Bol. Soc. Paran. Mat. (3s.) v. 2022 (40) : 1–7.
©SPM –ISSN-2175-1188 on line ISSN-0037-8712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.45542

Non-extremal Martingale with Brownian Filtration

Sakrani Samia

Abstract: Let $(\mathcal{B}_t)_{t\geq 0}$ be the filtration of a Brownian motion $(B_t)_{t\geq 0}$ on $(\Omega, \mathcal{B}, \mathbf{P})$. An example is given of an non-extremal martingale which generates the filtration $(\mathcal{B}_t)_{t\geq 0}$. We also discuss a property of pure martingales, we show here that it is a property of a filtration rather than a martingale.

Key Words: Extremal martingale, Brownian filtration, Pure martingale, Pure filtration.

Contents

1. Introduction

Among the series of questions asked at the end of the chap.V of [12]) (or also in [13] and [15]) is the following question: a filtration being given on a probability space, how to recognize if it is generated by a Brownian motion or not? This question is especially of interest for a weakly Brownian filtration (there exists an $\mathcal{F}$-Brownian motion which has the predictable representation property (PRP) with respect to $\mathcal{F}$, see [11] for application of this important property). In all generality, there are weakly Brownian filtrations, which are not Brownian, as it is shown in [6], paper that was followed by other examples of non-Brownian filtrations given in [4], [7], [14]. These works are important progress that raises many new questions, including how to establish the non-Brownian character of a weakly Brownian filtration?

In all the works above, it is the notion of non-cosiness (introduced by Tsirel'son in [14] and that we will not discuss in this paper) of these filtrations which serves as a criterion to show that they are non-Brownian, see [4], [10] for different types of cosiness: I-cosiness, D-cosiness and T-cosiness. One might think that a filtration generated by a non-pure extremal martingale or non-extremal martingale can not be Brownian. In fact we show in Section 3 that this is not true. The non-Brownian character of a weakly Brownian filtration is much more delicate. Section 4 shows that Brownian filtration can be generated by non-pure extremal martingale. In section 5, we discuss the following property denoted by (*) in [1]: If M is a continuous martingale and $\mathcal{F} = \mathcal{F}^M$, for every, $\mathcal{F}$-stopping time T finite a.s such that $\mathbf{P}(M_T = 0) = 0$,then

$$\mathcal{F}^+_{G_T} = \mathcal{F}^-_{G_T} \vee \sigma(M_T < 0),$$

where $G_T = \sup\{s \leq T, M_s = 0\}, T \in [0, \infty[$. Authors of [1] have shown that property (*) is satisfied by any pure martingale. It is understood here that (*) is a property of a filtration rather than a martingale.

2010 *Mathematics Subject Classification:* 60G44, 60J65.
Submitted November 25, 2018. Published May 25, 2019

2. Preliminaries

We will only consider completed probability spaces and right continuous filtrations. We denote $\int HdX$ the stochastic integral of H with respect to X and $\mathcal{F}^X$ the natural filtration of X. An $\mathcal{F}-$continuous local martingale X has the PRP (the predictable representation property) if for every $\mathcal{F}-$continuous local martingale M there exists an $\mathcal{F}-$predictable process H such that

$$M = M_0 + \int HdX,$$

X is called $\mathcal{F}-$extremal if $\mathcal{F}_0$ is trivial and X has the $\mathcal{F}-$PRP. If $\mathcal{F}^X = \mathcal{F}$ then X is called extremal martingale. (this terminology is justified by the fact that the law of an extremal martingale is an extremal point in the convex set of all probability measures on $W = C(\mathbb{R}^+ \cdot \mathbb{R})$, which make the coordinate process a local martingale). A continuous local martingale X with $\langle X\rangle_\infty = \infty$ is pure if $\mathcal{F}^X_\infty = \mathcal{F}^B_\infty$ where B is the Brownian motion of Dubins-Schwartz (DDS) associated with X, which is equivalent to say that for all t, $\langle X\rangle_t$ is $\mathcal{F}^B_\infty-$measurable.

Every pure martingale is extremal but the opposite is not true. Yor has given in [15] an example of an extremal martingale which is not pure; we will prove here that its natural filtration is Brownian.

