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Subdivisions of the Spectra for $D(r, 0, s, 0, t)$ Operator on Certain Sequence Spaces

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ABSTRACT: In this paper we have examined the approximate point spectrum, defect spectrum and compression spectrum of the operator $D(r, 0, s, 0, t)$ on the sequence spaces c_0, c and $bv_p (1 < p < \infty)$.

Key Words: Fine spectrum, Approximate point spectrum, Defect spectrum, Compression spectrum.

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1. Preliminaries and Definition

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on X into itself is denoted by $B(X)$. The adjoint $T^* : X^* \to X^*$ of T is defined by $(T^*\Phi)(x) = \Phi(Tx)$ for all $\Phi \in X^*$ and $x \in X$. Clearly, T^* is a bounded linear operator on the dual space X^*.

Let $T : D(T) \to X$ a linear operator, defined on $D(T) \subseteq X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T. Many properties of T_α and T_α^{-1} depend on α and spectral theory is concerned with those properties. We are interested in the set of all α in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another essential property. We also determine $\alpha's$ for which the domain of T_α^{-1} is dense in X.

A **regular value** is a complex number α of T such that
$(R_1) T_\alpha^{-1}$ exists,
$(R_2) T_\alpha^{-1}$ is bounded
and
$(R_3) T_\alpha^{-1}$ is defined on a set which is dense in X.

The **resolvent set** of T is the of all such regular values α of T, denoted by $\rho(T, X)$. Its complement is given by $C \setminus \rho(T, X)$ in the complex plane C is called the **spectrum** of T, denoted by $\sigma(T, X)$. Thus the spectrum $\sigma(T, X)$ consist of those values of $\alpha \in C$, for which T_α is not invertible.

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2. Subdivisions of the spectrum

In this section, we discuss about the point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular in quantum mechanics.

2.1. The point spectrum, continuous spectrum and residual spectrum

The spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:
(i) The **point (discrete) spectrum** $\sigma_p(T,X)$ is the set of complex numbers α such that T_α^{-1} does not exist. Further $\sigma_p(T,X)$ is called the eigen value of T.

(ii) The **continuous spectrum** $\sigma_c(T,X)$ is the set of complex numbers α such that T_α^{-1} exists and satisfies (R_3) but not (R_2) that is T_α^{-1} unbounded.

(iii) The **residual spectrum** $\sigma_r(T,X)$ is the set of complex numbers α such that T_α^{-1} exists (and may be bounded or not) but not satisfy (R_3), that is, the domain of T_α^{-1} is not dense in X.

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and equal to the empty set and the spectrum consists of only the point spectrum.

2.2. The approximate point spectrum, defect spectrum and compression spectrum

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a Weyl sequence for T if $||x_k|| = 1$ and $||Tx_k|| \to 0$, as $k \to \infty$.

Appell et al. [4], have been given three more classification of spectrum called the approximate point spectrum, defect spectrum and compression spectrum.

(a) The **approximate point spectrum**:

$\sigma_{ap}(T,X) = \{\alpha \in C\text{: there exist a Weyl sequence for } T - \alpha I\}$.

(b) The **defect spectrum**: $\sigma_\delta(T,X) = \{\alpha \in C : T - \alpha I \text{ is not surjective}\}$.

(c) The **compression spectrum**: $\sigma_{co}(T,X) = \{\alpha \in C : \overline{R(T - \alpha I)}\}$.

The two subspectra given by (a) and (b) form a (not necessarily disjoint) subdivisions $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$ of the spectrum.

The compression spectrum gives rise to another subdivisions (not necessarily disjoint) decomposition $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$ of the spectrum.

Clearly $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with $\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X)$

we note that $\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X)$ and $\sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_p(T,X) \cup \sigma_{co}(T,X)]$.

Proposition 2.3 [Appell et al. [4], *Proposition* 1.3, p.28] *Spectra and subspectra of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations*:
$(i)\sigma(T^*,X^*) = \sigma(T,X)$.
$(ii)\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.
$(iii)\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$.
$(iv)\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$.
$(v)\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.

$(vi)\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.
$(vii)\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

2.3. Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:
$(I)R(T) = X$,
$(II)R(T) \neq \overline{R(T)} = X$,
$(III)\overline{R(T)} \neq X$,

and

$(1)T^{-1}$ exists and is continuous,
$(2)T^{-1}$ exists but is discontinuous,
$(3)T^{-1}$ does not exist.
If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in the state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous.

