Non-extremal Martingale with Brownian Filtration

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ABSTRACT: Let $\langle B_t \rangle_{t \geq 0}$ be the filtration of a Brownian motion $\langle B_t \rangle_{t \geq 0}$ on $(\Omega, \mathcal{B}, P)$. An example is given of a non-extremal martingale which generates the filtration $\langle B_t \rangle_{t \geq 0}$. We also discuss a property of pure martingales, we show here that it is a property of a filtration rather than a martingale.

Key Words: Extremal martingale, Brownian filtration, Pure martingale, Pure filtration.

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1. Introduction

Among the series of questions asked at the end of the chap.V of [12]) (or also in [13] and [15]) is the following question: a filtration being given on a probability space, how to recognize if it is generated by a Brownian motion or not? This question is especially of interest for a weakly Brownian filtration (there exists an $\mathcal{F}$-Brownian motion which has the predictable representation property (PRP) with respect to $\mathcal{F}$, see [11] for application of this important property). In all generality, there are weakly Brownian filtrations, which are not Brownian, as it is shown in [6], paper that was followed by other examples of non-Brownian filtrations given in [4], [7], [14]. These works are important progress that raises many new questions, including how to establish the non-Brownian character of a weakly Brownian filtration?

In all the works above, it is the notion of non-cosiness (introduced by Tsirel’son in [14] and that we will not discuss in this paper) of these filtrations which serves as a criterion to show that they are non-Brownian, see [4], [10] for different types of cosiness: I-cosiness, D-cosiness and T-cosiness. One might think that a filtration generated by a non-pure extremal martingale or non-extremal martingale can not be Brownian. In fact we show in Section 3 that this is not true. The non-Brownian character of a weakly Brownian filtration is much more delicate. Section 4 shows that Brownian filtration can be generated by non-pure extremal martingale. In section 5, we discuss the following property denoted by $(\star)$ in [1]: If $M$ is a continuous martingale and $\mathcal{F} = \mathcal{F}^M$, for every, $\mathcal{F}$-stopping time $T$ finite a.s such that $\mathbb{P}(M_T = 0) = 0$, then

$$\mathcal{F}_{G_T}^+ = \mathcal{F}_{G_T}^- \lor \sigma(M_T < 0),$$

where $G_T = \sup\{s \leq T, M_s = 0\}, T \in [0, \infty[. Authors of [1] have shown that property $(\star)$ is satisfied by any pure martingale. It is understood here that $(\star)$ is a property of a filtration rather than a martingale.

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2. Preliminaries

We will only consider completed probability spaces and right continuous filtrations. We denote ∫ HdX the stochastic integral of H with respect to X and FX the natural filtration of X. An FX−continuous local martingale X has the PRP (the predictable representation property) if for every FX−continuous local martingale M there exists an FX−predictable process H such that

\[ M = M_0 + \int HdX, \]

X is called FX−extremal if FX is trivial and X has the FX−PRP. If FX = FX then X is called extremal martingale. (this terminology is justified by the fact that the law of an extremal martingale is an extremal point in the convex set of all probability measures on W = C(ℝ+;ℝ), which make the coordinate process a local martingale). A continuous local martingale X with (X)∞ = ∞ is pure if FX = FX∞ where B is the Brownian motion of Dubins-Schwartz (DDS) associated with X, which is equivalent to say that for all t, (X)t is FX∞ measurable.

Every pure martingale is extremal but the opposite is not true. Yor has given in [15] an example of an extremal martingale which is not pure; we will prove here that its natural filtration is Brownian.

**Definition 2.1.** A filtration FX is said to be immersed in a filtration G( defined on the same probability space) if any FX-martingale is G-martingale.

3. Example of non-extremal martingale with Brownian filtration

We have the following characterization of extremal martingales with respect to Brownian filtration:

**Lemma 3.1.** If B is a Brownian motion, B its natural filtration and M is a B− martingale, then M is B−extremal if and only if d(M) is equivalent to λ a.s, where λ is the Lebesgue measure on ℝ+.

**Proof.** M is a B− martingale, so there exists a B−predictable process H such that:

\[ M = M_0 + \int HdB \text{ and } H^2 = \frac{d(M)}{d\lambda}. \]

If M is B−extremal, then there exists a B−predictable process K such that B = ∫ KdM and dλ = K2d⟨M⟩, that is d⟨M⟩ is equivalent to λ. If now, d⟨M⟩ is equivalent to λ, it is enough to represent B as a stochastic integral with respect to M. We have H ≠ 0, λ ⊗ dP a.s so B = ∫ HdB.

