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Existence of Positive Solutions of Kirchhoff Hyperbolic Systems With Multiple Parameters

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abstract: In this paper, by using sub-super solutions method, we study the existence of weak positive solution of Kirrchoff hyperbolic systems in bounded domains with multiple parameters. These results extend and improve many results in the literature.

Key Words: Kirchhoff hyperbolic systems, Existence , Positive solutions, Sub-supersolution, Multiple parameters.

Contents

1. Introduction

In this paper, we consider the following system of hyperbolic differential equations

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - A\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = \lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u) \text{ in } Q_T = \Omega \times [0,T], \\ \dfrac{\partial^2 v}{\partial t^2} - B\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v = \lambda_2 \gamma(x) g(u) + \mu_2 \eta(x) \tau(v) \text{ in } Q_T = \Omega \times [0,T], \\ u = v = 0 \text{ on } \partial Q_T, \\ u(.,0) = \varphi_1 \text{ on } \Omega \\ u_t(x,0) = \varphi_2 \text{ on } \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N \ (N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and A, B :$\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$, $\lambda_1, \lambda_2, \mu_1$, and μ_2 are nonnegative parameters.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity, Therefore, it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [20]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883, see [17]. This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

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By using theta time scheme on (1.1), we obtain the following problems

$$\begin{cases} u_k+\tau'^2 A\left(\int\limits_{\Omega}\left|\nabla u_k\right|^2 dx\right)\triangle u_k=\dfrac{u_{k+1}+u_{k-1}}{2} \\ -\tau'^2\left[\lambda_1\alpha\left(x\right)f\left(v\right)+\mu_1\beta\left(x\right)h\left(u_k\right)\right] \text{ in } \Omega, \\ v_k+\tau'^2 B\left(\int\limits_{\Omega}\left|\nabla v\right|^2 dx\right)\triangle v=\dfrac{v_{k+1}+v_{k-1}}{2} \\ -\tau'^2\left[\lambda_2\gamma\left(x\right)g\left(u_k\right)+\mu_2\eta\left(x\right)\tau\left(v\right)\right] \text{ in } \Omega, \\ u_k=v=0 \text{ on } \partial\Omega, \\ u_0=\rho_1, u_.=\rho_2, \end{cases} \tag{1.3}$$

where $N\tau'=T$, $0<\tau'<1$, and for $1\leq k\leq N$.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to ([4], [7], [19], [21]-[23]), in which the authors have used different methods to get the existence of solutions for (1.1) in the single equation case. In the papers [7], Y. Bouizm et al. studied the existence of nontrivial singn-changing solutions for system (1.1) where $A\left(t\right)=B\left(t\right)=1$ via sub-supersolution method. Our paper is motivated by the recent results in ([1], [2], [3], [12], [13], [14] [15]). In the papers [2] (Theorem 2), Azzouz and Bensedik studied the existence of a positive solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int\limits_{\Omega}\left|\nabla u\right|^2 dx\right)\triangle u=\left|u\right|^{p-2}u+\lambda f\left(x\right) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N, N\geq 3$ and $p>1$, *i.e.* the nonlinear term at infinity and f is a sign-changing function.

Using the sub-supersolution method combining a comparison principle introduced in [1], the authors established the existence of a positive solution for (1.4), where the parameter $\lambda>0$ is small enough. In the present paper, we consider system (1.1) in the case when the nonlinearities are "sublinear" at infinity, see the condition (H 3). We are inspired by the ideas in the interesting paper [12], in which the authors considered system (1.1) in the case $A\left(t\right)=B\left(t\right)=1$. More precisely, under suitable conditions on f, g, we shall show that system (1.1) has a positive solution for $\lambda>\lambda^*$ large enough. To our best knowledge, this is a new research topic for nonlocal problems, see [15]. In current paper, motivated by previous works in ([2], [12]) and by using sub-super solutions method, we study the existence of weak positive solution for a class of Kirchhoff hyperbolic systems in bounded domains with multiple parameters.

2. Existence result

Lemma 2.1. *([1])Assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function satisfying*

$$\lim_{t\to 0^+} M\left(t\right)=m_0, \tag{2.1}$$

where m_0 is a positive constant. and assume that u,v are two non-negative functions such that

$$\begin{cases} -M\left(\int\limits_{\Omega}\left|\nabla u\right|^2 dx\right)\triangle u\geq -M\left(\int\limits_{\Omega}\left|\nabla v\right|^2 dx\right)\triangle v \text{ in } \Omega, \\ u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

then $u\geq v$ a.e. in Ω.