Table 1: Subdivisions of spectrum of a linear operator

		1	2	3
		T_α^{-1} exists and is bounded	T_α^{-1} exists and is unbounded	T_α^{-1} does not exist
I	$R(T-\alpha I) = X$	$\alpha \in \rho(T,X)$	-	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$
II	$\overline{R(T-\alpha I)} = X$	$\alpha \in \rho(T,X)$	$\alpha \in \sigma_c(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$
III	$\overline{R(T-\alpha I)} \neq X$	$\alpha \in \sigma_r(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$	$\alpha \in \sigma_r(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N = 0, 1, 2, \ldots$. Then, we say that A defines a matrix mapping from E into F, denote by $A : E \to F$, if for every sequence $x = (x_n) \in E$ the sequence $Ax = \{(Ax)_n\}$ is in F where $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k (n \in N$ and $x \in E)$, provided the right hand side converges for every $n \in N$ and $x \in E$.

Throughout the paper $w, \ell_\infty, c, c_0, \ell_p$ and bv_p denote the space of all, bounded, convergent, null, p-absolutely summable and p-bounded variation sequences respectively. The zero sequence is denoted by $\theta = (0, 0, ..., .)$.

Let $m, n \geq 0$ be fixed integers, then Esi, Tripathy and Sarma [8] has introduced the following type of difference sequence spaces.$Z(\triangle_m^n) = \{x = (x_k) \in w : \Delta_m^n x = (\Delta_m^n x_k) \in Z\}$ for $Z = \ell_\infty, c$ and c_0, where $\Delta_m^n x = (\Delta_m^n x_k) = (\triangle_m^{n-1} x_k - \triangle_m^{n-1} x_{k+m})$ and $\triangle_m^0 x_k = x_k$ for all $k \in N$.

Taking $n = 1$, we have the sequence spaces $\ell(\triangle_m), c(\triangle_m)$ and $c_0(\triangle_m)$ studied by Tripathy and Esi [15].

Taking $m = 1$, we have the sequence spaces $\triangle(\triangle^n), c(\triangle^n)$ and $c_0(\triangle^n)$ studied by Et and Colak [7].

Taking $m = 1$ and $n = 1$, we have the sequence spaces $\ell_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ studied by Kizmaz [10].

Our main focus in this paper is on the operator $D(r, 0, s, 0, t)$ represented by the following matrix

$$D(r,0,s,0,t) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & r & 0 & 0 & 0 & 0 & . & . & . \\ s & 0 & r & 0 & 0 & 0 & . & . & . \\ 0 & s & 0 & r & 0 & 0 & . & . & . \\ t & 0 & s & 0 & r & 0 & . & . & . \\ 0 & t & 0 & s & 0 & r & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix}$$

Here we assume that s and t are complex parameters which do not simultaneously vanish.

Remark: In particular if we consider $r = 1, s = -2$ and $t = 1$ then $D(1, 0, -2, 0, 1) = \triangle_2^2$.

The spectra of the difference operator has been investigated on different classes of sequences by various authors in the recent past. Altay and Basar ([1], [2], [3]) studied the spectra of difference operator and generalized difference operator on c_0, c and ℓ_p . Tripathy and Paul ([16],[18],[19]) studied the spectra of the difference type operators $D(r, 0, 0, s)$ and $D(r, 0, s, 0, t)$ over the sequence spaces c_0, c, ℓ_p and bv_p. Moreover, Paul and Tripathy ([11],[13]) have investigated the fine spectra of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p, bv_p and bv_0 respectively. Recently Tripathy and Paul [17] studied the spectrum of the operator $B(f, g)$ on the vector valued sequence space $c_0(X)$. Basar et.al ([5],[6]) have studied the subdivisions of the spectra for the generalized difference operator $B(r, s)$ and the triple band matrix $B(r, s, t)$ over the sequence spaces c_0, c and ℓ_p and bv_p. Paul and Tripathy [12] have investigated the subdivisions of the spectra for the operator $D(r, 0, 0, s)$ over the sequence spaces c_0, c, ℓ_p and bv_p. Das and Tripathy [14] studied the spectra of the lower triangular matrix $B(r, s, t)$ over the sequence space cs and Tripathy and Das [20] studied about upper triangular matrix $U(r, s)$ over the sequence space cs.