Lane [9], gave partial answers to the following question [12]: If B is a Brownian motion, f is borel function and M is the local martingale ∫ f(B)dB, under what conditions the filtration FM is Brownian?. An important example is when f ≥ 0 and μ(\{f = 0\}) > 0 but the set \{f = 0\} does not contain any interval (μ is the Lebesgue measure on ℝ). This case was studied by knight [8] with \( F = \{f = 0\} \) is a subset of \([0, 1]\), defined by the Cantor method: removing \([\frac{1}{3}, \frac{2}{3}]\) then \([\frac{1}{9}, \frac{2}{9}]\) and \([\frac{7}{9}, \frac{8}{9}]\) and so on. We define the set \( F_n^c \) by means of its complementary \( F_n^c \),

\[ F_1^c = [\frac{3}{8}, \frac{5}{8}], F_2^c = F_1^c \cup [\frac{5}{32}, \frac{7}{32}], F_3^c = F_2^c \cup [\frac{9}{32}, \frac{11}{32}], F_4^c = F_3^c \cup [\frac{13}{32}, \frac{15}{32}], \]

\[ F_n^c = F_{n-1}^c \cup \bigcup_{k=1}^{2^{n-1}} A_n^k, \quad n \geq 2, \]

where \( A_n^k = [a_n^k, b_n^k] \) are disjoint intervals of length \( \frac{1}{4^n} \). Finally

\[ F^c = \bigcup_n F_n^c = \bigcup_{n \geq 1} \bigcup_{k=1}^{\ell_n} A_n^k, \]

with \( \ell_n = \sum_{k=0}^{n-1} 2^k = 2^n - 1 \). Hence \( \mu(F^c) = \lim_{n \to \infty} \mu(F_n^c) = \sum_{n=1}^{\infty} \frac{2^n - 1}{4^n} = \frac{1}{2} \).
Theorem 3.2. Let $B$ be a Brownian motion, $\mathcal{B}$ its natural filtration and $M$ the martingale defined by

$$M = c' \int 1_{\{B<0\}} dB + c'' \int 1_{\{B>1\}} dB + \sum_{n \geq 1} \sum_{k=n}^{\ell_n} c_n^k \int 1_{A_n^k}(B) dB,$$

where the numbers $(c_n^k)$, $n \geq 1$, $k \in \{1, \ldots, \ell_n\}$, $c'$ and $c''$ are strictly positive and all different. The martingale $M$ is not extremal and we have $\mathcal{F}^M = \mathcal{B}$.

Remark 3.3. In order not to burden the proof of Theorem 1, at the end of this paper (in the appendix) we have gathered some non-detailed points.

Proof. The processes $B^-$ and $(B-1)^+$ are $\mathcal{F}^M$-adapted (Point 1), it remains to show that $B_t 1_{\{0<B_t<1\}}$ is $\mathcal{F}^M$-adapted. We consider the martingales

$$M_n^k = \int 1_{A_n^k}(B) dB$$

($M_n^k$) are also $\mathcal{F}^M$-adapted (Point 1). The stopping times $\{(S_n^k)^r, (T_n^k)^r\}_{r \geq 1}$ of the successive entries and exits of $B$ in the set $A_n^k$ are $\mathcal{F}^M_{\infty}$-measurable because these are the moments where $\Delta C_n^k > 0$, with $C_n^k$ the inverse of $< M_n^k >$.

Fix $n \in \mathbb{N}^*$, $k \in \{1, \ldots, \ell_n\}$ and for every $r \in \mathbb{N}^*$

$$S^r := (S_n^k)^r, \quad T^r := (T_n^k)^r, \quad A_n^k \subset ]a,b[, \quad N := M_n^k \quad \text{and} \quad \alpha := c_n^k.$$

(Attention! $a$, $b$, $N$ and $\alpha$ depend on $k$ and $n$).

