Three Nontrivial Solutions of Boundary Value Problems for Semilinear $\Delta_\gamma$–Laplace Equation *

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ABSTRACT: In this paper, we study the multiplicity of weak solutions to the boundary value problem

$$\Delta_\gamma u + f(x, u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 2$) and $\Delta_\gamma$ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j} \left( \gamma_j^2 \partial_{x_j} \right), \quad \gamma_j := \frac{\partial}{\partial x_j}, \quad \gamma = (\gamma_1, \gamma_2, ..., \gamma_N),$$

the nonlinearity $f(x, \xi)$ is subcritical growth and may be not satisfy the Ambrosetti-Rabinowitz (AR) condition.

We establish the existence of three nontrivial solutions by using Morse theory.

Key Words: Semilinear degenerate elliptic equations, Morse theory, Three solutions, Multiple solutions.

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1. Introduction

In the last decades, the boundary value problem for semilinear elliptic equations

$$-\Delta u = f(x, u), \quad x \in \Omega, \quad u \in H^1_0(\Omega),$$

has been studied by many authors, see, for example [1,20] and the references therein. The following (AR) condition introduced in [1]

(AR) For some $\theta > 2$ and $R > 0$, we have

$$\theta F(x, \xi) \leq f(x, \xi)\xi, \quad \forall \ |\xi| \geq R, \quad \forall \ x \in \Omega,$$

where $F(x, \xi) = \int_0^\xi f(x, \tau)\, d\tau$, plays an important role in their studies. Boundary value problems for nonlinear degenerate elliptic differential equations were treated in [10] and then subsequently in [8,5]. In [25,26], the critical exponent phenomenon was observed for a model of the Grushin type operators. The results were then generalized in [23] to a large class of semilinear degenerate elliptic differential equations. Recently, in [23,24] the second author of this paper and his collaborator have extended the research to a more complicated class of nonlinear degenerate elliptic differential operators. Very recently, the authors of [11] investigated the $\Delta_\gamma$–Laplace operator under the additional assumption that the operator is homogeneous of degree two with respect to a semigroup of dilations in $\mathbb{R}^N$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [27,28] (see also some recent results in [2,3,11,12,13,14,15,16,17,18,19,22,24,26]).

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In this paper, we study multiplicity of weak solutions to the following problem

\[
\begin{align*}
\Delta \gamma u + f(x, u) &= 0 & \text{in} & & \Omega, \\
u &= 0 & \text{on} & & \partial \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain with smooth boundary in \(\mathbb{R}^N\), \(\Delta\gamma\) (see the definition of this function space below) and \(f(x, \xi) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) such that \(f(x, 0) = 0\).

Let \(F(x, \xi) = \int_0^\xi f(x, \tau)d\tau\) and suppose that the non-linearity \(f\) satisfies the following conditions:

(A1) \(f \in C(\bar{\Omega} \times \mathbb{R})\) with \(f(x, 0) = 0\) and satisfies the improved subcritical polynomial growth condition, i.e.

\[\lim_{|\xi| \to \infty} \frac{f(x, \xi)}{|\xi|^{2\gamma^*} - 1} = 0\]

uniformly for \(x \in \Omega\),

where \(2^* := 2\bar{N}/(\bar{N} - 2)\);

(A2) \(\lim_{|\xi| \to 0} \frac{f(x, \xi)}{|\xi|} = p(x)\), uniformly for \(x \in \Omega\), where \(p \in L^\infty(\Omega)\) satisfies \(p(x) \leq \lambda_1\) for all \(x \in \Omega\) and \(p(x) < \lambda_1\) on some \(\Omega_0 \subset \Omega_1\) with \(|\Omega_0| > 0\), where \(\Omega_1 := \{x \in \Omega : \phi_1(x) \neq 0\}\) and \(\lambda_1 > 0\) that has an associated eigenfunction \(\phi_1\) is the first eigenvalue of \(-\Delta\gamma\) with homogeneous Dirichlet boundary data;

(A3) \(f(x, \xi)\) is superlinear at infinity, i.e.

\(\lim_{|\xi| \to +\infty} f(x, \xi)/|\xi| = +\infty\)

uniformly for all \(x \in \Omega\);

(A4) There exist \(\theta \geq 1\) and \(C(x) \in L^1_+(\Omega)\) such that \(\theta \mathcal{F}(x, \xi) \geq \mathcal{F}(x, s\xi) - C(x)\) for \((x, \xi) \in \Omega \times \mathbb{R}\) and \(s \in [0, 1]\), where \(\mathcal{F}(x, \xi) = f(x, \xi) - 2F(x, \xi)\).

