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# Note on the Fractional Mittag-Leffler Functions by Applying the Modified Riemann-Liouville Derivatives

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ABSTRACT: In this article, the fractional derivatives in the sense of the modified Riemann-Liouville derivative is employed for constructing some results related to Mittag-Leffler functions and established a number of important relationships between the Mittag-Leffler functions and the Wright function.

Key Words: Fractional calculus, Modified Riemann-Liouville derivative, Mittage-Leffler function.

## Contents



## 1. Introduction

It is well known that with the classical Riemann-Liouville definition of fractional derivative [2,5,15], the fractional derivative of a constant is not zero. The most useful alternative which has been proposed to cope with this feature is known Caputo derivative [6], but in this derivative fractional derivative would be defined for differentiable functions only. A modification of the Riemann-Liouville has been defined to deal with non-differentiable functions [3,4,9,21,16,23] and it is given as:

**Definition 1.1.** *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$, $x\longrightarrow f(x)$ denote a continuous function. The modified Riemann-Liouville derivative of order $\alpha$ is defined by the expression*

$$D^{\alpha}f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\int_a^x (x-\eta)^{-\alpha-1}f(\eta)d\eta & ;\alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\int_a^x (x-\eta)^{-\alpha}\left[f(\eta)-f(a)\right]d\eta & ;0<\alpha<1,\\ \left(f^{(\alpha-m)}(x)\right)^{(m)} & ;m\le\alpha<m+1. \end{cases}$$

Some important properties for this kind of derivatives were given in [20] as follows:

1. $D^{\alpha}x^{\mu}=\dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}x^{\mu-\alpha}, \mu>0,$
2. $D^{\alpha}\left(f(x)g(x)\right)=\left(D^{\alpha}f(x)\right)g(x)+f(x)\left(D^{\alpha}g(x)\right),$
3. $D^{\alpha}f(u(x))=D^{\alpha}f(u)\left(D(u)\right)^{\alpha},$
4. $D^{\alpha}(m)=0$ where $m$ is constant function.

There are some special functions which are studied their fractional derivative by several researchers ( Agarwal [1], Erdelyi [7] and Miller [18]). In this article, we deal with some of these functions such as Mittag-Leffler and Wright functions.

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The Mittag-Leffler function [7,8] of one parameter is denoted by $E_\alpha(x)$ and defined by:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \quad \alpha \in \mathbb{C}, \alpha > 0. \tag{1.1}$$

This function plays a crucial role in classical calculus for $\alpha = 1$, for $\alpha = 1$ it becomes the exponential function, that is $e^x = E_1(x)$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}.$$

The other important function which is a generalization of series is represented by:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta \in \mathbb{C}, \alpha > 0. \tag{1.2}$$

The functions (1.1) and (1.2) play important role in fractional calculus, also we note that when $\beta = 1$ in (1.2), then (1.1) is obtained which mean that $E_{\alpha,1}(x) = E_\alpha(x)$.

Another form which is generalization of (1.1) and (1.2) was introduced by Prabhakar [22] such as:

$$E^{\delta}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\alpha k+\beta)} x^k, \quad \alpha, \beta, \delta \in \mathbb{C}, \alpha > 0, \tag{1.3}$$

where $(\delta)_k$, the Pochhammer symbol, is defined by

$$(\delta)_k = \delta(\delta+1)...(\delta+k-1), \delta \in \mathbb{C}, k \in \mathbb{N},$$

while

$$(\delta)_0 = 1, \delta \neq 0.$$

There are some special cases of (1.3) such as:

1. $E^1_{\alpha,1}(x) = E_\alpha(x)$,
2. $E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$.

The second functions will be discussed is Wright function, which is defined as

$$W(x; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta) k!}.$$

This function plays an important role in the solution of a linear partial differential equation. Furthermore, there is an interesting link between the Wright function and the Mittag-Leffler function. Hence, some useful relationships between those functions have been obtained in this work.

## 2. Main Result

Now, we point out some formulas which do not hold for the classical Riemann-Liouville definition, but apply with the modified Riemann-Liouville definition.

**Theorem 2.1.** *Assume that $\alpha > 0$,$\beta > 0$ for $\lambda \in \mathbb{R}$, then the following formula holds*

$$D^\alpha \left[x^{\beta-1} E^{\delta}_{\alpha,\beta}(\lambda x^\alpha)\right] = \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + \lambda x^{\beta-1} E^{\delta}_{\alpha,\beta}(\lambda x^\alpha) \tag{2.1}$$

*Proof.*

$$\begin{aligned}
&D^{\alpha}\left[x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right]\\
=&D^{\alpha}\sum_{k=0}^{\infty}\frac{(\delta)_k\lambda^k}{\Gamma(\alpha k+\beta)k!}x^{\alpha k+\beta-1}\\
=&D^{\alpha}\left[\frac{x^{\beta-1}}{\Gamma(\beta)}+\frac{(\delta)_1\lambda}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1}+\frac{(\delta)_2\lambda^2}{\Gamma(2\alpha+\beta)2!}x^{2\alpha+\beta-1}+\frac{(\delta)_3\lambda^3}{\Gamma(3\alpha+\beta)3!}x^{3\alpha+\beta-1}+...\right]\\
=&\frac{1}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1}+\frac{(\delta)_1\lambda}{\Gamma(\beta)}x^{\beta-1}+\frac{(\delta)_2\lambda^2}{\Gamma(\alpha+\beta)2!}x^{\alpha+\beta-1}+\frac{(\delta)_3\lambda^3}{\Gamma(2\alpha+\beta)3!}x^{\alpha+\beta-1}+...\\
=&\frac{1}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1}+\lambda x^{\beta-1}\left[\frac{(\delta)_1}{\Gamma(\beta)}+\frac{(\delta)_2\lambda}{\Gamma(\alpha+\beta)2!}x^{\alpha}+\frac{(\delta)_3\lambda^2}{\Gamma(2\alpha+\beta)3!}x^{2\alpha}+...\right]\\
=&\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}+\lambda x^{\beta-1}\sum_{k=0}^{\infty}\frac{(\delta)_{k+1}\lambda^k}{(k+1)}W(x^{\alpha};\alpha,\beta).
\end{aligned}$$