Let us show that the sequence $(B_{S^r}, B_{T^r})_{r \geq 1}$ is $\mathcal{F}^M$-measurable. We have, $N_t = 0$ until $S^1$ and $B_{S^1} = a$. If $t \in [S^1, T^1]$, then

$$N_t = \int_{S^1}^t dB_s = B_t - a.$$ 

So, we know $B_{T^1}$ and for every $r \geq 1$ and $t \in [S^r, T^r]$ we have

$$M_t - M_{S^r} = \alpha(N_t - N_{S^r}) = \alpha(B_t - B_{S^r}) \quad \text{(1)}$$

Therefore

$$M_t - M_{S^r} = \alpha(B_{T^r} - B_{S^r})$$

Then, if we know $M$ and $B_{T^r}$, we can know $B_{S^r}$ (and the inverse is true).

If $M_{T^r} - M_{S^r} > 0$ then $B_{T^r} = b$ and $B_{S^r} = a$. If $M_{T^r} - M_{S^r} < 0$ then $B_{T^r} = a$ and $B_{S^r} = b$.

It remains the case where $M_{T^r} - M_{S^r} = 0$ so $B_{T^r} = b$ (and then $B_{T^r} = B_{S^r}$). Remark that

$$B_{T^r} = B_{S^r+1} \quad \text{(2)}$$

Indeed, if $B$ is above $]a,b[$ after $T^r$, then $B_{T^r} = b = B_{S^r+1}$, and if $B$ is below $]a,b[$ after $T^r$, then $B_{T^r} = a = B_{S^r+1}$.

Suppose we know $M$ until time $t$, since we know $B_{T^1}$, then, from (2), we can know $B_{S^2}$ and $B_{T^2}$ and so on, we can know the sequence $(B_{T^r}, B_{S^r})$ for $T^r, S^r \leq t$.

To finish the proof, let $t_0 \leq t$, the set $\{B_{t_0} \in F^c\}$ is $\mathcal{F}^M_{t_0}$-measurable (Point 2). If $B_{t_0} \in F^c$, then there exists $n$ and $k$ such that $B_{t_0} \in A_n^k$ and so, there exists $r$ such that $t_0 \in [S^r, T^r]$. We have

$$B_{t_0} = B_{t_0} - B_{S^r} + B_{S^r}.$$ 

and equality (1) gives

$$B_{t_0} = \frac{1}{\alpha}(M_{t_0} - M_{S^r}) + B_{S^r}.$$ 

Since $F^c$ is dense in $[0,1]$ (Point 3), we have

$$B_t 1_{\{0<B_t<1\}} = \limsup_{s \downarrow t} B_s 1_{\{B_s \in F^c\}} \quad \text{and} \quad \mathcal{F}^M = \mathcal{B}.$$ 

It remains to establish that $M$ is non-extremal. This follows easily from Lemma 1, since $\lambda(F) > 0$. □
4. Examples of extremal non-pure martingales with Brownian filtrations

We will now show that the filtration of the extremal non-pure martingale given in [15] is Brownian.

**Theorem 4.1.** Brownian filtration is generated by a non-pure extremal martingale.

**Proof.** Let \( B \) be a Brownian motion and \( \mathcal{B} \) its natural filtration. We start by considering the stochastic equation

\[
    dX_t = \varphi(X_t)dB_t, \quad X_0 = 0,
\]

where \( \varphi(x) = \frac{1}{\sqrt{2 + \frac{|x|}{t + |x|}}} \). We easily check that:

\[
    \left| \varphi(x) - \varphi(x') \right|^2 \leq c \left| \frac{1}{\varphi(x)} - \frac{1}{\varphi(x')} \right|^2 \leq c \left| \frac{x}{1 + |x|} - \frac{x'}{1 + |x'|} \right|
\]

and

\[
    \frac{1}{\sqrt{3}} \leq \varphi(x) \leq 1, \forall x, x' \in \mathbb{R}.
\]

The function \( \frac{x}{1 + |x|} \) is strictly increasing, we apply theorem 3.5(iii), chap.IX of [12] and we get \( \mathcal{F}^X = \mathcal{B} \).