The condition (A4) was first introduced by L. Jeanjean [7], there are many functions which satisfy (A4), but do not satisfy the (AR) condition. An example of such function is

\[f(x, \xi) = \xi \ln(1 + |\xi|).
\]

Our main result is given by the following theorem.

**Theorem 1.1.** Assume conditions (A1)-(A4) hold. Then the problem (1.1)-(1.2) has at least three nontrivial solutions.

The structure of our note is as follows: In Section 2, we give some preliminary results. In Section 3, we proved Theorem 1.1.

## 2. Preliminary results

First of all, let us collect some concepts and results of Morse theory that will be used below. For the details, we refer to [4]. Let \(X\) be a real Banach space and \(\Phi \in C^1(X, \mathbb{R})\). \(K = \{u \in X : \Phi'(u) = 0\}\) is the critical set of \(\Phi\). Let \(u \in K\) be an isolated critical point of \(\Phi\) with \(\Phi(u) = c \in \mathbb{R}\), and \(U\) be an isolated neighborhood of \(u\), i.e. \(K \cap U = \{u\}\). The group

\[C_m(\Phi, u) = H_m(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad m = 0, 1, 2, \ldots,
\]

is called the \(m\)-th critical group of \(\Phi\) at \(u\), where \(\Phi^c = \{u \in X : \Phi(u) \leq c\}\).

\(H_m(\cdot, \cdot)\) is the singular relative homology group of \(\Phi\) at infinity is defined by

\[C_m(\Phi, \infty) = H_m(X, \Phi^c), \quad m = 0, 1, 2, \ldots.
\]

We denote

\[P(u, t) = \sum_i \text{rank} \, C_i(\Phi, u)t^i, \quad P(\infty, t) = \sum_i \text{rank} \, C_i(\Phi, \infty)t^i.
\]
Let $\alpha < \beta$ be the regular values of $\Phi$ and set
\[
P(\alpha, \beta, t) = \sum_i \text{rank} C_i(\Phi, \infty) t^i.
\]
If $K = \{u_1, u_2, \ldots, u_k\}$, then there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that
\[
\sum_j P(u_j, t) = P(\infty, t) + (1 + t)Q(t),
\]
and for every $\gamma_j$.

By Definition 2.1.
\[
\sum_{\alpha < \Phi(u_j) < \beta} P(u_j, t) = P(\alpha, \beta, t) + (1 + t)Q(t).
\]

Throughout the paper $\Omega$ denotes a bounded domain with smooth boundary in $\mathbb{R}^N, N \geq 2$. As in [11], we consider the operators of the form
\[
\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j} \left( \gamma_j^2 \partial_{x_j} \right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, N.
\]
Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class $C^1$ in $\mathbb{R}^N\setminus \Pi$, where
\[
\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.
\]
Moreover, we assume the following properties:
i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that
\[
\delta_t : \mathbb{R}^N \to \mathbb{R}^N, \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), 1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N,
\]
such that $\gamma_j$ is $\delta_t$--homogeneous of degree $\varepsilon_j - 1$, i.e.,
\[
\gamma_j(\delta_t(x)) = t^{\varepsilon_j-1} \gamma_j(x), \forall x \in \mathbb{R}^N, \forall t > 0, j = 1, \ldots, N.
\]
The number
\[
\vec{N} := \sum_{j=1}^{N} \varepsilon_j
\]
is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

ii) $\gamma_1 = 1, \gamma_j(x) = \gamma_j(x_1, x_2, \ldots, x_{j-1}), j = 2, \ldots, N$.

iii) There exists a constant $\rho \geq 0$ such that
\[
0 \leq x_k \partial_{x_k} \gamma_j(x) \leq \rho \gamma_j(x), \forall k \in \{1, 2, \ldots, j-1\}, \forall j = 2, \ldots, N,
\]
and for every $x \in \mathbb{R}^N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \ldots, N\}$.

iv) Equalities $\gamma_j(x) = \gamma_j(x^*)$ $(j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where
\[
x^* = (|x_1|, \ldots, |x_N|) \text{ if } x = (x_1, x_2, \ldots, x_N).
\]