Then we obtain the following relation

$$D^{\alpha}\left[x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right]=\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}+\lambda x^{\beta-1}\sum_{k=0}^{\infty}\frac{(\delta)_{k+1}\lambda^k}{(k+1)}W(x^{\alpha};\alpha,\beta).$$

Also, the following formula is given

$$D^{\alpha}\left[x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right]=\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}+\lambda x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha}).$$

□

**Remark 2.2.** *1. Since*

$$x^{\beta-1}E^{-1}_{\alpha,\beta}(\lambda x^{\alpha})=\frac{x^{\beta-1}}{\Gamma(\beta)}-\frac{\lambda x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

*then*

$$\begin{aligned}
D^{\alpha}\left[x^{\beta-1}E^{-1}_{\alpha,\beta}(\lambda x^{\alpha})\right] &= \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}-\frac{\lambda x^{\beta-1}}{\Gamma(\beta)}\\
&= x^{\beta-\alpha-1}E^{0}_{\alpha,\beta-\alpha}(\lambda x^{\alpha})-\lambda x^{\beta-1}E^{0}_{\alpha,\beta}(\lambda x^{\alpha}).
\end{aligned}$$

*2. When $\delta=1$ in formula (2.1), then we obtain*

$$D^{\alpha}\left[x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})\right]=\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}+\lambda x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha}). \tag{2.2}$$

*3. When $\delta=1$ and $\beta=1$ in formula (2.1) and $1-\alpha\longrightarrow 0^{+}$, then we have the following intersting formula*

$$D^{\alpha}E_{\alpha}(\lambda x^{\alpha})=\lambda E_{\alpha}(\lambda x^{\alpha}). \tag{2.3}$$

*Also, we can show this formula by another method such as*

$$\begin{aligned} D^{\alpha}E_{\alpha}(\lambda x^{\alpha}) &= D^{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^{k}x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \sum_{k=1}^{\infty}\frac{\lambda^{k}x^{\alpha k-\alpha}}{\Gamma(\alpha(k-1)+1)} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{\alpha(k+1)-\alpha}}{\Gamma(\alpha k+1)} \\ &= \lambda\sum_{k=0}^{\infty}\frac{\lambda^{k}x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \lambda E_{\alpha}(\lambda x^{\alpha}). \end{aligned}$$

*The following figures show some modified Riemann-Liouville derivative of order closed to zero for* $E_{\alpha}(x^{\alpha})$.

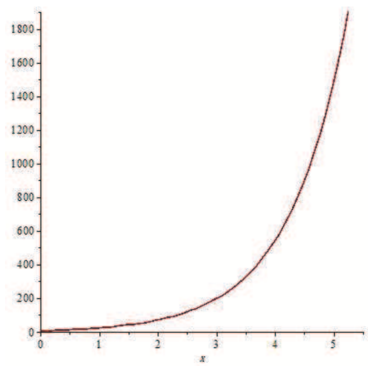


Figure 1: $D^{0.1}E_{0.1}(x^{0.1})$.

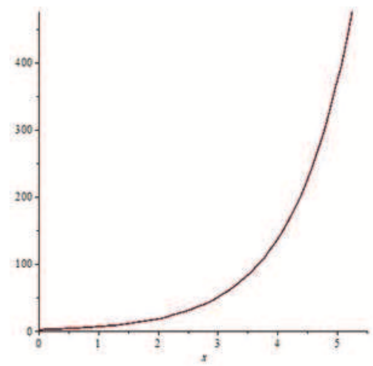


Figure 2: $D^{0.4}E_{0.4}(x^{0.4})$.

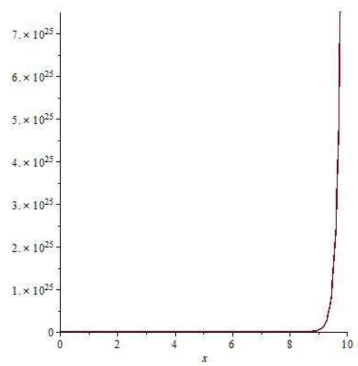


Figure 3: $D^{0.7}E_{0.7}(x^{0.7})$.

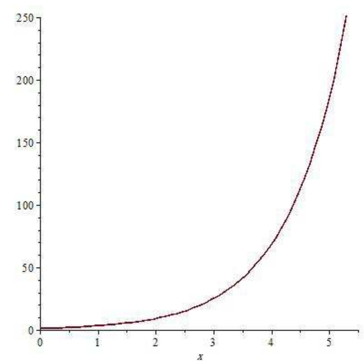


Figure 4: $D^{0.8}E_{0.8}(x^{0.8})$.

**Corollary 2.3.** *Let* $\alpha > 0, \beta > 0$ *and for* $\lambda \in \mathbb{R}$, *then the following formula holds*

$$D^{\alpha}E_{\alpha}\left(\lambda x\right) = \lambda\alpha^{-\alpha}x^{1-\alpha}E_{\alpha}\left(\lambda x\right). \tag{2.4}$$

*Proof.* We can write

$$D^{\alpha}E_{\alpha}\left(\lambda x\right) = D^{\alpha}E_{\alpha}\left(\left(\lambda^{\frac{1}{\alpha}}x^{\frac{1}{\alpha}}\right)^{\alpha}\right).$$

Let $u = \lambda^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}$ and by applying the fractional derivative properties, we get

$$\begin{aligned} D^{\alpha} E_{\alpha}\left(\lambda x\right) &= E_{\alpha}(u^{\alpha}) \left[\lambda^{\frac{1}{\alpha}} \alpha^{-1} x^{\frac{1}{\alpha}-1}\right]^{\alpha} \\ &= \lambda \alpha^{-\alpha} x^{1-\alpha} E_{\alpha}(\lambda x). \end{aligned}$$

□

In the following figures there are some modified Riemann-Liouville derivative of order closed to zero for $E_{\alpha}(x)$.