Lemma 2.5 [17]. *Let s be a complex number such that $\sqrt{s^2} = -s$ and defined the set by*

$$S = \left\{ \alpha \in C : \left| \frac{2(r-\alpha)}{-s + \sqrt{s^2 - 4t(r-\alpha)}} \right| \leq 1 \right\}.$$

Then $\sigma(D(r, 0, s, 0, t), c_0) = S$.

Lemma 2.6 [18] $\sigma_{pt}(D(r, 0, s, 0, t), c_0) = \emptyset$.

Lemma 2.7 [18] $\sigma_r(D(r, 0, s, 0, t), c_0) = S_1$, *where*

$$S_1 = \left\{ \alpha \in C : \left| \frac{2(r-\alpha)}{-s + \sqrt{s^2 - 4t(r-\alpha)}} \right| < 1 \right\}.$$

Lemma 2.8 [18] $\sigma(D(r, 0, s, 0, t), c) = S$, *where S is define as in Lemma* 2.5.

Lemma 2.9 [18] $\sigma_{pt}(D(r, 0, s, 0, t), c) = \emptyset$.

Lemma 2.10 [18] $\sigma_r(D(r, 0, s, 0, t), c) = S_1 \cup \{r + s + t\}$, *where S_1 is defined as in Lemma* 2.7.

Lemma 2.11 [19] $\sigma(D(r, 0, s, 0, t), \ell_p) = S$, *where S is define as in Lemma* 2.5.

Lemma 2.12 [19] $\sigma_{pt}(D(r, 0, s, 0, t), \ell_p) = \emptyset$.

Lemma 2.13 [19] $\sigma_r(D(r, 0, s, 0, t), \ell_p) = S_1$, *where S_1 is defined as in Lemma* 2.7.

Lemma 2.14 [19] *$\sigma(D(r,0,s,0,t),bv_p)=S$, where S is defined as in Lemma 2.5.*

Lemma 2.15 [19] *$\sigma_{pt}(D(r,0,s,0,t),bv_p)=\emptyset$.*

Lemma 2.16 [19] *$\sigma_r(D(r,0,s,0,t),bv_p)=S_1$, where S_1 is defined as in Lemma 2.7.*

3. Subdivisions of the spectrum of $D(r,0,s,0,t)$ over c_0

In this section, we give the subdivisions of the spectrum of the operator $D(r,0,s,0,t)$ over the sequence space c_0.

Theorem 3.1. *If $\alpha=r$, then $\alpha\in III_1\sigma(D(r,0,s,0,t),c_0)$.*

Proof: Let $\alpha=r$, then by Lemma 2.7, $D(r,0,s,0,t)-rI=D(0,0,s,0,t)$ is in state III_1 or III_2. The left inverse of $D(0,0,s,0,t)$ is given by

$$D(0,0,s,0,t)^{-1}=\begin{bmatrix} 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & & . & . \\ . & . & . & . & . & . & & & . \end{bmatrix}$$

Clearly $D(0,0,s,0,t)^{-1}\in B(c_0)$ for all t and s. That is, $D(0,0,s,0,t)$ has a continuous inverse for all t and s. Hence $\alpha\in III_1\sigma(D(r,0,s,0,t),c_0)$.

Theorem 3.2 *If $\alpha\neq r$ and $\alpha\in\sigma_r(D(r,0,s,0,t),c_0)$ then*

$$\alpha\in III_2\sigma(D(r,0,s,0,t),c_0).$$

Proof: Since

$$\sigma_r(D(r,0,s,0,t),c_0)=III_1\sigma(D(r,0,s,0,t),c_0)\cup III_2\sigma(D(r,0,s,0,t),c_0).$$

Now, $\alpha\in\sigma_r(D(r,0,s,0,t),c_0)$ implies either $\alpha\in III_1\sigma(D(r,0,s,0,t),c_0)$ or $\alpha\in III_2\sigma(D(r,0,s,0,t),c_0)$. Since from the Theorem 3.1, $\alpha\in III_1\sigma(D(r,0,s,0,t),c_0)$ if $\alpha=r$.