**Definition 2.1.** By $S^p_\gamma(\Omega)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^p(\Omega)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows
\[
\|u\|_{S^p_\gamma(\Omega)} = \left\{ \int_\Omega \left( \left| u \right|^p + \sum_{j=1}^{N} \left| \gamma_j \partial_{x_j} u \right|^p \right) dx \right\}^{\frac{1}{p}}.
\]
If $p = 2$ we can also define the scalar product in $S^2_\gamma(\Omega)$ as follows

$$(u, v)_{S^2_\gamma(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^N (\gamma_j \partial_{x_j} u, \gamma_j \partial_{x_j} v)_{L^2(\Omega)}.$$ 

The space $S^p_{\gamma,0}(\Omega)$ is defined as the closure of $C^1_0(\Omega)$ in the space $S^p_\gamma(\Omega)$.

Set

$$\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u),$$

$$|\nabla_\gamma u| := \left( \sum_{j=1}^N |\gamma_j \partial_{x_j} u|^2 \right)^{\frac{1}{2}}.$$ 

From Proposition 3.2 and Theorem 3.3 in [11], we have the following embedding result.

Proposition 2.1. Assume that $\tilde{N} > 2$. Then $S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega)$, where $1 \leq p \leq \frac{2\tilde{N}}{N - 2}$. Moreover, the number $2^*_\gamma = \frac{2\tilde{N}}{N - 2}$ is the critical Sobolev exponent of the embedding $S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega)$ and when $1 \leq p < 2^*_\gamma$, the embedding is compact.

We now give some examples of the $\Delta_\gamma$–Laplace operator. We use the following notations: we split $\mathbb{R}^N$ into

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

and write

$$x = (x^{(1)}, x^{(2)}, x^{(3)}), \quad x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_{N_i}^{(i)}) \in \mathbb{R}^{N_i},$$

$$|x^{(i)}|^2 = \sum_{j=1}^{N_i} |x_j^{(i)}|^2, \quad i = 1, 2, 3.$$ 

We denote the classical Laplace operator in $\mathbb{R}^{N_i}$ by

$$\Delta_{x^{(i)}} := \sum_{j=1}^{N_i} \partial_{x_j^{(i)}}^2.$$ 

Example 2.2. Let $\alpha$ be a real positive number. The operator

$$\Delta_\gamma := \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} (\Delta_{x^{(2)}} + \Delta_{x^{(3)}}),$$

where

$$\gamma = \underbrace{(1, 1, \ldots, 1)}_{N_1 \text{--times}}, \underbrace{|x^{(1)}|^{\alpha}, \ldots, |x^{(1)}|^{\alpha}}_{(N_2 + N_3) \text{--times}},$$

is called the Grushin operator (see [6]).

Example 2.3. Let $\alpha, \beta$ be nonnegative real numbers. The operator

$$\Delta_\gamma := \Delta_{x^{(1)}} + \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha} |x^{(2)}|^{2\beta} \Delta_{x^{(3)}},$$

where

$$\gamma = \underbrace{(1, 1, \ldots, 1)}_{(N_1 + N_2) \text{--times}}, \underbrace{|x^{(1)}|^{\alpha} |x^{(2)}|^{\beta}, \ldots, |x^{(1)}|^{\alpha} |x^{(2)}|^{\beta}}_{N_3 \text{--times}},$$

is called the strongly degenerate elliptic operators (see [24, 28]).
Definition 2.4. A function $u \in S^2_{\gamma,0}(\Omega)$ is called a weak solution of the problem \((1.1)-(1.2)\) if the identity
\[
\int_{\Omega} \nabla_{\gamma} u \cdot \nabla_{\gamma} \varphi \, dx - \int_{\Omega} f(x, u) \varphi \, dx = 0,
\]
holds for every $\varphi \in C_0^{\infty}(\Omega)$.

Definition 2.5. Let $X$ be a real Banach space with its dual space $X^*$ and $\Phi \in C^1(X, \mathbb{R})$. The functional $\Phi$ is said to satisfy Cerami condition at level $c \in \mathbb{R}$ \((C)_c\) condition for short) if for any sequence \(\{x_m\}_{m=1}^{\infty} \subset X\) with
\[
\Phi(x_m) \to c \quad \text{and} \quad (1 + \|x_m\|_X) \|\Phi'(x_m)\|_{X^*} \to 0,
\]
then there exists a subsequence \(\{x_{m_k}\}_{k=1}^{\infty}\) that converges strongly in $X$. $\Phi$ satisfies the \((C)_c\) condition if $\Phi$ satisfies \((C)_c\) condition at every $c \in \mathbb{R}$.