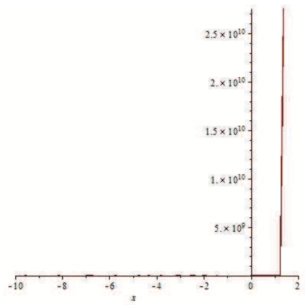


Figure 5: $D^{0.1} E_{0.1}(x)$.

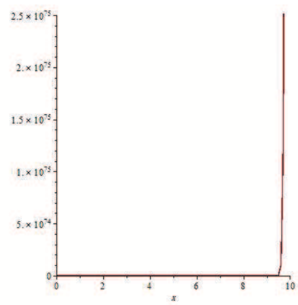


Figure 6: $D^{0.4} E_{0.4}(x)$.

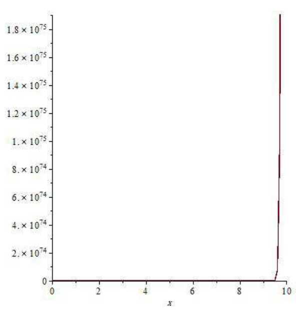


Figure 7: $D^{0.7} E_{0.7}(x)$.

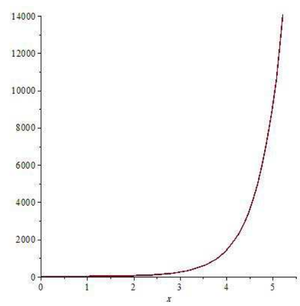


Figure 8: $D^{0.8} E_{0.8}(x)$.

**Theorem 2.4.** *Assume that $\alpha > 0$,$\beta > 0$ for $\lambda \in \mathbb{R}$, then the following formula holds*

$$D^{\gamma} \left[x^{\beta-1} E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right] = \frac{x^{\beta-1-\gamma}}{\Gamma(\beta-\gamma)} + \lambda x^{\alpha+\beta-\gamma-1} E^{\delta}_{\alpha,\alpha+\beta-\gamma}(\lambda x^{\alpha}). \tag{2.5}$$

*Proof.*

$$\begin{aligned} & D^{\gamma} \left[x^{\beta-1} E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right] \\ =& D^{\alpha} \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(\alpha k+\beta) k!} x^{\alpha k+\beta-1} \\ =& D^{\gamma} \left[\frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{(\delta)_1 \lambda}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} + \frac{(\delta)_2 \lambda^2}{\Gamma(2\alpha+\beta) 2!} x^{2\alpha+\beta-1} + \frac{(\delta)_3 \lambda^3}{\Gamma(3\alpha+\beta) 3!} x^{3\alpha+\beta-1} + ...\right] \\ =& x^{\beta-\gamma-1} \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(\alpha k+\beta-\gamma) k!} x^{\alpha k}. \end{aligned}$$

Hence, the relation with the Wright function is

$$D^{\gamma}\left[x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right] = x^{\beta-\gamma-1}\sum_{k=0}^{\infty}(\delta)_k\lambda^k W(x^{\alpha};\alpha,\beta-\gamma).$$

Also,

$$\begin{aligned} D^{\gamma}\left[x^{\beta-1}E^{\delta}_{\alpha,\beta}(\lambda x^{\alpha})\right] &= \frac{x^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \\ &\quad +\lambda x^{\alpha+\beta-\gamma-1}\sum_{k=0}^{\infty}\frac{(\delta)_{k+1}\lambda^k}{\Gamma(\alpha(k+1)+\beta-\gamma)(k+1)!}x^{\alpha k} \\ &= \frac{x^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} + \lambda x^{\alpha+\beta-\gamma-1}E^{\delta}_{\alpha,\alpha+\beta-\gamma}(\lambda x^{\alpha}). \end{aligned}$$

□

**Remark 2.5.** *1. If we set $\gamma = \alpha$ in formula (2.5), then the formula (2.1) is obtained.*

*2. Let $\delta = 1$ in (2.5), then*

$$D^{\gamma}\left[x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})\right] = \frac{x^{\beta-1-\gamma}}{\Gamma(\beta-\gamma)} + \lambda x^{\alpha+\beta-\gamma-1}E_{\alpha,\alpha+\beta-\gamma}(\lambda x^{\alpha}) \tag{2.6}$$

*Also, if $\beta = 1$ and $1-\gamma \longrightarrow 0^{+}$ then*

$$D^{\gamma}E_{\alpha}\left(\lambda x^{\alpha}\right) = \lambda x^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda x^{\alpha}). \tag{2.7}$$

*This formula is also true when $\alpha > 0$ and $0 < \gamma < 1$ for $\lambda \in \mathbb{R}$ by the following method:*

$$\begin{aligned} D^{\gamma}E_{\alpha}(\lambda x^{\alpha}) &= D^{\gamma}\sum_{k=0}^{\infty}\frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \sum_{k=1}^{\infty}\frac{\lambda^k x^{\alpha k-\gamma}}{\Gamma(\alpha k-\gamma+1)} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{\alpha k+\alpha-\gamma}}{\Gamma(\alpha k+\alpha-\gamma+1)} \\ &= \lambda x^{\alpha-\gamma}\sum_{k=0}^{\infty}\frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+\alpha-\gamma+1)} \\ &= \lambda x^{\alpha-\gamma}E_{\alpha,\alpha-\gamma+1}(\lambda x^{\alpha}). \end{aligned}$$