Definition 2.1. *A filtration $\mathcal{F}$ is said to be immersed in a filtration $\mathcal{G}$(defined on the same probability space) if any $\mathcal{F}$-martingale is $\mathcal{G}$-martingale.*

3. Example of non-extremal martingale with Brownian filtration

We have the following characterization of extremal martingales with respect to Brownian filtration:

Lemma 3.1. *If B is a Brownian motion, $\mathcal{B}$ its natural filtration and M is a $\mathcal{B}-$ martingale, then M is $\mathcal{B}$-extremal if and only if $d\langle M\rangle$ is equivalent to λ a.s, where λ is the Lebesgue measure on $\mathbb{R}^+$.*

Proof. M is a $\mathcal{B}-$ martingale, so there exists a $\mathcal{B}-$predictable process H such that:

$$M = M_0 + \int HdB \textit{ and } H^2 = \frac{d\langle M\rangle}{d\lambda}$$

If M is $\mathcal{B}-$extremal, then there exists a $\mathcal{B}-$predictable process K such that $B = \int KdM$ and $d\lambda = K^2 d\langle M\rangle$, that is $d\langle M\rangle$ is equivalent to λ. If now, $d\langle M\rangle$ is equivalent to λ, it is enough to represent B as a stochastic integral with respect to M. We have $H \neq 0$, $\lambda \otimes dP$ *a.s* so $B = \int \frac{1}{H} dM$. □

Lane [9], gave partial answers to the following question [12]: If B is a Brownian motion, f is borel function and M is the local martingale $\int f(B)\,dB$, under what conditions the filtration $\mathcal{F}^M$ is Brownian?.An important example is when $f \geq 0$ and $\mu(\{f=0\}) > 0$ but the set $\{f=0\}$ does not contain any interval (μ is the Lebesgue measure on $\mathbb{R}$). This case was studied by knight [8] with $F = \{f=0\}$ is a subset of $[0,1]$, defined by the Cantor method: removing $]\frac{3}{8}, \frac{5}{8}[$ then $]\frac{5}{32}, \frac{7}{32}[$ and $]\frac{19}{32}, \frac{21}{32}[$ and so on. We define the set F_n by means of its complementary F_n^c,

$$F_1^c =]\frac{3}{8}, \frac{5}{8}[, F_2^c = F_1^c \cup]\frac{5}{32}, \frac{7}{32}[\cup]\frac{19}{32}, \frac{21}{32}[\ ,$$

$$F_n^c = F_{n-1}^c \cup \bigcup_{k=1}^{2n-1} A_n^k, \quad n \geq 2,$$

where $A_n^k =]a_n^k, b_n^k[$ are disjoint intervals of length $\frac{1}{4^n}$. Finally

$$F^c = \bigcup_n F_n^c = \bigcup_{n\geq 1}\bigcup_{k=1}^{\ell_n} A_n^k,$$

with $\ell_n = \sum_{k=0}^{n-1} 2^k = 2^n - 1$. Hence $\mu(F^c) = \lim_{n\to\infty} \mu(F_n^c) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{1}{2}$.

Theorem 3.2. *Let B be a Brownian motion, $\mathcal{B}$ its natural filtration and M the martingale defined by*

$$M = c^{'} \int \mathbf{1}_{\{B<0\}} dB + c'' \int \mathbf{1}_{\{B>1\}} dB + \sum_{n\geq 1} \sum_{k=n}^{\ell_n} c_n^k \int \mathbf{1}_{A_n^k}(B) dB,$$

where the numbers (c_n^k), $n \geq 1$, $k \in \{1, ..., \ell_n\}$, c' and c'' are strictly positive and all different. The martingale M is not extremal and we have $\mathcal{F}^M = \mathcal{B}$.

Remark 3.3. *In order not to burden the proof of Theorem 1, at the end of this paper (in the appendix) we have gathered some non-detailed points.*

Proof. The processes B^- and $(B-1)^+$ are $\mathcal{F}^M-$adapted (Point 1), it remains to show that $B_t \, \mathbf{1}_{\{0<B_t<1\}}$ is $\mathcal{F}^M-$adapted. We consider the martingales

$$M_n^k = \int \mathbf{1}_{A_n^k}(B) dB$$

(M_n^k) are also $\mathcal{F}^M-$adapted (Point 1). The stopping times $\{(S_n^k)^r, (T_n^k)^r\}_{r\geq 1}$ of the successive entries and exits of B in the set A_n^k are $\mathcal{F}_\infty^{M_n^k}-$measurable because these are the moments where $\Delta C_n^k > 0$, with C_n^k the inverse of $\prec M_n^k \succ$.