As $\alpha\neq r$, hence $\alpha\in III_2\sigma(D(r,0,s,0,t),c_0)$.

Theorem 3.3. *$III_3\sigma(D(r,0,s,0,t),c_0)=\emptyset$..*

Proof: $III_3\sigma(D(r,0,s,0,t),c_0)=\sigma_p(D(r,0,s,0,t),c_0)=\emptyset$ is obtained by Lemma 2.6.

Theorem 3.4. *$\sigma_{co}(D(r,0,s,0,t),c_0)=S_1$, where S_1 is defined as in Lemma 2.7.*

Proof:

$$\begin{aligned}\sigma_{co}(D(r,0,s,0,t),c_0) &= III_1\sigma(D(r,0,s,0,t),c_0)\cup III_2\sigma(D(r,0,s,0,t),c_0)\\ &\quad \cup III_3\sigma(D(r,0,s,0,t),c_0).\end{aligned}$$

Now, $III_1\sigma(D(r,0,s,0,t),c_0)\cup III_2\sigma(D(r,0,s,0,t),c_0)=\sigma_r(D(r,0,s,0,t),c_0)=S_1$ is obtained by Lemma 2.7. Again, $III_3\sigma(D(r,0,s,0,t),c_0)=\emptyset$ is obtained by Theorem 3.3.

Hence, $\sigma_{co}(D(r,0,s,0,t),c_0)=S_1$.

Theorem 3.5. $\sigma_{ap}(D(r,0,s,0,t),c_0) = S \setminus \{r\}$, *where S is define as in Lemma* 2.5.

Proof: Since

$$\sigma_{ap}(D(r,0,s,0,t),c_0) = \sigma(D(r,0,s,0,t),c_0) \setminus III_1\sigma(D(r,0,s,0,t),c_0),$$

$\sigma_{ap}(D(r,0,s,0,t),c_0) = S \setminus \{r\}$ is obtained by Lemma 2.5 and Theorem 3.1.

Theorem 3.6. $\sigma_\delta(D(r,0,s,0,t),c_0) = S$, *where S is as define in Lemma* 2.5.

Proof: Since

$$\sigma_\delta(D(r,0,s,0,t),c_0) = \sigma(D(r,0,s,0,t),c_0) \setminus I_3\sigma(D(r,0,s,0,t),c_0).$$

Now, $I_3\sigma(D(r,0,s,0,t),c_0) \cup II_3\sigma(D(r,0,s,0,t),c_0) \cup III_3\sigma(D(r,0,s,0,t),c_0) = \sigma_p(D(r,0,s,0,t),c_0) = \emptyset$ is obtained by Lemma 2.6 and hence

$$I_3\sigma(D(r,0,s,0,t),c_0) = \emptyset.$$

Thus, $\sigma_\delta(D(r,0,s,0,t),c_0) = \emptyset$.

As a consequence of proposition 2.3, we have the following results.

Corollary 3.7. *The following results hold*:
$(i)\sigma_{ap}(D(r,0,s,0,t)^*, c_0^*) = S$
$(ii)\sigma_\delta(D(r,0,s,0,t)^*, c_0^*) = S \setminus \{r\}$ *where S is define as in Lemma* 2.5.

4. Subdivisions of the spectrum of $D(r,0,s,0,t)$ over c

In this section, we give the subdivisions of the spectrum of the operator $D(r,0,s,0,t)$ over the sequence space c.

Theorem 4.1 *If* $\alpha = r$, *then* $\alpha \in III_1\sigma(D(r,0,s,0,t),c)$.

Proof: This theorem can be established in a way similar to that of the proof of Theorem 3.1.

Theorem 4.2 *If* $\alpha \neq r$ *and* $\alpha \in \sigma_r(D(r,0,s,0,t),c)$ *then*

$$\alpha \in III_2\sigma(D(r,0,s,0,t),c).$$

Proof: This is obtained in the similar way that is used in the proof of Theorem 3.2.

Theorem 4.3 $III_3\sigma(D(r,0,s,0,t),c) = \emptyset$..

Proof: This is obtained in the similar way that is used in the proof of Theorem 3.3.

Theorem 4.4 $\sigma_{co}(D(r,0,s,0,t),c) = S_1 \cup \{r+s+t\}$,*where* S_1 *is defined as in Lemma* 2.7.