*In the following figures show $D^{\gamma}E_{\alpha}(x^{\alpha}), \lambda = 1$.*

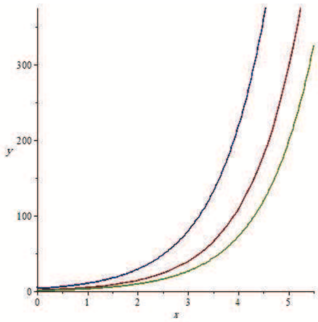

Figure 9: $D^{0.5}E_\alpha(x^\alpha)$
$\alpha = 0.5, 0.25, 0.75.$

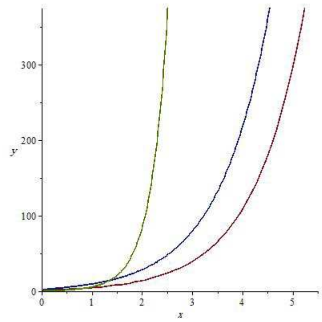

Figure 10: $D^{0.25}E_\alpha(x^\alpha)$
$\alpha = 0.5, 0.25, 0.75.$

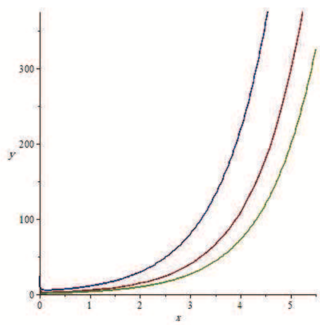

Figure 11: $D^{0.75}E_\alpha(x^\alpha)$
$\alpha = 0.5, 0.25, 0.75.$

*Moreover, we note that* $\dfrac{x^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \longrightarrow 0$ *when* $\beta - \gamma \longrightarrow 0^+$, *then*

$$D^\beta \left[x^{\beta-1} E_{\alpha,\beta}\left(\lambda x^\alpha\right)\right] = \lambda x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha).$$

*3. Asume that* $\alpha = \gamma$, $\beta = 1$ , $\delta = 1$ *and* $1 - \gamma \longrightarrow 0^+$ *in (2.5), then the formula (2.3) is given.*

**Corollary 2.6.** *We can write*

$$D^\beta E_\alpha\left(\lambda x\right) = D^\beta E_\alpha\left(\left(\lambda^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}\right)^\alpha\right).$$

*Let* $u = \lambda^{\frac{1}{\alpha}}.x^{\frac{1}{\alpha}}$ *and by applying the fractional derivative properties, we get*

$$\begin{aligned} D^\beta E_\alpha\left(\lambda x\right) &= u^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(u^\alpha)\left[\lambda^{\frac{1}{\alpha}} \alpha^{-1} x^{\frac{1}{\alpha}-1}\right]^\beta \\ &= \lambda \alpha^{-\beta} x^{1-\beta} E_{\alpha,\alpha-\beta+1}(\lambda x). \end{aligned}$$

*Then*

$$D^\beta E_\alpha\left(\lambda x\right) = \lambda \alpha^{-\beta} x^{1-\beta} E_{\alpha,\alpha-\beta+1}(\lambda x) \tag{2.8}$$

*Let* $\beta = \alpha$ *in the above formula, then*

$$D^\alpha E_\alpha\left(\lambda x\right) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_\alpha(\lambda x).$$

In the following figures show $D^\beta E_\alpha(x), \lambda = 1$.

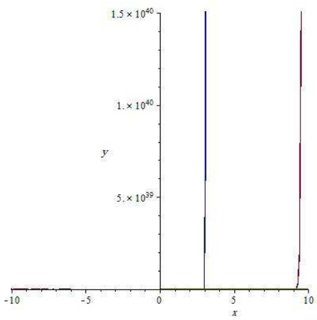


Figure 12: $D^{0.5}E_\alpha(x)$
$\alpha = 0.5, 0.25, 0.75.$

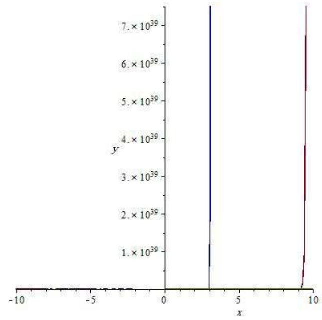


Figure 13: $D^{0.25}E_\alpha(x)$
$\alpha = 0.5, 0.25, 0.75.$

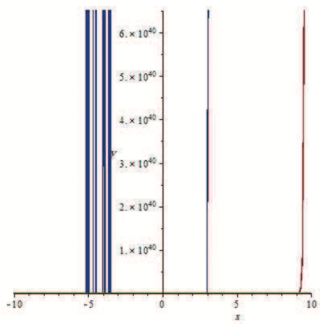


Figure 14: $D^1E_\alpha(x)$
$\alpha = 0.5, 0.25, 0.75.$

**Theorem 2.7.** *Assume that $\alpha > 0, \beta > 0$ for $\lambda \in \mathbb{R}$, then the following formula holds*

$$D^\beta E^\delta_{\alpha,\beta}(\lambda x^{-\alpha}) = (-1)^\beta \lambda x^{-\alpha-\beta} E^\delta_{\alpha,\alpha}(\lambda x^{-\alpha}) \tag{2.9}$$

*Proof.*

$$\begin{aligned} D^\beta E^\delta_{\alpha,\beta}(\lambda x^{-\alpha}) &= D^\beta \sum_{k=0}^\infty \frac{(\delta)_k \lambda^k}{\Gamma(\alpha k+\beta)k!} x^{-\alpha k} \\ &= \sum_{k=0}^\infty \frac{(-1)^\beta (\delta)_{k+1} \lambda^{k+1}}{\Gamma(\alpha k+\alpha)(k+1)!} x^{-\alpha k-\alpha-\beta}. \end{aligned}$$

Then

$$D^\beta E^\delta_{\alpha,\beta}(\lambda x^{-\alpha}) = (-1)^\beta \sum_{k=0}^\infty \frac{(\delta)_{k+1}\lambda^{k+1}x^{-\alpha-\beta}}{k+1} W(x^{-\alpha};\alpha,\alpha).$$

