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Korovkin-type Approximation Theorem for Bernstein Stancu Operator of Rough Statistical Convergence of Triple Sequence

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ABSTRACT: We obtain a Korovkin-type approximation theorem for Bernstein Stancu polynomials of rough statistical convergence of triple sequences of positive linear operators of three variables from $H_{\omega}(K)$ to $C_B(K)$, where $K = [0,\infty) \times [0,\infty) \times [0,\infty)$ and ω is non-negative increasing function on K.

Key Words: Triple sequences, rough convergence, Bernstein Stancu operator, Korovkin-type approximation theorem.

Contents

1. Introduction

The notion of statistical convergence was introduced by Fast [19] and Schoenberg [38] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Fridy [20], Šalát [36]). The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

The idea of rough convergence was first introduced by Phu [31,32,33] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also

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investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [3] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [4] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dundar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. In [25], Malik and Maity introduced rough statistical convergence of double sequences in normed linear spaces. A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [35], Esi et al. [12,17], Esi and Catalbas [13], Esi [18], Esi and Savas [14], Esi and Subramanian [15,16], Subramanian and Esi [39], Hazarika, Subramanian and Esi [23], Mursaleen and Kilicman [29], Mursaleen et al. [30], Datta et al. [10], Debnath et al. [6], Savas and Esi [37], Tripathy and Goswami [40], and references therein. The notion of statistical convergence of a triple sequence spaces which is based on the natural density of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ was introduced by Sahiner et al. [34].

Korovkin approximation theory which deals with the problem of approximation of function f by the sequence $(T_m(f,x))$ where (T_m) is a sequence of positive linear operators [21], [24]. This theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations, etc, (see [2]. We prove an analogue of the classical Korovkin theorem by using of ideal lambda summability. Recently such type of approximation theorems proved in [7] for ideal convergence, in [22] for statistical convergence. The classical Korovkin approximation theorem states as follows [24]:

Let $C[a,b]$ be the set of all functions f continuous on $[a,b]$. Suppose that (T_m) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then

$$\lim_m ||T_m(f,x) - f(x)||_{C[a,b]} = 0 \text{ for all } f \in C[a,b]$$

if and only if

$$\lim_m ||T_m(f_j,x) - f_j(x)||_{C[a,b]} = 0, \text{ for } j = 0,1,2,$$

where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

We know that $C[a,b]$ is a Banach space with the norm $||f|| := ||f||_{C[a,b]} = \sup_{x\in[a,b]} |f(x)|$, $f \in C[a,b]$. We write $T_m(f,x)$ for $T_m(f(t),x)$ and we say that T is positive if $T(f,x) \geq 0$ for all $f(x) \geq 0$.

Korovkin-type approximation theorems extended by using various test functions in several setups, including Banach spaces, abstract Banach lattices, function spaces, and Banach algebras. Firstly, Gadjiev and Orhan [22] established classical

Korovkin theorem through statistical convergence and display an interesting example in support of our result. Recently, Korovkin-type theorems have been obtained by Mohiuddine [26] for almost convergence. Korovkin-type theorems were also obtained in [11] for λ-statistical convergence. The authors of [1] established these types of approximation theorem in weighted L_p spaces, where $1 \leq p < \infty$, through A-summability which is stronger than ordinary convergence. For these types of approximation theorems and related concepts, one can be referred to [5,8,27,28], and references therein.The sum of this paper was presented in International Conference of Mathematical Sciences (ICMS, 2018), [41].

2. Definitions and Preliminaries

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional case with the usual metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3$; $m, n, k \in \mathbb{N}$.

Let A be a six dimensional summability matrix. For a given triple sequence $x = (x_{mnk})$, the A-transform of x, denoted by $Ax := \Big((Ax)_{ij\ell}\Big)$, given by

$$(Ax)_{i,j,\ell} = \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k}x_{mnk} \tag{2.1}$$

provided the triple series converges in Pringsheim's sense for every $(i, j, \ell) \in \mathbb{N}^3$.