3. Proof of the main result

First, we observe that the problem \((1.1)-(1.2)\) has a variational structure. Indeed it is the Euler-Lagrange equation of the functional $\Phi : S^2_{\gamma,0}(\Omega) \to \mathbb{R}$ defined as follows:
\[
\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\gamma} u|^2 \, dx - \int_{\Omega} F(x, u) \, dx,
\]
By the hypotheses on $f$, we can see that the functional $\Phi$ is Fréchet differentiable in $S^2_{\gamma,0}(\Omega)$ and for any $\varphi \in S^2_{\gamma,0}(\Omega)$,
\[
\langle \Phi'(u), \varphi \rangle = \int_{\Omega} \nabla_{\gamma} u \cdot \nabla_{\gamma} \varphi \, dx - \int_{\Omega} f(x, u) \varphi \, dx.
\]
Thus, critical points of $\Phi$ are solutions of problem \((1.1)-(1.2)\).

Let
\[
f_\pm(x, \xi) = \begin{cases} f(x, \xi), & \xi > 0, \\ 0, & \xi \leq 0; \end{cases}
\]
\[
\Phi_\pm(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\gamma} u|^2 \, dx - \int_{\Omega} F_\pm(x, u) \, dx,
\]
where $F_\pm(x, \xi) = \int_0^\xi f_\pm(x, \tau) \, d\tau$. Now, we prove the following compactness condition for $\Phi$ and $\Phi_\pm$.

Lemma 3.1. Let \((A1)-(A4)\) be satisfied. Then the functionals $\Phi$ and $\Phi_\pm$ satisfies the \((C)\) condition on $S^2_{\gamma,0}(\Omega)$.

Proof. We only give the proof for $\Phi_+$, the cases of $\Phi$ and $\Phi_-$ are similar. Let \(\{u_n\}_{n=1}^{\infty} \subset S^2_{\gamma,0}(\Omega)\) be a sequence such that
\[
\Phi_+(u_n) \to c, \quad \left(1 + \|u_n\|_{S^2_{\gamma,0}(\Omega)}\right) \|\Phi'_+(u_n)\|_{(S^2_{\gamma,0}(\Omega))^*} \to 0, \quad \text{as } n \to \infty. \tag{3.1}
\]
The proof of this lemma, we divide two steps:

Step 1. We first prove that \(\{u_n\}_{n=1}^{\infty}\) is bounded in $S^2_{\gamma,0}(\Omega)$. Let $u_n^+ = \max\{u_n, 0\}$, $u_n^- = \min\{u_n, 0\}$. From (3.1), we obtain
\[
||\Phi'_+(u_n), \varphi|| \leq \epsilon_n ||\varphi||_{S^2_{\gamma,0}(\Omega)} \quad \text{for any } \varphi \in S^2_{\gamma,0}(\Omega),
\]
where $\epsilon_n \to 0$ as $n \to \infty$, then the boundedness of $u_n^-$ can be directly obtained. For the case of $u_n^+$, by contradiction, we assume that $||u_n^+||_{S^2_{\gamma,0}(\Omega)} \to \infty$ as $n \to \infty$. Let $v_n = ||u_n^+||^{-1}_{S^2_{\gamma,0}(\Omega)} u_n^+$, then
∥v_n∥_{S^2_{γ,0}(Ω)} = 1. By Proposition 2.1, up to a subsequence, we have

\[ v_n \rightharpoonup v \quad \text{weakly in } S^2_{γ,0}(Ω) \text{ as } n \to ∞, \]

\[ v_n \to v \quad \text{strongly in } L^q(Ω) \text{ as } n \to ∞, \]

\[ v_n \to v \quad \text{a.e. in } Ω \text{ as } n \to ∞. \]

**Case 1.** If \( v \neq 0 \) then the Lebesgue measure of \( Ω_0 = \{x \in Ω : v(x) \neq 0\} \) is positive. Using (3.1), we obtain

\[ \langle Φ'_+(u_n), u_n^+ \rangle = o(1), \]

which implies that

\[ \int Ω \frac{f_+(x, u_n^+)}{|u_n^+|^2} dx = \int Ω \frac{f_+(x, u_n^+)}{|u_n^+|^2} |v_n|^2 dx = 1 + o(1). \] (3.3)

