(2+t) + \frac{4^8}{8} + (\frac{v}{10^4})^3 & u \in [4,+\infty], \end{cases}$$

with

$$\sigma_1 = 0.0528, \sigma_2 = 0.0469, \sigma_3 = 0.839, \sigma_4 = 0.0469, \rho = 0.2817$$

and

$$M_1 = 0.444,\ M_2 = 5.480,\ K = 4.075,\ L = 0.333,\ N = 1.084.$$

Setting $a' = 1,\ b' = 5,\ d' = 10^4$, then $f(t,u,v)$ satisfies

$$\begin{cases} f(t,u,v) \leq \frac{d'}{K} = 2453.987, & for \quad (t,u,v) \in [1,2] \times [0, 2 \times 10^4] \times [-10^4, 10^4]; \\ f(t,u,v) \geq \frac{b'}{L} = 15.151, & for \quad (t,u,v) \in [\frac{9}{8}, \frac{15}{8}] \times [5, \frac{45}{8}] \times [-10^4, 10^4]; \\ f(t,u,v) \leq \frac{a'}{N} = 0.9225, & for \quad (t,u,v) \in [1,2] \times [0,1] \times [-10^4, 10^4]. \end{cases}$$

All conditions in Theorem(3.5) hold; then problem(4.1) has at least three positive solutions.

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