Fix $n \in \mathbb{N}^*, k \in \{1, ..., \ell_n\}$ and for every $r \in \mathbb{N}^*$

$$S^r := (S_n^k)^r \quad , \quad T^r := (T_n^k)^r \quad , \quad A_n^k =]a, b[\quad , \; N := M_n^k \; \text{ and } \alpha := c_n^k.$$

(Attention! a, b, N and α depend on k and n).

Let us show that the sequence $(B_{S^r}, B_{T^r})_{r\geq 1}$ is $\mathcal{F}_\infty^M$ - measurable.We have, $N_t = 0$ until S^1 and $B_{S^1} = a$. If $t \in [S^1, T^1]$, then

$$N_t = \int_{S^1}^t dB_s = B_t - a.$$

So, we know B_{T^1} and for every $r \geq 1$ and $t \in [S^r, T^r]$ we have

$$M_t - M_{S^r} = \alpha(N_t - N_{S^r}) = \alpha(B_t - B_{S^r}) \tag{1}$$

Therefore

$$M_t - M_{S^r} = \alpha(B_{T^r} - B_{S^r})$$

Then, if we know M and B_{T^r}, we can know B_{S^r} (and the inverse is true).

If $M_{T^r} - M_{S^r} > 0$ then $B_{T^r} = b$ and $B_{S^r} = a$. If $M_{T^r} - M_{S^r} < 0$ then $B_{T^r} = a$ and $B_{S^r} = b$.

It remains the case where $M_{T^r} - M_{S^r} \; = 0$ so $B_{T^r} = b$ (and then $B_{T^r} = B_{S^r}$).Remark that

$$B_{T^r} = B_{S^{r+1}} \tag{2}$$

Indeed, if B is above $]a, b[$ after T^r, then $B_{T^r} = b = B_{S^{r+1}}$, and if B is below $]a, b[$ after T^r, then $B_{T^r} = a = B_{S^{r+1}}$.

Suppose we know M until time t, since we know B_{T^1}, then, from (2), we can know B_{S^2} and B_{T^3} and so on, we can know the sequence (B_{T^r}, B_{S^r}) for $T^r, S^r \leq t$.

To finish the proof, let $t_0 \leq t$, the set $\{B_{t_0} \in F^c\}$ is $\mathcal{F}_{t_0}^M$-measurable (Point 2). If $B_{t_0} \in F^c$,then there exists n and k such that $B_{t_0} \in A_n^k$ and so, there exists r such that $t_0 \in]S^r, T^r[$. We have

$$B_{t_0} = B_{t_0} - B_{S^r} + B_{S^r}.$$

and equality (1) gives

$$B_{t_0} = \frac{1}{\alpha}(M_{t_0} - M_{S^r}) + B_{s^r}$$

Since F^c is dense in $[0, 1]$ (Point 3), we have

$$B_t \mathbf{1}_{\{0<B_t<1\}} = \lim_{s\downarrow t} \sup B_s \mathbf{1}_{\{B_s\in F^c\}} \text{ and } \mathcal{F}^M = \mathcal{B}.$$

It remains to establish that M is non-extremal. This follows easily from Lemma 1, since $\lambda(F) > 0$. □

4. Examples of extremal non-pure martingales with Brownian filtrations

We will now show that the filtration of the extremal non-pure martingale given in [15] is Brownian.