Moreover,

$$D^\beta E^\delta_{\alpha,\beta}(\lambda x^{-\alpha}) = (-1)^\beta \lambda x^{-\alpha-\beta} E^\delta_{\alpha,\alpha}(\lambda x^{-\alpha}).$$

□

When $\delta = 1$, then

$$D^\beta E_{\alpha,\beta}(\lambda x^{-\alpha}) = (-1)^\beta \lambda x^{-\alpha-\beta} E_{\alpha,\alpha}(\lambda x^{-\alpha})$$

**Corollary 2.8.** *Let $u = \lambda^{\frac{-1}{\alpha}} x^{\frac{1}{\alpha}}$, then by using formula (2.9) we have*

$$D^\beta E^\delta_{\alpha,\beta}(\lambda x^{-1}) = (-1)^\beta \lambda^{\frac{\beta}{\alpha}} \alpha^{-\alpha} x^{-\alpha-\frac{\beta}{\alpha}} E^\delta_{\alpha,\alpha}(\lambda x^{-1}). \tag{2.10}$$

*Here when* $\delta = 1$*, then*

$$D^{\beta}E_{\alpha,\beta}(\lambda x^{-1}) = (-1)^{\beta}\lambda^{\frac{\beta}{\alpha}}\alpha^{-\alpha}x^{-\alpha-\frac{\beta}{\alpha}}E_{\alpha,\alpha}(\lambda x^{-1}).$$

**Theorem 2.9.** *Assume that* $\alpha > 0$,$\beta > 0$ *and* $\lambda \in \mathbb{R}$*, then the following formula holds*

$$D^{\alpha n}E_{\alpha}\left(\lambda x^{\alpha}\right) = \lambda^{n}E_{\alpha}\left(\lambda x^{\alpha}\right) \tag{2.11}$$

*where* $n = 1, 2, 3, \cdots$

*Proof.*

$$\begin{aligned} D^{\alpha n}E_{\alpha}(\lambda x^{\alpha}) &= D^{\alpha n}\sum_{k=0}^{\infty}\frac{\lambda^{k}x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \sum_{k=n}^{\infty}\frac{\lambda^{k}x^{\alpha(k-n)}}{\Gamma(\alpha(k-n)+1)} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^{k+n}x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \lambda^{n}\sum_{k=0}^{\infty}\frac{\lambda^{k}x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \lambda^{n}E_{\alpha}(\lambda x^{\alpha}). \end{aligned}$$

□

Note that, if $n = 1$, then we obtain formula (2.3).

**Corollary 2.10.** *Assume that* $\alpha > 0$,$\beta > 0$ *and* $\lambda \in \mathbb{R}$*, then the following formula holds*

$$D^{\alpha n}E_{\alpha}\left(\lambda x\right) = \lambda^{n}\alpha^{-\alpha n}x^{(1-\alpha)n}E_{\alpha}\left(\lambda x\right) \tag{2.12}$$

*where* $n = 1, 2, 3, \cdots$

*Proof.* Let

$$D^{\alpha n}E_{\alpha}\left(\lambda x\right) = D^{\alpha n}E_{\alpha}\left(\left(\lambda^{\frac{1}{\alpha}}x^{\frac{1}{\alpha}}\right)^{\alpha}\right)$$

and put $u = \lambda^{\frac{1}{\alpha}}x^{\frac{1}{\alpha}}$, then by applying the fractional derivative properties, we get

$$\begin{aligned} D^{\alpha n}E_{\alpha}\left(\lambda x\right) &= E_{\alpha}(u^{\alpha})\left[\lambda^{\frac{1}{\alpha}}\alpha^{-1}x^{\frac{1}{\alpha}-1}\right]^{\alpha n} \\ &= \lambda^{n}\alpha^{-\alpha n}x^{(1-\alpha)n}E_{\alpha}(\lambda x). \end{aligned}$$

Let $n = 1$, then formula (2.4) is obtained . □

**Theorem 2.11.** *Assume that* $\alpha > 0$,$\beta > 0$ *and* $\lambda \in \mathbb{R}$*, then the following formula holds*

$$D^{\beta n}E_{\alpha}\left(\lambda x^{\alpha}\right) = \lambda^{n}x^{(\alpha-\beta)n}E_{\alpha,\alpha n-\beta n+1}(\lambda x^{\alpha}) \tag{2.13}$$

*where* $n = 1, 2, 3, \cdots$

*Proof.*

$$\begin{aligned} D^{\beta n} E_\alpha(\lambda x^\alpha) &= D^{\beta n} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \sum_{k=n}^{\infty} \frac{\lambda^k x^{\alpha k-\beta n}}{\Gamma(\alpha k-n\beta+1)} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^{k+n} x^{\alpha k+\alpha n-\beta n}}{\Gamma(\alpha k+\alpha n-\beta n+1)} \\ &= \lambda^n x^{\alpha n-\beta n} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+\alpha n-\beta n+1)} \\ &= \lambda^n x^{\alpha n-\beta n} E_{\alpha,\alpha n-\beta n+1}(\lambda x^\alpha). \end{aligned}$$

Here, when $n=1$, then formula (2.7) is obtained. □

**Corollary 2.12.** *Assume that $\alpha>0$,$\beta>0$ and $\lambda\in\mathbb{R}$, then the following formula holds*

$$D^{\beta n} E_\alpha(\lambda x) = \lambda^n \alpha^{-\beta n} x^{(1-\beta)n} E_{\alpha,\alpha n-\beta n+1}(\lambda x) \tag{2.14}$$

*where $n=1,2,3,\cdots$*

*Proof.* Assume that

$$D^{\beta n} E_\alpha(\lambda x) = D^{\beta n} E_\alpha\left(\left(\lambda^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}\right)^\alpha\right)$$

and let $u=\lambda^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}$, then by applying the fractional derivative properties, we get

$$\begin{aligned} D^{\beta n} E_\alpha(u^\alpha) &= u^{\alpha n-\beta n} E_{\alpha,\alpha n-\beta n+1}(u^\alpha)\left[\lambda^{\frac{1}{\alpha}} \alpha^{-1} x^{\frac{1}{\alpha}-1}\right]^{\beta n} \\ &= \lambda^n \alpha^{-\beta n} x^{(1-\beta)n} E_{\alpha,\alpha n-\beta n+1}(\lambda x). \end{aligned}$$

□

Also, when $n=1$, then formula (2.8) is obtained.