A six dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is said to be RH-regular it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. A three dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is RH-regular if and only if

(i) $P - \lim_{i,j} a_{i,j,\ell,m,n,k} = 0$ for each $(m, n, k) \in \mathbb{N}^3$,

(ii) $P - \lim_{i,j,\ell} \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k} = 1$,

(iii) $P - \lim_{i,j,\ell} \sum_{m\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $n, k \in \mathbb{N}$,

(iv) $P - \lim_{i,j,\ell} \sum_{n\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $m, k \in \mathbb{N}$,

(v) $P - \lim_{i,j,\ell} \sum_{k\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $m, n \in \mathbb{N}$,

(vi) $\sum_{(m,n,k)\in\mathbb{N}^3} |a_{i,j,\ell,m,n,k}|$ is $P-$convergent for every $(i, j, \ell) \in \mathbb{N}^3$,

(vii) there exist finite positive integers A and B such that
$\sum_{m,n,k>B} |a_{i,j,\ell,m,n,k}| < A$ holds for every $(i, j, \ell) \in \mathbb{N}^3$.

Now let $A = (a_{i,j,\ell,m,n,k})$ be a non-negative RH-regular summability matrix, and $K \subset \mathbb{N}^3$. Then the A-density of K is given by

$$\delta_3^A\{K\} := P - \lim_{i,j,\ell} \sum_{(m,n,k)\in K(\epsilon)} a_{i,j,\ell,m,n,k},$$

where

$$K(\epsilon) := \left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - L| \geq \epsilon\right\}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real triple sequence $x = (x_{mnk})$ is said to be A-statistically convergent to a number L if, for every $\epsilon > 0$,

$$\delta_3^A \left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - L| \geq \epsilon\right\} = 0.$$

In this case, we write $st_2^A - \lim_{m,n,k} x_{mnk} = L$.

In this paper we investigate some basic properties of Korovkin-type approximation theorem for rough statistical convergence of a triple sequences in six dimensional matrix which are not discussed earlier.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m,n,k) \in K : m \geq i, n \geq j, k \geq \ell\}$. Then the natural density of K is given by

$$\delta_3(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $u,v,w \in \mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [uvw]_{p,q} \text{ if } u,v,w \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}} \frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}} \frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k \in \mathbb{N}$ with $(u,v,w) \geq (m,n,k)$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax+by)_{p,q}^{uvw} =$$
$$\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$
$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},$$

$$(x+y)_{p,q}^{uvw} =$$
$$(x+y)(px+qy)\left(p^2x+q^2y\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)_{p,q}^{uvw} =$$
$$(1-x)\left(p-qx\right)\left(p^2-q^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)_{p,q}^{mnk} = x\left(px\right)\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}\left(f,x\right) =$$
$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}\left(1-x\right)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

(p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}\left(f,x\right) =$$
$$\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k}.$$
$$\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)\cdot \tag{2.2}$$
$$f\left(\frac{p^{(r-m)}\left[m\right]_{p,q}+p^{(s-n)}\left[n\right]_{p,q}p^{(t-k)}\left[k\right]_{p,q}}{\left[r\right]_{p,q}+\left[s\right]_{p,q}+\left[t\right]_{p,q}}\right), x\in[0,1]$$

Also, we have

$$(1-x)^{rst}_{p,q} =$$
$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{2}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\cdot$$
$$\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}.$$

$$S_{rst,p,q}(f,x) =$$
$$\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}\cdot$$
$$\prod_{u_1=0}^{(r-m-1)}(p^{u_1}-q^{u_1}x)\prod_{u_2=0}^{(s-n-1)}(p^{u_2}-q^{u_2}x)\prod_{u_3=0}^{(t-k-1)}(p^{u_3}-q^{u_3}x)\cdot \tag{2.3}$$
$$f\left(\frac{p^{(r-m)}[m]_{p,q}+p^{(s-n)}[n]_{p,q}p^{(t-k)}[k]_{p,q}+\eta}{[r]_{p,q}+[s]_{p,q}+[t]_{p,q}+\mu}\right), x\in[0,1]$$

Note that for $\eta=\mu=0$, (p,q)-Bernstein-Stancu operators given by (2.3) reduces into (p,q)-Bernstein operators. Also for $p=1$, (p,q)-Bernstein-Stancu operators given by (2.2) turn out to be q-Bernstein-Stancu operators.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein Stancu polynomials $(S_{rst,p,q}(f,x))$ such that $(S_{rst,p,q}(f,x))\in\mathbb{R}$, $m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(S_{rst,p,q}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st_3-\lim S_{rst,p,q}(f,x)=f(x)$, provided that the set

$$K_\epsilon := \left\{(m,n,k)\in\mathbb{N}^3 : |S_{rst,p,q}(f,x)-(f,x)|\geq\epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein Stancu polynomials. i.e., $\delta_3(K_\epsilon)=0$. That is,

$$\lim_{rst\to\infty}\frac{1}{rst}\left|\{(m,n,k)\leq(r,s,t) : |S_{rst,p,q}(f,x)-(f,x)|\geq\epsilon\}\right|=0.$$