Theorem 4.1. *Brownian filtration is generated by a non-pure extremal martingale.*

Proof. Let B be a Brownian motion and $\mathcal{B}$ its natural filtration. We start by considering the stochastic equation

$$dX_t = \varphi(X_t)dB_t \quad, X_0 = 0,$$

where $\varphi(x) = \frac{1}{\sqrt{2+\frac{x}{1+|x|}}}$. We easily check that:

$$\begin{aligned} |\varphi(x) - \varphi(x')|^2 &\leq c\left|\frac{1}{\varphi(x)} - \frac{1}{\varphi(x')}\right|^2 \\ &\leq c\left|\frac{x}{1+|x|} - \frac{x'}{1+|x'|}\right| \end{aligned}$$

and

$$\frac{1}{\sqrt{3}} \leq \varphi(x) \leq 1, \forall x, x' \in \mathbb{R}.$$

The function $\frac{x}{1+|x|}$ is strictly increasing, we apply theorem 3.5(iii), chap.IX of [12] and we get $\mathcal{F}^X = \mathcal{B}$. We have, $\langle X\rangle = \int \varphi^2(X_t)dt$, since φ^2 is continuous and strictly decreasing

$$\mathcal{F}^{\langle X\rangle} = \mathcal{F}^X$$

We define the martingale

$$M_t = \widetilde{\gamma}_{\langle X\rangle_t}$$

where $\widetilde{\gamma}_t = \int_0^t sgn\gamma_s d\gamma_s$ and γ is the DDS Brownian motion associated to X. We have $\langle X\rangle = \langle M\rangle$ then

$$\mathcal{F}^{\langle M\rangle} = \mathcal{F}^M = \mathcal{B}.$$

It remains to show that M is extremal but non-pure. Since φ is strictly positive, $d\langle M\rangle$ is equivalent to Lebesgue measure and $\mathcal{F}^M$ is a Brownian filtration, therefore, using Lemma 1, we deduce that M is extremal. M is non-pure because

$$\mathcal{F}^{\widetilde{\gamma}}_\infty \subsetneqq \mathcal{F}^\gamma_\infty = \mathcal{F}^M_\infty.$$

□

Here is an other example of non-pure extremal martingale with Brownian filtration :

Theorem 4.2. *Let B be a Brownian motion. There exists a strictly positive predictable process H such that $N_t = \int_0^t H(B_u, u \leq s)dB_s$ is non-pure extremal martingale.*

Proof. Let (T_t) be absolutely continuous and strictly increasing time change of Theorem 4.1 of [7]. Then $M_t := (B_{T_t})$ generates non-Brownian filtration. We have $M_t = \int_0^t f(M_u, u \leq s)d\gamma_s$ (see Proposition 3.8, Chap V of [12]), for γ a Brownian motion and f predictable process which can be choose strictly positive. Since M is pure by construction (so $\mathcal{F}^M_C = \mathcal{F}^B$), $B_t = \int_0^t g(B_u, u \leq s)d\gamma_{C_s}$, where g is $\mathcal{F}^B-$predictable process and C the inverse of T, so

$$\gamma_{C_t} = \int_0^t H_s dB_s,$$

with $H = \frac{1}{g}$. Since the filtration of M is non Brownian, $\mathcal{F}^M \neq \mathcal{F}^\gamma$ and the martingale $N = \gamma_C$ is not pure. But $\mathcal{F}^N = \mathcal{F}^B$ and H is strictly positive, then N is extremal by Lemma 1. □

Remark 4.3. *Theorem 3 responds affirmatively to the following question asked at the end of Chap V of [12]: is there a strictly positive predictable process H such that the martingale $N_t = \int_0^t H_s dB_s$ is not pure?*

5. A martingale class that satisfy property $(\star)$

In [1], authors discussed a property $(\star)$ verified by all pure martingales and gave some examples of non-pure extremal martingales and non-extremal martingales that nevertheless satisfy property $(\star)$. In [2], we better understand this property that we reset here: Let M be a continuous martingale and $\mathcal{F} = \mathcal{F}^M$, for every, $\mathcal{F}$-stopping time T finite $a.s$ such that $\mathbf{P}(M_T = 0) = 0$, we have

$$\mathcal{F}^+_{G_T} = \mathcal{F}^-_{G_T} \vee \sigma(M_T < 0),$$

where $G_T = \sup\{s \leq T, M_s = 0\}, T \in [0, \infty[$. The example given in [1] of non-pure extremal martingale satisfying property $(\star)$ is in fact the example of Yor [15]. We have shown that its filtration is Brownian and therefore, it is obvious that this martingale satisfies $(\star)$ using Barlow's property proven in [2]. In the same way, our non-extremal martingale of Theorem 1, satisfies $(\star)$.