Proof:$\sigma_{co}(D(r,0,s,0,t),c) = III_1\sigma(D(r,0,s,0,t),c) \cup III_2\sigma(D(r,0,s,0,t),c) \cup III_3\sigma(D(r,0,s,0,t),c)$. Now,

$$\begin{aligned} III_1\sigma(D(r,0,s,0,t),c) \cup III_2\sigma(D(r,0,s,0,t),c) &= \sigma_r(D(r,0,s,0,t),c) \\ &= S_1 \cup \{r+s+t\}, \end{aligned}$$

is obtained by Lemma 2.10. Again $III_3\sigma(D(r,0,s,0,t),c) = \emptyset$ is obtained by Theorem 4.3. Hence, $\sigma_{co}(D(r,0,s,0,t),c) = S_1 \cup \{r+s+t\}$

Theorem 4.5 $\sigma_{ap}(D(r,0,s,0,t),c) = S \setminus \{r\}$, *where* S *is define as in Lemma* 2.5.

Proof: Since

$$\sigma_{ap}(D(r,0,s,0,t),c) = \sigma(D(r,0,s,0,t),c) \setminus III_1\sigma(D(r,0,s,0,t),c),$$

$\sigma_{ap}(D(r,0,s,0,t),c) = S \setminus \{r\}$ is obtained by Lemma 2.8 and Theorem 4.1.

Theorem 4.6 $\sigma_\delta(D(r,0,s,0,t),c) = S$, *where* S *is define as in Lemma* 2.5.

Proof: Since, $\sigma_\delta(D(r,0,s,0,t),c) = \sigma(D(r,0,s,0,t),c) \setminus I_3\sigma(D(r,0,s,0,t),c)$.

Now, $I_3\sigma(D(r,0,s,0,t),c) \cup II_3\sigma(D(r,0,s,0,t),c) \cup III_3\sigma(D(r,0,s,0,t),c) = \sigma_p(D(r,0,s,0,t),c) = \emptyset$ is obtained by Lemma 2.9 and hence

$$I_3\sigma(D(r,0,s,0,t),c) = \emptyset.$$

Thus, $\sigma_\delta(D(r,0,s,0,t),c) = S$.

As a consequence of proposition 2.3, we have the following results.

Corollary 4.7. *The following results hold*:

(i)$\sigma_{ap}(D(r,0,s,0,t)^*, \ell_1) = S$

(ii)$\sigma_\delta(D(r,0,s,0,t)^*, \ell_1) = S \setminus \{r\}$ *where* S *is define as in Lemma* 2.5.

5. Subdivisions of the spectrum of $D(r,0,s,0,t)$ on $(1 < p < \infty)$

In this section, we give the subdivisions of the spectrum of the operator $D(r,0,s,0,t)$ over the sequence space where $1 < p < \infty$.

Theorem 5.1. *If* $\alpha = r$, *then* $\alpha \in III_1\sigma(D(r,0,s,0,t),\ell_p)$.

Proof: If $\alpha = r$, then by Lemma 2.13, $D(r,0,s,0,t) - rI = D(0,0,s,0,t)$ is in state III_1 or III_2. The left inverse of $D(0,0,s,0,t)$ is given by

$$D(0,0,s,0,t)^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & & . & . \\ . & . & . & . & . & . & & & . \end{bmatrix}$$

Then, $D(0,0,s,0,t)^{-1} \in (\ell_1 : \ell_1) \cap (\ell_\infty : \ell_\infty)$ that is, $D(0,0,s,0,t)^{-1} \in (\ell_p : \ell_p)$ for all t and s. Thus, $D(0,0,s,0,t)$ has a continuous inverse for all t and s. Hence $\alpha \in III_1\sigma(D(r,0,s,0,t),\ell_p)$.

Theorem 5.2. *If* $\alpha \neq r$ *and* $\alpha \in \sigma_r(D(r,0,s,0,t),\ell_p)$ *then*

$$\alpha \in III_2\sigma(D(r,0,s,0,t),\ell_p).$$

Proof: This is obtained in the similar way that is used in the proof of Theorem 3.2.

Theorem 5.3.$III_3\sigma(D(r,0,s,0,t),\ell_p)=\emptyset$..