In this section, we shall state and prove the main result of this paper. Let us assume the following assumptions:
$(H1)$ Assume that $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exists $a_i, b_i > 0, i = 1, 2,$ such that

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2 \text{ for all } t \in \mathbb{R}^+;$$

$(H2)$ $\alpha, \beta, \gamma, \eta \in C(\overline{\Omega})$ and

$$\alpha(x) \geq \alpha_0 > 0, \beta(x) \geq \beta_0 > 0,$$

$$\gamma(x) \geq \gamma_0 > 0, \eta(x) \geq \eta_0 > 0$$

for all $x \in \Omega$,
$(H3)$ f, g, h, and τ are continuous on $[0, +\infty[$, C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t\to+\infty} f(t) = +\infty, \lim_{t\to+\infty} g(t) = +\infty,$$

$$\lim_{t\to+\infty} h(t) = +\infty = \lim_{t\to+\infty} \tau(t) = +\infty;$$

$(H4)$ It holds that

$$\lim_{t\to+\infty} \frac{f(K(g(t)))}{t} = 0, \text{ for all } K > 0;$$

$(H5)$

$$\lim_{t\to+\infty} \frac{h(t)}{t} = \lim_{t\to+\infty} \frac{\tau(t)}{t} = 0.$$

Theorem 2.2. *Assume that the conditions $(H1) - (H5)$ hold, and M is a nonincreasing function satisfying (2.1). Then for $\lambda_1\alpha_0 + \mu_1\beta_0$ and $\lambda_2\gamma_0 + \mu_2\eta_0$ are large then problem (1.1) has a large positive weak solution.*

We give the following two definitions before we give our main result.

Definition 2.3. *Let $(u_k, v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, (u_k, v) is said a weak solution of (1.3) if it satisfies*

$$A\left(\int_\Omega |\nabla u_k|^2 dx\right) \int_\Omega \nabla u_k \nabla \phi dx = \int_\Omega \left(\lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u_k) - \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2}\right) \phi dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla v|^2 dx\right) \int_\Omega \nabla v \nabla \psi dx = \int_\Omega \left(\lambda_2 \gamma(x) g(u_k) \psi + \mu_2 \eta(x) \tau(v) - \frac{v_{k+1} - 2v_k + v_{k-1}}{2\tau'^2}\right) \psi dx \text{ in } \Omega,$$

for all $(\phi, \psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Definition 2.4. *A pair of nonnegative functions $(\underline{u_k}, \underline{v}), (\overline{u_k}, \overline{v})$ in $(H_0^1(\Omega) \times H_0^1(\Omega))$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\underline{u_k}, \underline{v}), (\overline{u_k}, \overline{v}) = (0, 0)$ on $\partial\Omega$*

$$A\left(\int_\Omega |\nabla \underline{u_k}|^2 dx\right) \int_\Omega \nabla \underline{u_k} \nabla \phi dx \leq \int_\Omega \left(\lambda_1 \alpha(x) f(\underline{v}) + \mu_1 \beta(x) h(\underline{u_k}) - \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2}\right) \phi \, dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right) \int_\Omega \nabla \underline{v} \nabla \psi dx \leq \int_\Omega \left(\lambda_2 \gamma(x) g(\underline{u_k}) + \mu_2 \eta(x) \tau(\underline{v}) - \frac{v_{k+1} - 2v_k + v_{k-1}}{2\tau'^2}\right) \psi \, dx \text{ in } \Omega$$

and

$$A\left(\int_{\Omega}\left|\nabla\overline{u_k}\right|^2dx\right)\int_{\Omega}\nabla\overline{u_k}\nabla\phi dx\geq\int_{\Omega}\left(\lambda_1\alpha(x)f(\overline{v})+\mu_1\beta(x)h(\overline{u_k})-\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2}\right)\phi\ dx\text{ in }\Omega,$$

$$B\left(\int_{\Omega}\left|\nabla\overline{v}\right|^2dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\int_{\Omega}\left(\lambda_2\gamma(x)g(\overline{u_k})+\mu_2\eta(x)\tau(\overline{v})-\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2}\right)\psi\ dx\text{ in }\Omega$$

for all $(\phi,\psi)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

Proof of Theorem 1. Let σ be the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\|=1$. Let $k_0,m_0,\delta>0$ such that $f(t),g(t),h(t),\tau(t)\geq -k_0$ for all $t\in\mathbb{R}^+$ and $|\nabla\phi_1|^2-\sigma\phi_1^2\geq m_0$ on $\overline{\Omega}_\delta=\{x\in\Omega: d(x,\partial\Omega)\leq\delta\}$.