In general, the following proposition can be stated:

Proposition 5.1. *Let $\mathcal{F}$ be a filtration such that all $\mathcal{F}$-martingales are continuous and $SpMult[\mathcal{F}] \leq 2$ (see the definition below), then all martingales generating $\mathcal{F}$ satisfy property $(\star)$.*

Before proving the proposition, we recall the following definition:

Definition 5.2. *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be probability space and $\mathcal{T}$ a sub-field of $\mathcal{A}$. Let $\mathcal{Q}$ be the set of all finite measurable partitions of $(\Omega, \mathcal{A})$, for $Q \in \mathcal{Q}, \mid Q \mid$is the cardinal of Q. The conditional multiplicity of $\mathcal{A}$ with respect to $\mathcal{T}$ is the random variable with values in $\mathbb{N}^* \cup \{\infty\}$*

$$Mult[\mathcal{A} \mid \mathcal{T}] = \underset{Q. \in \mathcal{Q}}{ess \sup} \mid Q \mid \ \mathbf{1}_{S_B(Q)}$$

where $S_B(Q.) = \{\forall A \in Q, P(A \mid \mathcal{T}) > 0\}$. The splitting multiplicity of a filtration $\mathcal{F}$, $SpMult[\mathcal{F}]$ is the smallest integer n such that: $Mult[\mathcal{F}_{L^+} \mid \mathcal{F}_L] \leq n$, for any honest time L of $\mathcal{F}$.

Proof. Using proposition 1 of [1], it is enough to show $(\star)$ for $T = t$.

Let $A = \{M_t > 0\}$, we have $\mathbf{E}[M_t \mid \mathcal{F}_{G_t}] = 0$ $a.s$, because $M_{G_t} = 0$ $a.s$ (by Theorem XX-35 of [5]). Then $a.s$

$$\mathbf{E}[M_t \mathbf{1}_A \mid \mathcal{F}_{G_t}] = -\mathbf{E}[M_t \mathbf{1}_{A^c} \mid \mathcal{F}_{G_t}]. \tag{3}$$

We define the sets $C_1 = \{\mathbf{P}(A \mid \mathcal{F}_{G_t}) = 0\}$ and $C_2 = \{\mathbf{P}(A^c \mid \mathcal{F}_{G_t}) = 0\}$ which are in $\mathcal{F}_{G_t}$. We have $\mathbf{P}(A \cap C_1) = 0$ and $\mathbf{P}(A^c \cap C_2) = 0$.

And for every $n \in \mathbb{N}$:

$$\mathbf{E}[\mathbf{1}_{C_1} M_t \mathbf{1}_{\{0 < M_t < n\}} \mid \mathcal{F}_{G_t}] \leq n\mathbf{P}(A \cap C_1 \mid \mathcal{F}_{G_t}) = 0,$$

then

$$\mathbf{1}_{C_1} \mathbf{E}[M_t \mathbf{1}_A \mid \mathcal{F}_{G_t}] = 0$$

and from (3), we have

$$\mathbf{1}_{C_1} \mathbf{E}[M_t \mathbf{1}_{A^c} \mid \mathcal{F}_{G_t}] = 0.$$

So, $\mathbf{E}[M_t \mathbf{1}_{c_1 \cap A^c}] = 0$ and $C_1 \subset \{M_t = 0\}$.

Similarly, we have $C_2 \subset \{M_t = 0\}$ Applying hypothesis $\mathbf{P}\{M_t = 0\}$ is null, we get $\mathbf{P}(C_1 \cup C_2) = 0$ So

$$\mathcal{F}^+_{G_t} = \ \mathcal{F}_{G_t} \vee \sigma(M_t > 0),$$

according to proposition 3 of [2] (see also Lemma 4.3 ,Chap . I of [3]). □

Here is an example of a filtration with $SpMult \leq 2$.