We have, \( \langle X \rangle = \int \varphi^2(X_t)dt \), since \( \varphi^2 \) is continuous and strictly decreasing

\[
    \mathcal{F}^{\langle X \rangle} = \mathcal{F}^X
\]

We define the martingale

\[
    M_t = \tilde{\gamma}(X_t),
\]

where \( \tilde{\gamma}_t = \int_0^t \text{sgn}\gamma_s d\gamma_s \) and \( \gamma \) is the DDS Brownian motion associated to \( X \). We have \( \langle X \rangle = \langle M \rangle \) then

\[
    \mathcal{F}^{\langle M \rangle} = \mathcal{F}^M = \mathcal{B}.
\]

It remains to show that \( M \) is extremal but non-pure. Since \( \varphi \) is strictly positive, \( d\langle M \rangle \) is equivalent to Lebesgue measure and \( \mathcal{F}^M \) is a Brownian filtration, therefore, using Lemma 1, we deduce that \( M \) is extremal. \( M \) is non-pure because

\[
    \mathcal{F}^{\tilde{\gamma}}_\infty \subsetneq \mathcal{F}^{\langle M \rangle}_\infty = \mathcal{F}_\infty^M.
\]

Here is another example of non-pure extremal martingale with Brownian filtration:

**Theorem 4.2.** Let \( B \) be a Brownian motion. There exists a strictly positive predictable process \( H \) such that \( N_t = \int_0^t H(B_u, u \leq s)dB_s \) is non-pure extremal martingale.

**Proof.** Let \( (T_t) \) be absolutely continuous and strictly increasing time change of Theorem 4.1 of [7]. Then \( M_t := (B_{T_t}) \) generates non-Brownian filtration. We have \( M_t = \int_0^t f(M_u, u \leq s)d\gamma_s \) (see Proposition 3.8, Chap V of [12]), for \( \gamma \) a Brownian motion and \( f \) predictable process which can be choose strictly positive. Since \( M \) is pure by construction (so \( \mathcal{F}^M_B = \mathcal{F}^B \)), \( B_t = \int_0^t g(B_u, u \leq s)d\gamma_{C_t} \), where \( g \) is \( \mathcal{F}^B \)-predictable process and \( C \) the inverse of \( T \), so

\[
    \gamma_{C_t} = \int_0^t H_s dB_s,
\]

with \( H = \frac{1}{\sqrt{3}} \). Since the filtration of \( M \) is non Brownian, \( \mathcal{F}^M \neq \mathcal{F}^\gamma \) and the martingale \( N = \gamma_{C} \) is not pure. But \( \mathcal{F}^N = \mathcal{F}^B \) and \( H \) is strictly positive, then \( N \) is extremal by Lemma 1.

**Remark 4.3.** Theorem 3 responds affirmatively to the following question asked at the end of Chap V of [12]: is there a strictly positive predictable process \( H \) such that the martingale \( N_t = \int_0^t H_s dB_s \) is not pure?
5. A martingale class that satisfy property (⋆)

In [1], authors discussed a property (⋆) verified by all pure martingales and gave some examples of non-pure extremal martingales and non-extremal martingales that nevertheless satisfy property (⋆). In [2], we better understand this property that we reset here: Let \( M \) be a continuous martingale and \( \mathcal{F} = \mathcal{F}^M \), for every, \( \mathcal{F} \)-stopping time \( T \) finite \( a.s \) such that \( P(M_T = 0) = 0 \), we have

\[
\mathcal{F}^+_{G_T} = \mathcal{F}^-_{G_T} \vee \sigma(M_T < 0),
\]

where \( G_T = \sup\{s \leq T, M_s = 0\}, T \in [0, \infty[. \) The example given in [1] of non-pure extremal martingale satisfying property (⋆) is in fact the example of Yor [15]. We have shown that its filtration is Brownian and therefore, it is obvious that this martingale satisfies (⋆) using Barlow’s property proven in [2]. In the same way, our non-extremal martingale of Theorem 1, satisfies (⋆).

In general, the following proposition can be stated:

**Proposition 5.1.** Let \( \mathcal{F} \) be a filtration such that all \( \mathcal{F} \)-martingales are continuous and \( \text{SpMult}[\mathcal{F}] \leq 2 \) (see the definition below), then all martingales generating \( \mathcal{F} \) satisfy property (⋆).