For each $\lambda_1\alpha_0+\mu_1\beta_0$ and $\lambda_2\gamma_0+\mu_2\ \eta_0$ large, let us define

$$\underline{u_k}=\left(\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{2m_0a_1}\right)\phi_1^2$$

and

$$\underline{v}=\left(\frac{(\lambda_2\gamma_0+\mu_2\eta_0)k_0}{2m_0b_1}\right)\phi_1^2,$$

where a_1 and b_1 are given by the condition $(H1)$. We shall verify that $(\underline{u_k},\underline{v})$ is a subsolution of problem (1.3) for $\lambda_1\alpha_0+\mu_1\beta_0$ and $\lambda_2\gamma_0+\mu_2\ \eta_0$ large enough. Indeed, let $\phi\in H_0^1(\Omega)$ with $\phi\geq 0$ in Ω. By $(H1)-(H3)$, a simple calculation shows that

$$\begin{aligned}A\left(\int_{\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\overline{\Omega}_\delta}\nabla\underline{u_k}.\nabla\phi dx&=A\left(\int_{\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}\int_{\overline{\Omega}_\delta}\phi_1\nabla\phi_1.\nabla\phi dx\\&=\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}A\left(\int_{\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\left(\int_{\overline{\Omega}_\delta}\nabla\phi_1\nabla(\phi_1.\phi)dx-\int_{\overline{\Omega}_\delta}|\nabla\phi_1|^2\phi dx\right)\\&=\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}A\left(\int_{\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx.\end{aligned}$$

On $\overline{\Omega}_\delta$ we have $|\nabla\phi_1|^2-\sigma\phi_1^2\geq m_0$, then by $(H3)$

$$f(\underline{v}),\ h(\underline{u_k}),\ g(\underline{u_k}),\ \tau(\underline{v})\geq\frac{k_0}{m_0}$$

that

$$\begin{aligned}&A\left(\int_{\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\overline{\Omega}_\delta}\nabla\underline{u_k}\nabla\phi dx\\&\leq\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0}\int_{\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx\\&\leq\int_{\Omega}\left[\lambda_1\alpha(x)f(\underline{v})+\mu_1\beta(x)h(\underline{u_k})-\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2}\right]\phi\ dx.\end{aligned}\tag{2.3}$$

Next, on $\Omega\backslash\overline{\Omega}_\delta$, we have $\phi_1\geq r$ for some $r>0$. Therefore, under the conditions $(H1)-(H3)$ and the definition of $\underline{v}$, it follows that

$$\begin{aligned}
&\int_{\Omega}\left[\lambda_1\alpha(x)f(\underline{v})+\mu_1\beta(x)h(\underline{u_k})-\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2}\right]\phi\,dx\\
&\geq(\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0a_2}{m_0a_1}\sigma\int_{\Omega\backslash\overline{\Omega}_\delta}\phi dx\\
&\geq(\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\sigma\int_{\Omega\backslash\overline{\Omega}_\delta}\phi dx\\
&\geq(\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx\\
&=A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\nabla\underline{u_k}\nabla\phi dx,
\end{aligned}\tag{2.4}$$

for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough.

Relation (2.3) and (2.4) imply that

$$A\left(\int_{\Omega}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega}\nabla\underline{u_k}\nabla\phi dx\leq\int_{\Omega}\left[\begin{array}{c}\lambda_1\alpha(x)f(\underline{v})+\mu_1\beta(x)h(\underline{u_k})\\-\dfrac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2}\end{array}\right]\phi dx\text{ in }\Omega,\tag{2.5}$$

for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough and any $\phi\in H_0^1(\Omega)$ with $\phi\geq0$ in Ω.