Definition 5.3. *A filtration generated by a pure martingale is called pure filtration.*

Proposition 5.4. *Let $\mathcal{F}$ be a filtration, $C = (C_t)$ time change for $\mathcal{F}$ and $\widehat{\mathcal{F}} = (\mathcal{F}_{C_t})$. We have:*

(a) *$SpMult(\mathcal{F}) \leq SpMult(\widehat{\mathcal{F}})$. If moreover C is strictly increasing, we have: $SpMult(\mathcal{F}) = SpMult(\widehat{\mathcal{F}})$. In particular, if $\mathcal{F}$ is pure(non trivial), then $SpMult(\mathcal{F}) = 2$.*

(b) *Let $\mathcal{F}$ be the natural filtration of a continuous martingale M and C the inverse of $\langle M \rangle$.we suppose that $\langle M \rangle$ is strictly increasing and $\langle M \rangle_\infty = \infty$. If $\widehat{\mathcal{F}}$ is Brownian, then M is extremal and $\mathcal{F}$ is pure.*

Proof. **(a)** Suppose $SpMult(\widehat{\mathcal{F}}) = n \in \mathbb{N}^*$.

Let M be $\mathcal{F}$-spider martingale of multiplicity $n+1$, bounded and $M_0 = 0$. Then $M_c = \mathbf{E}[M_\infty \mid \widehat{\mathcal{F}}]$ is $\widehat{\mathcal{F}}$-spider martingale of multiplicity $n+1$ vanishing at the origin, Proposition 13 of [2] gives $M_\infty = 0$ $a.s$ and $SpMult(\mathcal{F}) \leq n$. If C is strictly increasing and if τ is its inverse, then by Lemma 5.9 of [13], we have

$$\widehat{\mathcal{F}}_\tau = \mathcal{F}_{C_\tau} = \mathcal{F}.$$

If $\mathcal{F}$ is pure, then there exists a time change which we also note C, such that $\mathcal{F}_c$ is Brownian, then $SpMult(\widehat{\mathcal{F}}) = 2$ and $SpMult(\mathcal{F}) \leq 2$.

(b) Let W be a Brownian motion that generates $\widehat{\mathcal{F}}$ and X the martingale $W_{\langle M \rangle}$ (by construction, X is pure).

Let us show that M is extremal: let B be the DDS Brownian motion of M, B is $\widehat{\mathcal{F}}-$ Brownian motion that has $\widehat{\mathcal{F}} - PRP$ (because $\widehat{\mathcal{F}}$ is Brownian), as $\mathcal{F}_{C_0}$ is trivial, $\mathcal{F}_0$ is too, and M is extremal. Notice now that

$$\mathcal{F}^X_\infty = \mathcal{F}^W_\infty = \widehat{\mathcal{F}}_\infty = \mathcal{F}_\infty. \tag{4}$$

and

$$M_t = \int_0^t \varepsilon_{\langle M \rangle_s} dX_s,$$

with $\varepsilon_t = \frac{d\langle B, W \rangle_t}{dt}$. Hence X is $\mathcal{F}$-extremal (and since it is extremal), Proposition 7.1 of [13], gives us that $\mathcal{F}^X$ is immersed in $\mathcal{F}$. So we have $\mathcal{F} = \mathcal{F}^X$ using (4). ☐

The next question naturally arises: *The reciprocal of proposition 1 is it true? i.e if all the martingales that generate a filtration $\mathcal{F}$ satisfy the property ($\star$), do we have $SpMult(\mathcal{F}) = 2$?*

For now, we do not have a general answer to this question. In any case, let us note that the following example given in [1] section 6, does not give a negative answer, let

$$M_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{(X_s^2 + Y_s^2)^\alpha},$$

where $(X_t + iY_t)$ is a planar Brownian motion starting from $z \in \mathbb{C}^*$ and $\alpha \in]-\infty, \frac{1}{2}]$. Let $\mathcal{F}$ be the filtration of M, C the inverse of $\langle M \rangle$ and $\widehat{\mathcal{F}} = (\mathcal{F}_{C_t})_{t \geq 0}$, $\widehat{\mathcal{F}}$ is Brownian, so $\mathcal{F}$ is pure and according to proposition 1, M satisfy property ($\star$).