Kiryakova introduced and studied the multi-index Mittag-Leffler function as their typical representatives, including many interesting special cases that have already proven their usefulness in FC and its applications [12].

**Definition 2.13.** *Assume that $n>1$ is an integer, $\eta_1,...,\eta_n>0$ and $\beta_1,...,\beta_n$ are arbitrary real numbers. The multi-idex Mittag-Leffler function is given as*

$$E_{(\frac{1}{\eta_1}),(\beta_i)}(x) = E^{(n)}_{(\frac{1}{\eta_1}),(\beta_i)}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma\left(\frac{k}{\eta_1}+\beta_1\right)\cdots\Gamma\left(\frac{k}{\eta_n}+\beta_n\right)}.$$

The same function was given by Lunchko [17], called by him Mittag-Leffler function of vector index.

Futhermore, the Wright generalized hypergeometric function ${}_m\bar{W}_n$ is defined as

$${}_m\bar{W}_n\left[\begin{array}{c|c} (a_i,A_i)_i^m & \\ (b_j,B_j)_i^n & x \end{array}\right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1k+A_1)\cdots\Gamma(a_mk+A_m)}{\Gamma(b_1k+b_1)\cdots\Gamma(b_nk+B_n)} \frac{x^k}{k!}.$$

The ${}_m\bar{W}_n$ function is special case of the Fox $H$-function

$$H^{p,q}_{m,n}\left[\ x \ \middle|\ \begin{matrix}(a_i,A_i)_i^m\\(b_j,B_j)_i^n\end{matrix}\ \right].$$

In particlar, when $A_i = B_j = 1, \forall i, j$, then Meijer's G-function is obtained

$$H^{p,q}_{m,n}\left[\ x \ \middle|\ \begin{matrix}(a_i,1)_i^m\\(b_j,1)_i^n\end{matrix}\ \right] \quad = \quad G^{p,q}_{m,n}\left[\ x \ \middle|\ \begin{matrix}(a_i)_i^m\\(b_j)_i^n\end{matrix}\ \right].$$

For more detils see [10,11,13,14].

There are some interested properties related to multi- Mittag-Leffler function which were proven in [12]:

1. $E_\alpha ={}_1 \bar{W}_1\left[\begin{matrix}(1,1)\\(\alpha,1)\end{matrix}\ \middle|\ x\right]$.
2. $E_{\alpha,\beta} ={}_1 \bar{W}_1\left[\begin{matrix}(1,1)\\(\alpha,\beta)\end{matrix}\ \middle|\ x\right]$.
3. $E_{\left(\frac{1}{\eta_i}\right),(\beta_i)} ={}_1 \bar{W}_n\left[\begin{matrix}(1,1)\\(\frac{1}{\eta_i},\beta_i)_1^n\end{matrix}\ \middle|\ x\right]$.
4. $E^{\delta}_{\alpha,\beta} = \frac{1}{\Gamma(\delta)}\ {}_1\bar{W}_1\left[\begin{matrix}(1,\delta)\\(\alpha,\beta)\end{matrix}\ \middle|\ x\right]$.

In the same paper, the author showed Wright function as a case of multi- Mittag-Leffler function with $n = 2$:

$$\begin{aligned}W(x;\alpha,\beta) &= \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)k!}\\ &= {}_0\bar{W}_1\left[\begin{matrix}-\\(\alpha,\beta)\end{matrix}\ \middle|\ x\right]\\ &= E^{(2)}_{(\alpha,1),(\beta,1)}(x).\end{aligned}$$

Indeed, the multi-idex Mittaag-Leffler function when $\beta_i = 1, \forall i$ can be written as

$$E_{\left(\frac{1}{\eta_i}\right)}(x^{\eta_i}) = \sum_{k=0}^{\infty}\frac{x^{\eta_i k}}{\prod_{i=1}^{n}\Gamma\left(\frac{k}{\eta_i}+1\right)}.$$

Then

$$\begin{aligned}D^{\frac{1}{\eta_i}}E_{\left(\frac{1}{\eta_i}\right)}(x^{\frac{1}{\eta_i}}) =& D^{\eta_i}\sum_{k=0}^{\infty}\frac{x^{\eta_i k}}{\prod_{i=1}^{n}\Gamma\left(\frac{k}{\eta_i}+1\right)}\\ =&\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k}{\eta_i}+\frac{1}{\eta_i}+1\right)}{\Gamma\left(\frac{k}{\eta_i}+1\right)\prod_{i=1}^{n}\Gamma\left(\frac{k}{\eta_i}+\frac{1}{\eta_1}+1\right)}x^{\frac{k}{\eta_i}}\\ =&\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k}{\eta_i}+\frac{1}{\eta_i}+1\right)}{\Gamma\left(\frac{k}{\eta_i}+1\right)}\ {}_1\bar{W}_n\left[\begin{matrix}(1,1)\\\left(\frac{1}{\eta_i},\frac{1}{\eta_i}+1\right)_1^n\end{matrix}\ \middle|\ x^{\frac{1}{\eta_i}}\right].\end{aligned} \tag{2.15}$$

Here, if we set $\alpha = \frac{1}{\eta_i}$ and $n = 1$, then we obtain formula (2.3), $\lambda = 1$.

$$D^{\frac{1}{\eta_i}} E_{\left(\frac{1}{\eta_i}\right)}(x) = \eta_i^{\frac{1}{\eta_i}} x^{1-\frac{1}{\eta_i}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + 1\right)} \ {}_1\bar{W}_n \left[ \begin{array}{c} (1,1) \\ \left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n \end{array} \middle| \, x \right]. \tag{2.16}$$

The above formula can be obtained by putting $u = (x^{\eta_i})^{\frac{1}{\eta_i}}$ and then applying formula (2.15). Especially, if $\alpha = \frac{1}{\eta_i}$ and $n = 1$, formula (2.16) yields to the formula (2.4) when $\lambda = 1$.