Similarly,

$$B\left(\int_{\Omega}|\nabla\underline{v}|^2dx\right)\int_{\Omega}\nabla\underline{v}\nabla\psi dx\leq\int_{\Omega}\left[\lambda_2\gamma(x)g(u_k)\psi+\mu_2\eta(x)\tau(v)-\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2}\right]\psi dx\text{ in }\Omega,\tag{2.6}$$

for $\lambda_2\gamma_0+\mu_2\eta_0>0$ large enough and any $\psi\in H_0^1(\Omega)$ with $\psi\geq0$ in Ω. From (2.5) and (2.6), $(\underline{u_k},\underline{v})$ is a subsolution of problem (1.3). Moreover, we have $\underline{u_k}>0$, $\underline{v}>0$ in Ω, $\underline{u}\to+\infty$ and $\underline{v}\to+\infty$ also $\lambda_1\alpha_0+\mu_1\beta_0\to+\infty$ and $\lambda_2\gamma_0+\mu_2\eta_0\to+\infty$.

Next, we shall construct a supersolution of problem (1.3). Let ω be the solution of the following problem:

$$\begin{cases}-\triangle e=1\text{ in }\Omega,\\ e=0\text{ on }\partial\Omega.\end{cases}\tag{2.7}$$

Let

$$\overline{u_k}=Ce,\ \overline{v}=\left(\frac{\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty}{b_1}\right)\left[g\left(C\|e\|_\infty\right)\right]e,$$

where e is given by (2.7) and $C>0$ is a large positive real number to be chosen later. We shall verify that $(\overline{u_k},\overline{v})$ is a supersolution of problem (1.3). Let $\phi\in H_0^1(\Omega)$ with $\phi\geq0$ in Ω. Then, we obtain from (2.7) and the condition $(H1)$ that

$$\begin{aligned}
A\left(\int_{\Omega}|\nabla\overline{u_k}|^2dx\right)\int_{\Omega}\nabla\overline{u_k}.\nabla\phi dx&=A\left(\int_{\Omega}|\nabla\overline{u_k}|^2dx\right)C\int_{\Omega}\nabla\omega.\nabla\phi dx\\
&=A\left(\int_{\Omega}|\nabla\overline{u_k}|^2dx\right)C\int_{\Omega}\phi dx\\
&\geq a_1C\int_{\Omega}\phi dx.
\end{aligned}$$

By $(H4)$ and $(H5)$, we can choose C large enough, thus

$$\begin{aligned} a_1 C &\geq \lambda_1 \|\alpha\|_\infty f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C\|e\|_\infty\right)\|e\|_\infty\right) \\ &\quad + \mu_1 \|\beta\|_\infty h\left(C\|e\|_\infty\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right) \int_\Omega \nabla \overline{u_k}.\nabla \phi dx \\ &\geq \left[\lambda_1 \|\alpha\|_\infty f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C\|e\|_\infty\right)\|e\|_\infty\right) + \mu_1 \|\beta\|_\infty h\left(C\|e\|_\infty\right)\right] \\ &- \int_\Omega \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} \phi dx \\ &\geq \lambda_1 \|\alpha\|_\infty \int_\Omega f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C\|e\|_\infty\right)\|e\|_\infty\right) \phi dx + \mu_1 \int_\Omega h\left(C\|e\|_\infty\right) \phi dx \\ &- \int_\Omega \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} \phi dx \\ &\geq \int_\Omega \left[\lambda_1 \alpha(x) f\left(\underline{v}\right) + \mu_1 \beta(x) h\left(\underline{u_k}\right) - \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2}\right] \phi dx. \end{aligned} \tag{2.8}$$

Also,

$$\begin{aligned} &B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right) \int_\Omega \nabla \overline{v} \nabla \psi dx \geq \left(\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty\right) \int_\Omega g\left(C\|e\|_\infty\right) \psi dx \\ &= \lambda_2 \int_\Omega \gamma(x) g\left(\overline{u_k}\right) \psi dx + \mu_2 \int_\Omega \eta(x) g\left(C\|e\|_\infty\right) \psi dx - \int_\Omega \frac{v_{k+1} - 2v_k + v_{k-1}}{2\tau'^2} \psi dx. \end{aligned} \tag{2.9}$$

Again by $(H4)$ and $(H5)$ for C large enough, we have

$$g\left(C\|e\|_\infty\right) \geq \tau\left[\frac{\left(\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty\right)}{b_1} g\left(C\|e\|_\infty\right)\|e\|_\infty\right] \geq \tau\left(\overline{v}\right). \tag{2.10}$$

From (2.9) and (2.10), we have

$$\begin{aligned} &B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right) \int_\Omega \nabla \overline{v} \nabla \psi dx \geq \lambda_2 \int_\Omega \gamma(x) g\left(\overline{u_k}\right) \psi dx \\ &+ \mu_2 \int_\Omega \eta(x) \tau\left(\overline{v}\right) \psi dx - \int_\Omega \frac{v_{k+1} - 2v_k + v_{k-1}}{2\tau'^2} \psi dx. \end{aligned} \tag{2.11}$$

From (2.8) and (2.11), we have$(\overline{u}, \overline{v})$ is a subsolution of problem (1.3) with $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ for C large.