By (A3), there is a constant \( M > 0 \) such that

\[ f_+(x, u_n^+) > 0, \quad \text{as } |u_n| > M, \]

then we have

\[ \int Ω,Ω_0 \frac{f_+(x, u_n^+)}{(u_n^+)^2} |v_n|^2 dx \geq -C. \] (3.4)

On the other hand, for \( x \in Ω_0, u_n^+ \to ∞ \) as \( n \to ∞ \). Then by the Fatou’s lemma and (A3) we have

\[ \int Ω_0 \frac{f_+(x, u_n^+)}{(u_n^+)^2} |v_n|^2 dx \to ∞, \quad \text{as } n \to ∞. \]

Combining this with (3.4) gives

\[ \int Ω \frac{f_+(x, u_n^+)}{(u_n^+)^2} |v_n|^2 dx \to ∞, \quad \text{as } n \to ∞. \] (3.5)

This contradicts (3.3). Then this case is impossible.

**Case 2.** If \( v ≡ 0 \) then for any \( n \in N \) there exists \( t_n \in [0,1] \) such that

\[ Φ_+(t_n u_n^+) = \max_{t \in [0,1]} Φ_+(tu_n^+). \]

For any \( R > 0 \), we assume that \( w_n = 2\sqrt{R}v_n \). Then \( w_n \to 0 \) in \( L^q(\mathbb{R}^N) \). So from conditions (A1) and (A2), for every \( ε > 0 \), we can find a constant \( C(ε) > 0 \) such that

\[ F(x, w_n) \leq C(ε)(w_n)^2 + ε(w_n)^2, \] (3.6)

which implies

\[ \lim_{n \to ∞} \int Ω F_+(x, w_n)dx = 0. \] (3.7)

Since \( 2\sqrt{R}∥u_n^+∥_{S^2_{γ,0}(Ω)} = 0,1 \) for \( n \) large enough, by (3.7) we obtain

\[ Φ_+(t_n u_n^+) \geq Φ_+(w_n) = 2R - \int Ω F_+(x, w_n)dx \geq R, \]

which implies

\[ Φ_+(t_n u_n^+) \to ∞, \quad \text{as } n \to ∞. \] (3.8)
From $\Phi_+(0) = 0$ and $\Phi_+(u_n^+) \to c$ we have $t_n \in (0, 1)$, then

$$\langle \Phi'_+(t_n u_n^+), t_n u_n^+ \rangle = t_n \frac{d}{dt} \Phi_+(t u_n) = 0.$$  

Then, from (A4) it follows that

$$\frac{1}{\theta} \Phi_+(t_n u_n^+) = \frac{1}{\theta} \left( \Phi_+(t_n u_n^+) - \frac{1}{2} \langle \Phi'_+(t_n u_n^+), t_n u_n^+ \rangle \right)$$

$$= \frac{1}{2} \int_{\Omega} f(x, t_n u_n^+) dx$$

$$\leq \frac{1}{2} \int_{\Omega} f(x, u_n^+) dx + \frac{1}{2\theta} \int_{\Omega} C(x) dx$$

$$= \Phi_+(u_n^+) - \frac{1}{2} \langle \Phi'_+(u_n^+), u_n^+ \rangle + c \to C.$$

This contradicts that $\Phi_+(t_n u_n^+) \to \infty$. Hence $\{u_n\}_{n=1}^{\infty}$ is bounded; that is, there exists a positive constant $M$ such that

$$\|u_n\|_{S^2_{\gamma,0}(\Omega)} \leq M, \text{ for all } n \in \mathbb{N}.$$

**Step 2.** We prove $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence. In fact, we can suppose that

$$u_n \rightharpoonup u \text{ weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n \to \infty,$$

$$u_n \to u \text{ strongly in } L^q(\Omega) \text{ as } n \to \infty,$$

$$u_n \to u \text{ a.e. in } \Omega \text{ as } n \to \infty.$$

Now, since $\Omega$ is a bounded set, for every $\epsilon > 0$, we can find a constant $C(\epsilon) > 0$ such that

$$f_+(x, s) \leq C(\epsilon) + |s|^{2^*_\gamma-1}, \quad \forall (x, s) \in \Omega \times \mathbb{R},$$