Proof: This is obtained in the similar way that is used in the proof of Theorem 3.3.

Theorem 5.4. *$\sigma_{co}(D(r,0,s,0,t),\ell_p)=S_1$, where S_1 is as defined in Lemma* 2.7.

Proof:

$$\begin{aligned}\sigma_{co}(D(r,0,s,0,t),\ell_p) &= III_1\sigma(D(r,0,s,0,t),\ell_p)\cup III_2\sigma(D(r,0,s,0,t),\ell_p)\\ &\quad \cup III_3\sigma(D(r,0,s,0,t),\ell_p).\end{aligned}$$

Now, $III_1\sigma(D(r,0,s,0,t),\ell_p)\cup III_2\sigma(D(r,0,s,0,t),\ell_p)=\sigma_r(D(r,0,s,0,t),\ell_p)=S_1$, is obtained by Lemma 2.13. Again, $III_3\sigma(D(r,0,s,0,t),\ell_p)=\emptyset$, is obtained by Theorem 5.3.

Hence $\sigma_{co}(D(r,0,s,0,t),\ell_p)=S_1$.

Theorem 5.5. *$\sigma_{ap}(D(r,0,s,0,t),\ell_p)=S\setminus\{r\}$,where S is as defined in Lemma* 2.5.

Proof: Since

$$\sigma_{ap}(D(r,0,s,0,t),\ell_p)=\sigma(D(r,0,s,0,t),\ell_p)\setminus III_1\sigma(D(r,0,s,0,t),\ell_p),$$

$\sigma_{ap}(D(r,0,s,0,t),\ell_p)=S\setminus\{r\}$ is obtained by Lemma 2.11 and Theorem 5.1.

Theorem 5.6. *$\sigma_\delta(D(r,0,s,0,t),\ell_p)=S$, where S is as defined in Lemma* 2.5.

Proof: Since $\sigma_\delta(D(r,0,s,0,t),\ell_p)=\sigma(D(r,0,s,0,t),\ell_p)\cup I_3\sigma(D(r,0,s,0,t),\ell_p)$. Now,

$$I_3\sigma(D(r,0,s,0,t),\ell_p)\cup II_3\sigma(D(r,0,s,0,t),\ell_p)\cup III_3\sigma(D(r,0,s,0,t),\ell_p)=\sigma_p(D(r,0,s,0,t),\ell_p)=\emptyset,$$

is obtained by Lemma 2.12 and hence

$$I_3\sigma(D(r,0,s,0,t),\ell_p)=\emptyset.$$

Thus, $\sigma_\delta(D(r,0,s,0,t),\ell_p)=S$

As a consequence of proposition 2.3, we have the following results.

Corollary 5.7. *Let $p^{-1}+q^{-1}=1$ then, the following results hold*:

(i)$\sigma_{ap}(D(r,0,s,0,t)^*,\ell_q)=S$

(ii)$\sigma_\delta(D(r,0,s,0,t)^*,\ell_q)=S\setminus\{r\}$ *where S is as defined in Lemma* 2.5.

6. Subdivisions of the spectrum of $D(r,0,s,0,t)$ on $bv_p(1<p<\infty)$

In this section, we give the subdivisions of the spectrum of the operator $D(r,0,s,0,t)$ over the sequence space bv_p.

Since the subdivisions of the spectrum of the operator $D(r,0,s,0,t)$ on the sequence space bv_p can be derived by analogy to that space ℓ_p, we omit the detail and give the related results without proof.

Theorem 6.1. *The following results hold*:

$(i)\sigma_{ap}(D(r,0,s,0,t),bv_p)=S\setminus\{r\}$

(ii)$\sigma_{\delta}(D(r,0,s,0,t), bv_p) = S$,

(iii)$\sigma_{co}(D(r,0,s,0,t), bv_p) = S_1$, *where* S *and* S_1 *are defined as in Lemma* 2.5 *and Lemma* 2.7 *respectively.*

As a consequence of proposition 2.3, we have the following results.

Corollary 6.2. *The following results hold*:
(i)$\sigma_{ap}(D(r,0,s,0,t)^*, bv_p^*) = S$
(ii)$\sigma_{\delta}(D(r,0,s,0,t)^*, bv_p^*) = S \setminus \{r\}$ *where* S *is as defined in Lemma* 2.5.

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