**Theorem 2.14.** *Assume that $\eta_i > 0$ are arbitrary real numbers and $0 < \gamma < 1$, then the following formula holds*

$$D^{\gamma} E_{\left(\frac{1}{\eta_i}\right)} \left(x^{\frac{1}{\eta_i}}\right) = x^{\frac{1}{\eta_i} - \gamma} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \beta + 1\right)} \ {}_1\bar{W}_n \left[ \begin{array}{c} (1,1) \\ \left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n \end{array} \middle| \, x^{\frac{1}{\eta_i}} \right]. \tag{2.17}$$

*Proof.*

$$\begin{aligned} D^{\gamma} E_{\left(\frac{1}{\eta_i}\right)} \left(x^{\frac{1}{\eta_i}}\right) &= D^{\gamma} \sum_{k=0}^{\infty} \frac{x^{\frac{k}{\eta_i}}}{\prod_{i=1}^{n} \Gamma\left(\frac{k}{\eta_i} + 1\right)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1)}{\Gamma(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma + 1) \prod_{i=1}^{n} \Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)} x^{\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma}. \end{aligned}$$

which is the result. ☐

As expected when $\alpha = \frac{1}{\eta_i}$ and $n = 1$,the last formula turns to be the formula (2.7) when $\lambda = 1$.

Since $D^{\gamma} E_{\left(\frac{1}{\eta_i}\right)}(x)$ can be written as $D^{\gamma} E_{\left(\frac{1}{\eta_i}\right)} \left((x^{\eta_i})^{\frac{1}{\eta_i}}\right)$ and by appling (2.17), the following formula is given:

$$D^{\gamma} E_{\left(\frac{1}{\eta_i}\right)}(x) = \eta_i^{\gamma} x^{1-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma + 1\right)} \ {}_1\bar{W}_n \left[ \begin{array}{c} (1,1) \\ \left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n \end{array} \middle| \, x \right]. \tag{2.18}$$

We would like to mention that if $\alpha = \frac{1}{\eta_i}$ and $n = 1$ in formula (2.18), then (2.8) is obtained.

**Corollary 2.15.** *For arbitrary $n \geq 2$, let $\forall \eta_i = \infty$ and $\forall \beta_i = 1, i = 1, \cdots, n$. Then*

$$D^{\gamma} E^{(n)}_{(0,0,\cdots,0),(1,1,\cdots,1)}(x) = x^{-\gamma} \sum_{k=0}^{\infty} \Gamma(k+1) \ {}_0\bar{W}_1 \left[ \begin{array}{c} - \\ (1, 1-\gamma) \end{array} \middle| \, x \right]$$

Now, we study modified Riemann-Liouville derivitive of fractional Sine and Cosine function. Since

$$\cos_{\alpha}(t^{\alpha}) = \frac{1}{2} \left[ E_{\alpha}(it^{\alpha}) + E_{\alpha}(-it^{\alpha}) \right],$$

then

$$\begin{aligned} D^{\alpha} \cos_{\alpha}(t^{\alpha}) &= \frac{1}{2} \left[ iE_{\alpha}(it^{\alpha}) - iE_{\alpha}(-it^{\alpha}) \right] \\ &= -\sin_{\alpha}(t^{\alpha}). \end{aligned}$$

Hence, we get a very useful relation

$$D^{\alpha} \cos_{\alpha}(t^{\alpha}) = -\sin_{\alpha}(t^{\alpha}).$$

By using the same technique we can write

$$D^{\alpha}\sin_{\alpha}(t^{\alpha}) = \cos_{\alpha}(t^{\alpha}).$$

Moreover, since $\cos_{\alpha}(t) = \frac{1}{2}\left[E_{\alpha}(it) + E_{\alpha}(-it)\right]$, then

$$\begin{aligned} D^{\alpha}\cos_{\alpha}(t) &= \frac{1}{2}\left[i\alpha^{-\alpha}t^{1-\alpha}E_{\alpha}(it) - i\alpha^{-\alpha}t^{1-\alpha}E_{\alpha}(-it)\right] \\ &= -\alpha^{-\alpha}t^{1-\alpha}\sin_{\alpha}(t). \end{aligned}$$

The following figures show $D^{\alpha}\cos_{\alpha}(x)$ when $\alpha = 0.3, 0.5$ and $0.75$:

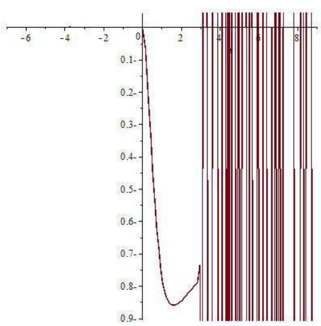

Figure 15: $D^{0.3}\cos_{0.3}(x)$

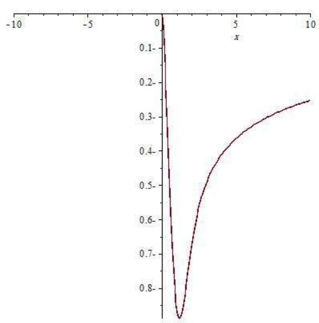

Figure 16: $D^{0.5}\cos_{0.5}(x)$

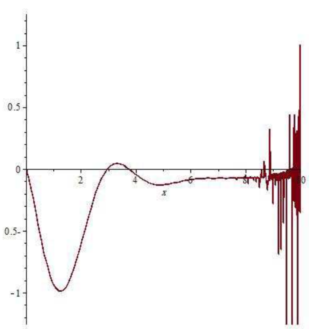

Figure 17: $D^{0.75}\cos_{0.75}(x)$

Also, we can write

$$D^{\alpha}\sin_{\alpha}(t) = \alpha^{-\alpha}t^{1-\alpha}\cos_{\alpha}(t).$$

The next figures show $D^{\alpha}\sin_{\alpha}(x)$ when $\alpha = 0.3, 0.5$ and $0.75$:

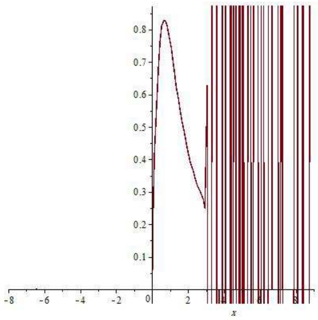

Figure 18: $D^{0.3}\sin_{0.3}(x)$

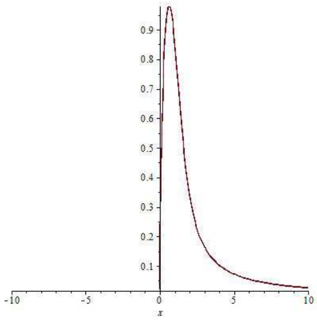

Figure 19: $D^{0.5}\sin_{0.5}(x)$

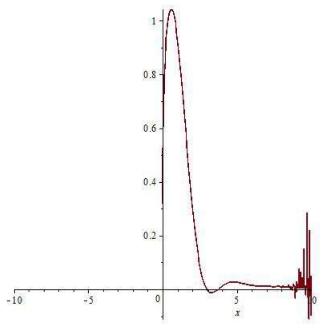

Figure 20: $D^{0.75}\sin_{0.75}(x)$

The next step we study $D^{\beta}\cos_{\alpha}(t^{\alpha})$ and $D^{\beta}\cos_{\alpha}(t)$.

$$\begin{aligned} D^{\beta}\cos_{\alpha}(t^{\alpha}) &= \frac{1}{2}\left[it^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(it^{\alpha}) - it^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-it^{\alpha})\right] \\ &= -t^{\alpha-\beta}\sin_{\alpha,\alpha-\beta+1}(t^{\alpha}), \end{aligned}$$

where

$$\sin_{\alpha,\alpha-\beta+1}(t^{\alpha}) = \frac{t^{\alpha}}{\Gamma(2\alpha-\beta+1)} - \frac{t^{3\alpha}}{\Gamma(4\alpha-\beta+1)} + \frac{t^{5\alpha}}{\Gamma(6\alpha-\beta+1)} - \cdots .$$

Similarly we can show that

$$D^{\beta}\sin_{\alpha}(t^{\alpha}) = t^{\alpha-\beta}\cos_{\alpha,\alpha-\beta+1}(t^{\alpha}),$$

where

$$\cos_{\alpha,\alpha-\beta+1}(t^{\alpha}) = \frac{1}{\Gamma(\beta)} - \frac{t^{2\alpha}}{\Gamma(3\alpha-\beta+1)} + \frac{t^{4\alpha}}{\Gamma(5\alpha-\beta+1)} - \cdots .$$

$$\begin{aligned} D^{\beta}\cos_{\alpha}(t) &= \frac{1}{2}\left[i\alpha^{-\beta}t^{1-\beta}E_{\alpha,\alpha-\beta+1}(it) - i\alpha^{-\beta}t^{1-\beta}E_{\alpha,\alpha-\beta+1}(-it)\right] \\ &= -\alpha^{-\beta}t^{1-\beta}\sin_{\alpha,\alpha-\beta+1}(t). \end{aligned}$$

Similarly

$$D^{\beta}\sin_{\alpha}(t) = \alpha^{-\beta}t^{1-\beta}\cos_{\alpha,\alpha-\beta+1}(t).$$

**Theorem 2.16.** *The fractional derivative of hyperbolic function of order $m$ is given as*

$$D^{\alpha}\left[h_v(x,m)\right] = \frac{x^{v-\alpha-1}}{\Gamma(v-\alpha)} + x^{v+m-\alpha-1}E_{m,v+m-\alpha}(x^m), v = 1, 2, \cdots .$$

*when $v-\alpha \longrightarrow 0^{+}$, then*

$$D^{\alpha}\left[h_v(x,m)\right] = x^{m-1}E_{m,m}(x^m).$$

*Proof.* Since hyperbolic function of order $m$ is defined as

$$h_v(x,m) = \sum_{k=0}^{\infty} \frac{x^{mk+v-1}}{\Gamma(mk+v)} = x^{v-1} E_{m,v}(x^m), v = 1, 2, \cdots,$$

then by using formula (2.6) we get the result. □

**Theorem 2.17.** *The fractional derivative of Mellin- Ross function,*

$$R_\alpha(\beta, x) = x^\alpha \sum_{k=0}^{\infty} \frac{\beta^k x^{k(\alpha+1)}}{\Gamma((1+\alpha)(k+1)} = x^\alpha E_{\alpha+1,\alpha+1}(\beta x^{\alpha+1}),$$

*is given by*

$$D^\alpha \left[ x^\alpha E_{\alpha+1,\alpha+1}(\beta x^{\alpha+1}) \right] = \lambda x^\alpha E_{\alpha+1,\alpha+1}(x^{\alpha+1})$$

The proof is directed by using formula (2.2).

## 3. Conclusion

In this note, some useful formulas have been established by using modified Riemann-Liouville definition of fractional derivative. These formulas can be used to solve some linear fractional differential equations which are useful in several physical problems.

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