6. Appendix

Point 1. We have

$$\int \mathbf{1}_{\{B<0\}} dB = \frac{1}{c'} \int \mathbf{1}_{\{B<0\}} dM$$

and

$$\int \mathbf{1}_{\{B>1\}} dB = \frac{1}{c''} \int \mathbf{1}_{\{B>1\}} dM.$$

Hence, by applying Skorokhod's Lemma (Lemma 2.1 , Chap.VI of [12]) it is sufficient to see that the sets $\{B_t < 0\}$ and $\{B_t > 1\}$ are $\mathcal{F}^M_t-$ measurable:

$$\{B_t < 0\} = \{\frac{d\langle M \rangle}{dt}(t) = c'\} \text{ and } \{B_t > 1\} = \{\frac{d\langle M \rangle}{dt}(t) = c''\},$$

and similarly for martingales $(M_n^k), n \geq 1, k \in \{1, ..., \ell_n\}$.

Point 2 . According to Point 1, the martingale $\int \mathbf{1}_{F^c}(B)dB = \sum_n \sum_k M_n^k$ is $\mathcal{F}^M-$adapted, so that's its quadratic variation.

Point 3 . We will only show that $0 \in \overline{F^c}$, more precisely $\inf F^c = 0$.

Let $x_n = \inf F_n^c$. We have

$$x_n = \frac{x_{n-1}}{2} - \frac{1}{2 \times 4^n}, n \geq 2$$

and $x_1 = \frac{3}{8}$.

Hence

$$x_n = \frac{x_1}{2^{n-1}} - \sum_{k=2}^{n} \frac{1}{2^{n+1-k} \times 4^k}.$$

But

$$\sum_{k=2}^{n} \frac{1}{2^{-k} \times 4^k} = \frac{1}{2^n \times 4}(1 - (\frac{1}{2})^{n-1}),$$

and then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2^{n+1}}(1 - \frac{1}{2^n}) = 0.$$

References

1. J. Azéma, C. Rainer, and M. Yor. Une propriété des martingales pures. *Séminaire de probabilités de Strasbourg*, 30:243–254, 1996.
2. M. T. Barlow, M. Émery, F. B. Knight, S. Song, and M. Yor. Autour d'un théorème de tsirelson sur des filtrations Browniannes et non Browniannes. In *Séminaire de Probabilités XXXII*, pages 264–305. Springer, 1998.
3. S. Beghdadi Sakrani. *Martingales continues, Filtrations faiblement Browniennes et Mesures signées*. PhD thesis, Paris 6, 2000.
4. S. Beghdadi-Sakrani and M. Emery. On certain probabilities equivalent to coin-tossing, d'après schachermayer. In *Séminaire de Probabilités XXXIII*, pages 240–256. Springer, 1999.
5. C. Dellacherie. *Probabilités et potentiel: Tome 5, Processus de Markov (fin): Compléments de calcul stochastique*, volume 5. Hermann, 2008.
6. L. Dubins, J. Feldman, M. Smorodinsky, B. Tsirelson, et al. Decreasing sequences of *sigma*-fields and a measure change for Brownian motion. *The Annals of Probability*, 24(2):882–904, 1996.
7. M. Émery and W. Schachermayer. Brownian filtrations are not stable under equivalent time-changes. *Séminaire de probabilités de Strasbourg*, 33:267–276, 1999.
8. F. B. Knight. On invertibility of martingale time changes. In *Seminar on Stochastic Processes, 1987*, pages 193–221. Springer, 1988.
9. D. A. Lane. On the fields of some Brownian martingales. *The Annals of Probability*, pages 499–508, 1978.
10. S. Laurent. On standardness and i-cosiness. In *Séminaire de Probabilités XLIII*, pages 127–186. Springer, 2011.
11. L. PETROVIĆ and D. VALJAREVIĆ. Statistical causality and martingale representation property with application to stochastic differential equations. *Bulletin of the Australian Mathematical Society*, 90(2):327–338, 2014.
12. D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.
13. D. Stroock and M. Yor. On extremal solutions of martingale problems. In *Annales scientifiques de l'École Normale Supérieure*, volume 13, pages 95–164, 1980.
14. B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. *Geometric and Functional Analysis*, 7(6):1096–1142, 1997.
15. M. Yor. Sur l'étude des martingales continues extrémales. *Stochastics: An International Journal of Probability and Stochastic Processes*, 2(1-4):191–196, 1979.

Sakrani Samia,
Department of Mathematics,
University of 8 Mai 1945, Guelma,
Algeria.
E-mail address: `sakrani.samia@univ-guelma.dz`