Before proving the proposition, we recall the following definition:

**Definition 5.2.** Let \( (\Omega, \mathcal{A}, P) \) be probability space and \( \mathcal{F} \) a sub-field of \( \mathcal{A} \). Let \( \Omega \) be the set of all finite measurable partitions of \( (\Omega, \mathcal{A}) \), for \( Q \in \Omega, |Q| \) is the cardinal of \( Q \). The conditional multiplicity of \( \mathcal{A} \) with respect to \( \mathcal{F} \) is the random variable with values in \( \mathbb{N}^* \cup \{\infty\} \)

\[
\text{Mult}[\mathcal{A} \mid \mathcal{F}] = \text{ess sup}_{Q \in \Omega} |Q| \cdot 1_{S_B(Q)}
\]

where \( S_B(Q.) = \{\forall A \in Q, P(A \mid \mathcal{F}) > 0\} \). The splitting multiplicity of a filtration \( \mathcal{F} \), \( \text{SpMult}[\mathcal{F}] \) is the smallest integer \( n \) such that: \( \text{Mult}[\mathcal{F}_{L^n} \mid \mathcal{F}_L] \leq n \), for any honest time \( L \) of \( \mathcal{F} \).

**Proof.** Using proposition 1 of [1], it is enough to show (⋆) for \( T = t \).

Let \( A = \{M_t > 0\} \), we have \( E[M_t \mid \mathcal{F}_{G_t}] = 0 \) a.s, because \( M_{G_t} = 0 \) a.s (by Theorem XX-35 of [5]). Then a.s

\[
E[M_t1_A \mid \mathcal{F}_{G_t}] = -E[M_t1_{A^c} \mid \mathcal{F}_{G_t}].
\]

(3)

We define the sets \( C_1 = \{P(A \mid \mathcal{F}_{G_t}) = 0\} \) and \( C_2 = \{P(A^c \mid \mathcal{F}_{G_t}) = 0\} \) which are in \( \mathcal{F}_{G_t} \). We have \( P(A \cap C_1) = 0 \) and \( P(A^c \cap C_2) = 0 \).

And for every \( n \in \mathbb{N} \):

\[
E[1_{C_1}M_t1_{\{0<M_t<n\}} \mid \mathcal{F}_{G_t}] \leq nP(A \cap C_1 \mid \mathcal{F}_{G_t}) = 0,
\]

then

\[
1_{C_1}E[M_t1_A \mid \mathcal{F}_{G_t}] = 0
\]

and from (3), we have

\[
1_{C_1}E[M_t1_{A^c} \mid \mathcal{F}_{G_t}] = 0.
\]

So, \( E[M_t1_{C_1 \cap A^c}] = 0 \) and \( C_1 \subset \{M_t = 0\} \).

Similarly, we have \( C_2 \subset \{M_t = 0\} \) Applying hypothesis \( P\{M_t = 0\} \) is null, we get \( P(C_1 \cup C_2) = 0 \) So

\[
\mathcal{F}^+_{G_t} = \mathcal{F}_{G_t} \vee \sigma(M_t > 0),
\]

according to proposition 3 of [2] (see also Lemma 4.3 ,Chap . I of [3]).

Here is an example of a filtration with \( \text{SpMult} \leq 2 \).

**Definition 5.3.** A filtration generated by a pure martingale is called pure filtration.
Proposition 5.4. Let $\mathcal{F}$ be a filtration, $C = (C_t)$ time change for $\mathcal{F}$ and $\mathcal{F}' = (\mathcal{F}_C)$. We have:

(a) $\text{SpMult}(\mathcal{F}) \leq \text{SpMult}(\mathcal{F}')$. If moreover $C$ is strictly increasing, we have: $\text{SpMult}(\mathcal{F}) = \text{SpMult}(\mathcal{F}')$. In particular, if $\mathcal{F}$ is pure (non trivial), then $\text{SpMult}(\mathcal{F}) = 2$.

(b) Let $\mathcal{F}$ be the natural filtration of a continuous martingale $M$ and $C$ the inverse of $\langle M \rangle$, we suppose that $\langle M \rangle$ is strictly increasing and $\langle M \rangle_\infty = \infty$. If $\mathcal{F}'$ is Brownian, then $M$ is extremal and $\mathcal{F}$ is pure.

Proof. (a) Suppose $\text{SpMult}(\mathcal{F}') = n \in \mathbb{N}^*$.

Let $M$ be $\mathcal{F}$-spider martingale of multiplicity $n + 1$, bounded and $M_0 = 0$. Then $M_c = \mathbb{E}[M_\infty \mid \mathcal{F}]$ is $\mathcal{F}$-spider martingale of multiplicity $n + 1$ vanishing at the origin, Proposition 13 of [2] gives $M_\infty = 0$ a.s and $\text{SpMult}(\mathcal{F}) \leq n$. If $C$ is strictly increasing and if $\tau$ is its inverse, then by Lemma 5.9 of [13], we have

$$\mathcal{F}_\tau = (\mathcal{F}_C)_\tau = \mathcal{F}.$$  

If $\mathcal{F}$ is pure, then there exists a time change which we also note $C$, such that $\mathcal{F}_C$ is Brownian, then $\text{SpMult}(\mathcal{F}) = 2$ and $\text{SpMult}(\mathcal{F}') \leq 2$.