In order to obtain a weak solution of problem (1.3), we shall use the arguments by Azzouz and Bensedik [2] (observe that f, g, h, and τ does not depend on x). For this purpose, we define a sequence $\{(u_n, v_n)\} \subset \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$ as follows: $u_0 = \overline{u}, v_0 = \overline{v}$ and (u_n, v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2 dx\right) \triangle u_n = \lambda_1 \alpha(x) f\left(v_{n-1}\right) + \mu_1 \beta(x) h\left(U_{n-1}\right) \\ -\frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} \text{ in } \Omega, \\ -B\left(\int_\Omega |\nabla v_n|^2 dx\right) \triangle v_n = \lambda_2 \gamma(x) g\left(u_{n-1}\right) + \mu_2 \eta(x) \tau\left(v_{n-1}\right) \\ -\frac{v_{k+1} - 2v_k + v_{k-1}}{2\tau'^2} \text{ in } \Omega, \\ u_n = v_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.12}$$

(2.12) is (A,B)-linear in the sense that, if $(u_{n-1},v_{n-1}) \in \left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$, the right hand sides of (2.12) is independent of u_n and v_n.

Setting

$$A(t) = tA\left(t^2\right), B(t) = tB\left(t^2\right).$$

Since $A(\mathbb{R}) = \mathbb{R}$, $B(\mathbb{R}) = \mathbb{R}$, $f(v_{n-1})$, $h(u_{n-1})$, $g(u_{n-1})$, and $\tau(v_{n-1}) \in L^2(\Omega)$, we deduce from the result in [1], that system (2.12) has a unique solution $(u_n,v_n) \in \left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

By using (2.12) and the fact that (u_0,v_0) is a supersolution of (1.3), we have

$$\begin{cases} -A\left(\int_\Omega |\nabla u_0|^2 dx\right)\triangle u_0 \geq \lambda_1\alpha(x) f(v_0) + \mu_1\beta(x) h(u_0) \\ \quad -\dfrac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} = -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\triangle u_1, \\ -B\left(\int_\Omega |\nabla v_0|^2 dx\right)\triangle v_0 \geq \lambda_2\gamma(x) g(u_0) + \mu_2\eta(x)\tau(v_0) \\ \quad -\dfrac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2} = -B\left(\int_\Omega |\nabla v_1| dx\right)\triangle v_1 \end{cases}$$

and by Lemma 1, we also have $u_0 \geq u_1$ and $v_0 \geq v_1$. In addition, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and under the monotonicity condition of f, h, g, and τ, it can be deduced

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\triangle u_1 &= \lambda_1\alpha(x) f(v_0) + \mu_1\beta(x) h(u_0) \\ &\quad -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} \\ &\geq \lambda_1\alpha(x) f(\underline{v}) + \mu_1\beta(x) h(\underline{u}) - \frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} \\ &\geq -A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\triangle \underline{u} \end{aligned}$$

and

$$\begin{aligned} -B\left(\int_\Omega |\nabla v_1|^2 dx\right)\triangle v_1 &= \lambda_2\gamma(x) g(u_0) + \mu_2\eta(x)\tau(v_0) \\ &\quad -\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2} \\ &\geq \lambda_2\gamma(x) g(\underline{u}) + \mu_2\eta(x)\tau(\underline{v}) \\ -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} &\geq -B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\triangle \underline{v}. \end{aligned}$$

According to Lemma 1, we have $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$ for any u_2, v_2, thus we can write

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2 dx\right)\triangle u_1 &= \lambda_1\alpha(x) f(v_0) + \mu_1\beta(x) h(u_0) \\ &\quad -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} \\ &\geq \lambda_1\alpha(x) f(v_1) + \mu_1\beta(x) h(u_0) \\ -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} &= -A\left(\int_\Omega |\nabla u_2|^2 dx\right)\triangle u_2, \end{aligned}$$

and

$$\begin{aligned}
-B\left(\int_{\Omega}|\nabla v_1|\,dx\right)\triangle v_1 &= \lambda_2\gamma(x)\,g(u_0)+\mu_2\eta(x)\,\tau(v_0) \\
&\quad -\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2} \\
&\geq \lambda_1\alpha(x)\,g(u_1)+\mu_2\beta(x)\,\tau(v_1) \\
-\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2} &= -B\left(\int_{\Omega}|\nabla v_2|^2\,dx\right)\triangle v_2.
\end{aligned}$$

Then $u_1 \geq u_2$, $v_1 \geq v_2$.

Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned}
-A\left(\int_{\Omega}|\nabla u_2|^2\,dx\right)\triangle u_2 &= \lambda_1\alpha(x)\,f(v_1)+\mu_1\beta(x)\,h(u_1) \\
&\quad -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} \\
&\geq \lambda_1\alpha(x)\,f(\underline{v})+\mu_1\beta(x)\,h(\underline{u}) \\
-\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} &\geq -A\left(\int_{\Omega}|\nabla\underline{u}|^2\,dx\right)\triangle\underline{u}, \\
-B\left(\int_{\Omega}|\nabla v_2|^2\,dx\right)\triangle v_2 &= \lambda_2\gamma(x)\,g(u_1)+\mu_2\eta(x)\,\tau(v_1) \\
&\quad -\frac{u_{k+1}-2u_k+u_{k-1}}{2\tau'^2} \\
&\geq \lambda_2\gamma(x)\,g(\underline{u})+\mu_2\eta(x)\,\tau(\underline{v})-\frac{v_{k+1}-2v_k+v_{k-1}}{2\tau'^2} \\
&\geq -B\left(\int_{\Omega}|\nabla\underline{v}|^2\,dx\right)\triangle\underline{v}.
\end{aligned}$$

Repeating this argument, we get a bounded monotone sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ satisfying

$$\overline{u}=u_0\geq u_1\geq u_2\geq\ldots\geq u_n\geq\ldots\geq\underline{u}>0 \tag{2.13}$$

and

$$\overline{v}=v_0\geq v_1\geq v_2\geq\ldots\geq v_n\geq\ldots\geq\underline{v}>0. \tag{2.14}$$

Using the continuity of the functions $f,h,\,g,\,\tau$ and the definition of the sequences $\{u_n\},\{v_n\}$,there exist constants $C_i>0$, $i=1,\ldots,4$ independent of n such that

$$|f(v_{n-1})|\leq C_1,\quad |h(u_{n-1})|\ \leq C_2,\ |g(u_{n-1})|\leq C_3 \tag{2.15}$$

and

$$|\tau(u_{n-1})|\ \leq C_4 \text{ for all } n.$$

Multiplying the first equation of (2.12 by u_n, integrating, using the Holder inequality and Sobolev embedding, we can show that

$$\begin{aligned}
a_1 \int_\Omega |\nabla u_n|^2 dx &\leq A\left(\int_\Omega |\nabla u_n|^2 dx\right) \int_\Omega |\nabla u_n|^2 dx \\
&= \lambda_1 \int_\Omega \alpha(x) f(v_{n-1}) u_n dx + \mu_1 \int_\Omega \beta(x) h(u_{n-1}) u_n dx \\
&\quad - \int_\Omega \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} u_n dx \\
&\leq \lambda_1 \|\alpha\|_\infty \int_\Omega |f(v_{n-1})| \ |u_n| dx + \mu_1 \|\beta\|_\infty \int_\Omega |h(u_{n-1})| \ |u_n| dx \\
&\quad - \int_\Omega \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} |u_n| dx \\
&\leq C_1 \lambda_1 \int_\Omega |u_n| dx + C_2 \mu_1 \int_\Omega |u_n| dx \\
&\quad - \int_\Omega \frac{u_{k+1} - 2u_k + u_{k-1}}{2\tau'^2} |u_n| dx \\
&\leq C_5 \|u_n\|_{H_0^1(\Omega)},
\end{aligned}$$

or

$$\|u_n\|_{H_0^1(\Omega)} \leq C_5, \ \forall n, \tag{2.16}$$

where $C_5 > 0$ is a constant independent of n. Similarly, there exist $C_6 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_6, \quad \forall n. \tag{2.17}$$

From (2.16) and (2.17), we infer that $\{(u_n, v_n)\}$ has a subsequence which weakly converges in $H_0^1(\Omega)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also by using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v).

Now, passing the limit in (2.12), we deduce that (u, v) is a positive solution of system (1.3).

The proof of theorem is completed. □

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