then

$$\left| \int_{\Omega} f_+(x, u_n)(u_n - u) dx \right|$$

$$\leq C(\epsilon) \int_{\Omega} |u_n - u| dx + \epsilon \int_{\Omega} |u_n - u|^{2^*_\gamma-1} dx$$

$$\leq C(\epsilon) \int_{\Omega} |u_n - u| dx + \epsilon \left( \int_{\Omega} |u_n|^{2^*_\gamma-1} \right)^{\frac{2^*_\gamma-1}{2^*_\gamma}} \left( \int_{\Omega} |u_n - u|^{2^*_\gamma} \right)^{\frac{1}{2^*_\gamma}}$$

$$\leq C(\epsilon) \int_{\Omega} |u_n - u| dx + \epsilon C(\Omega).$$

Similarly, since $u_n \rightharpoonup u$ in $S^2_{\gamma,0}(\Omega)$, it follows that $\int_{\Omega} |u_n - u| dx \to 0$. Since $\epsilon > 0$ is arbitrary, we can conclude that

$$\int_{\Omega} (f_+(x, u_n) - f_+(x, u))(u_n - u) dx \to 0 \quad \text{as } n \to \infty. \quad (3.9)$$

By (3.9), we have

$$\langle \Phi'_+(u_n) - \Phi'_+(u), (u_n - u) \rangle \to 0 \quad \text{as } n \to \infty. \quad (3.10)$$

From (3.9) and (3.10), we obtain $\|u_n\|_{S^2_{\gamma,0}(\Omega)} \to \|u\|_{S^2_{\gamma,0}(\Omega)}$ as $n \to \infty$. Thus we have

$$\|u_n - u\|_{S^2_{\gamma,0}(\Omega)} \to 0, \text{ as } n \to \infty,$$

which means that $\Phi_+$ satisfies condition (C). □
Lemma 3.2. Assume that conditions (A1), (A3), (A4) hold. Then we have

\[ C_m(\Phi, \infty) = C_m(\Phi_{\pm}, \infty) = \{0\}, \quad m = 0, 1, 2, \ldots. \]

Proof. We only give the proof of \( \Phi_{+} \); the others are similar. Let \( S = \{ u \in S_{2,0}^{2}(\Omega) : \| u \|_{S_{2,0}^{2}(\Omega)} = 1, u^{+} \neq 0 \} \) and \( B^{\infty} = \{ u \in S_{2,0}^{2}(\Omega) : \| u \|_{S_{2,0}^{2}(\Omega)} \leq 1 \} \). By (A3), for any \( M > 0 \) there exists \( c > 0 \), such that \( F(x, t) \geq Mt^{2} - c \), for \( (x, t) \in \Omega \times \mathbb{R} \), which implies \( \Phi_{+}(tu) \to -\infty \), as \( t \to +\infty \), for any \( u \in S \). Using (A4), we have

\[ f_{+}(x, t)t - 2F_{+}(x, t) \geq -\frac{C(x)}{\theta}, \quad \text{for} \quad (x, t) \in \Omega \times \mathbb{R}. \]  

Choose

\[ a < \min \left\{ \inf_{u \in B^{\infty}} \Phi_{+}(u), -\frac{C_*}{2\theta} \right\}, \]

where \( C_* = \int_{\Omega} C(x)dx \). Then for any \( u \in S \), there exists \( t > 1 \) such that \( \Phi_{+}(tu) \leq a \), that is

\[ \Phi_{+}(tu) = \frac{t^{2}}{2} - \int_{\Omega} F_{+}(x, tu)dx \leq a, \]

which (3.11) implies

\[ \frac{d}{dt}\Phi_{+}(tu) = t - \int_{\Omega} f_{+}(x, tu)u \leq \frac{1}{t}(2a + \frac{C_*}{\theta}) < 0. \]

Therefore, by the implicit function theorem, there exists a unique \( T \in C(S, \mathbb{R}) \) such that

\[ \Phi_{+}(T(u)u) = a, \quad \text{for} \quad u \in S. \]