(b) Let $W$ be a Brownian motion that generates $\mathcal{F}'$ and $X$ the martingale $W_\langle M \rangle$ (by construction, $X$ is pure).

Let us show that $M$ is extremal: let $B$ be the DDS Brownian motion of $M$, $B$ is $\mathcal{F}'$-Brownian motion that has $\mathcal{F}' - \text{PRP}$ (because $\mathcal{F}$ is Brownian), as $\mathcal{F}_C_0$ is trivial, $\mathcal{F}_0$ is too, and $M$ is extremal. Notice now that

$$\mathcal{F}_\infty^X = \mathcal{F}_\infty^W = \mathcal{F}_\infty = \mathcal{F}_\infty.$$  

(4)

and

$$M_t = \int_0^t \varepsilon_{\langle M \rangle} dX_s,$$

with $\varepsilon_t = \frac{d(B,W)_t^2}{dW_t^2}$. Hence $X$ is $\mathcal{F}$-extremal (and since it is extremal), Proposition 7.1 of [13], gives us that $\mathcal{F}_X$ is immersed in $\mathcal{F}$. So we have $\mathcal{F} = \mathcal{F}_X$ using (4). \qed

The next question naturally arises: The reciprocal of proposition 1 is it true? i.e if all the martingales that generate a filtration $\mathcal{F}$ satisfy the property $(\star)$, do we have $\text{SpMult}(\mathcal{F}) = 2$?

For now, we do not have a general answer to this question. In any case, let us note that the following example given in [1] section 6, does not give a negative answer, let

$$M_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{(X_s^2 + Y_s^2)^{\alpha}},$$

where $(X_t + iY_t)$ is a planar Brownian motion starting from $z \in \mathbb{C}^+$ and $\alpha \in [-\infty, \frac{1}{2}]$. Let $\mathcal{F}$ be the filtration of $M$, $C$ the inverse of $\langle M \rangle$ and $\mathcal{F}' = (\mathcal{F}_C)_t \geq 0$, $\mathcal{F}'$ is Brownian, so $\mathcal{F}$ is pure and according to proposition 1, $M$ satisfy property $(\star)$.

6. Appendix

**Point 1.** We have

$$\int 1_{\{B < 0\}} dB = \frac{1}{c'} \int 1_{\{B < 0\}} dM$$

and

$$\int 1_{\{B > 1\}} dB = \frac{1}{c''} \int 1_{\{B > 1\}} dM.$$ 

Hence, by applying Skorokhod’s Lemma (Lemma 2.1, Chap.VI of [12]) it is sufficient to see that the sets $\{B_t < 0\}$ and $\{B_t > 1\}$ are $\mathcal{F}_t^M$-measurable:

$$\{B_t < 0\} = \left\{\frac{d\langle M \rangle}{dt}(t) = c'\right\}$$

and

$$\{B_t > 1\} = \left\{\frac{d\langle M \rangle}{dt}(t) = c''\right\},$$
and similarly for martingales \((M^k_n), n \geq 1, k \in \{1, \ldots, \ell_n\}\).

**Point 2.** According to Point 1, the martingale \(\int 1_{F^c}(B)dB = \sum_n \sum_k M^k_n\) is \(\mathcal{F}^M\)-adapted, so that’s its quadratic variation.

**Point 3.** We will only show that \(0 \in \overline{F^c}\), more precisely \(\inf F^c = 0\).

Let \(x_n = \inf F^c_n\). We have

\[
x_n = x_{n-1} - \frac{1}{2} \times 4^n, n \geq 2
\]

Hence \(x_1 = \frac{3}{7}\).

But

\[
\sum_{k=2}^{n} \frac{1}{2^{k-1}} = \frac{1}{2} \times 4 \left(1 - \frac{1}{2}^{n-1}\right),
\]

and then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{2^{n+1}} (1 - \frac{1}{2^n}) = 0.
\]

### References


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