Let \( S_1 = \{ u \in S_{2,0}^{2}(\Omega) : \| u \|_{S_{2,0}^{2}(\Omega)} \geq 1, u^{+} \neq 0 \} \). We construct a strong deformation retract \( \tau : [0, 1] \times S_1 \to S_1 \) which satisfies \( \tau(s, u) = (1 - s)u + sT\left(n\| u \|_{S_{2,0}^{2}(\Omega)}\right)\| u \|_{S_{2,0}^{2}(\Omega)} \) if \( \Phi_{+}(u) \geq a \) and \( \tau(s, u) = u \) if \( \Phi_{+}(u) < a \). Hence, It follows from the construction of \( \tau \) that \( \Phi_{+}^{a} \) is a strong deformation retract of \( S_1 \), which is homotopy equivalent to the set \( S \). By the homotopy invariance of homology group, we have

\[ C_m(\Phi_{+}, \infty) = H_m(S_{2,0}^{2}(\Omega), \Phi_{+}^{a}) = H_m(S_{2,0}^{2}(\Omega), S) = H_m(S_{2,0}^{2}(\Omega), S_{2,0}^{2}(\Omega) \setminus \{0\}) = 0. \]

Proof of Theorem 1.1. By Lemma 3.1, we know that \( \Phi \) and \( \Phi_{\pm} \) satisfy the (C) condition. By conditions (A1) and (A2), we can easily prove that 0 is a local minimum of \( \Phi \) and \( \Phi_{\pm} \). So, we have

\[ C_m(\Phi, 0) = C_m(\Phi_{\pm}, 0) = \delta_{m,0}G. \]  

(3.12)

Using the mountain pass theorem in [21], we obtain \( \Phi_{+} \text{ (} \Phi_{-} \text{)} \) has a critical point \( u_{+} > 0 \text{ (} u_{-} < 0 \text{)} \), and \( u_{\pm} \) are also the nontrivial critical points of the functional \( \Phi \). Without loss of generality, we assume that \( u_{\pm} \) are isolated and the only nontrivial critical points of the functional \( \Phi \). Now we claim that

\[ C_m(\Phi_{\pm}, u_{\pm}) = \delta_{m,1}G. \]

(3.13)

Indeed, using the methods of [9], we let \( \Phi_{+}(u_{+}) = c > 0 \). It follows from the homology exact sequence of the triple \( \Phi_{+}^{A} \subset \Phi_{+}^{B} \subset S_{2,0}^{2}(\Omega) \), we have

\[ \cdots \to H_m(S_{2,0}^{2}(\Omega), \Phi_{+}^{A}) \to H_m(S_{2,0}^{2}(\Omega), \Phi_{+}^{B}) \to H_{m-1}(\Phi_{+}^{B}, \Phi_{+}^{A}) \to H_{m-1}(S_{2,0}^{2}(\Omega), \Phi_{+}^{A}) \to \cdots. \]

(3.14)
where \( A < 0 \) is a constant. Since 0 is the only critical point of \( \Phi_+ \) in the set \( \Phi_+^\delta \), by (3.12), we obtain

\[
H_m(\Phi_+^\delta, \Phi_+^\delta) = C_m(\Phi_+, 0) = \delta_{m,0} G. \tag{3.15}
\]

Similarly, since \( u_+ \) is the only critical point of \( \Phi_+ \) in the set \( \{u \in S^2_{\gamma,0}(\Omega) | \Phi_+(u) \geq \frac{c}{2} \} \), we have

\[
H_m(S^2_{\gamma,0}(\Omega), \Phi_+^\delta) = C_m(\Phi_+, u_1), \quad m = 0, 1, 2, \ldots. \tag{3.16}
\]

From Lemma 3.2, we have

\[
H_m(S^2_{\gamma,0}(\Omega), \Phi_+^\delta) = C_m(\Phi_+, \infty) = 0, \quad m = 0, 1, 2, \ldots. \tag{3.17}
\]

From (3.14) to (3.17), we deduce that

\[
C_m(\Phi_+, u_+) = C_{m-1}(\Phi_+, 0) = \delta_{m,1} G.
\]

The case for \( u_- \) is similar, that is

\[
C_m(\Phi_-, u_-) = C_{m-1}(\Phi_-, 0) = \delta_{m,1} G.
\]

Hence

\[
C_m(\Phi, u_\pm) = \delta_{m,1} G.
\]

The Morse equality (2.1) with \( t = -1 \) implies that

\[
(-1)^0 + (-1)^1 + (-1)^1 = 0,
\]

which is a contradiction. Then the problem (1.1)–(1.2) has at least three nontrivial solutions.

References


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