In this case, we write $\delta_3-\lim S_{rst,p,q}(f,x)=(f,x)$ or $S_{rst,p,q}(f,x)\xrightarrow{st_3}(f,x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A- the characteristic function of $A\subset\mathbb{N}$, $\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d_3(A)=\lim_{ij\ell\to\infty}\frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}\chi_A(K).$$

Definition 2.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers and $A=(a_{i,j,\ell,m,n,k})$ be a non-negative RH-regular summability matrix is said to be rough statistically A-summable to $f(x)$ if for every $\epsilon>0$,*

$$\delta_3\left(\left\{(i,j,\ell)\in\mathbb{N}^3:|S_{ij\ell mnk,p,q}(f,Ax)-(f,x)|\geq r+\epsilon\right\}\right)=0,$$

i.e.,

$$P-\lim_{mnk}\frac{1}{mnk}\left|\{i\leq m,j\leq n,\ell\leq k:|S_{ij\ell mnk,p,q}(f,Ax)-(f,x)|\geq r+\epsilon\}\right|=0,$$

where $(Ax)_{ij\ell}$ is as in (2.1).

3. A Korovkin-type approximation theorem

Let $C_B(K)$ the space of all continuous and bounded real valued functions on $K=[0,\infty)\times[0,\infty)\times[0,\infty)$. This space is equipped with the supremum norm

$$\|f\|=\sup_{(x,y,z)\in K}S_{rst,p,q}\left|f,(x,y,z)\right|,(f\in C_B(K)).$$

Consider the triple space of $H_\omega(K)$ of all real valued functions of Bernstein Stancu polynomials of f on K satisfying

$$\begin{aligned}&|S_{rst,p,q}(f,(u,v,w))-S_{rst,p,q}(f,(x,y,z))|\\ \leq&\omega\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|,\left|\frac{v}{1+v}-\frac{y}{1+y}\right|,\left|\frac{w}{1+w}-\frac{z}{1+z}\right|\right)\end{aligned}$$

where w be a function of the type of the modulus of continuity given by, for $\delta,\delta_1,\delta_2,\delta_3>0$,

(1) ω is non-negative increasing function on K with respect to $\delta_1,\delta_2,\delta_3$,

(2) $\omega(\delta,\delta_1+\delta_2+\delta_3)\leq\omega(\delta,\delta_1)+\omega(\delta,\delta_2)+\omega(\delta,\delta_3)$

(3) $\omega(\delta_1+\delta_2+\delta_3,\delta)\leq\omega(\delta_1,\delta)+\omega(\delta_2,\delta)+\omega(\delta_3,\delta)$

(4) $\lim_{\delta_1,\delta_2\delta_3\to 0}\omega(\delta_1,\delta_2,\delta_3)=0.$

The Bernstein Stancu polynomials of $S_{rst,p,q}(f)\in H_\omega(K)$ satisfies the inequality

$$S_{rst,p,q}\left|(f,(x,y,z))\right|\leq S_{rst,p,q}(f,(0,0,0))+w(1,1,1),x,y,z\geq 0$$

and hence it is bounded on K. Therefore $H_\omega(K)\subset C_B(K)$.

We also use the following Bernstein Stancu polynomials of test functions

$$S_{rst,p,q}\left(f_{000},(u,v,w)\right)=1,$$
$$S_{rst,p,q}\left(f_{111},(u,v,w)\right)=\frac{u}{1+u},$$
$$S_{rst,p,q}\left(f_{222},(u,v,w)\right)=\frac{v}{1+v},$$
$$S_{rst,p,q}\left(f_{333},(u,v,w)\right)=\frac{w}{1+w}$$
$$S_{rst,p,q}\left(f_{444},(u,v,w)\right)=\left(\frac{u}{1+u}\right)^2+\left(\frac{v}{1+v}\right)^2+\left(\frac{w}{1+w}\right)^2.$$

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu polynomials of $\left(S_{rst,p,q}\left(f,x\right)\right)$ of real numbers from $H_w\left(K\right)$ into $C_B\left(K\right)$ and let $A=\left(a_{i,j,\ell,m,n,k}\right)$ be a nonnegative RH-regularsummability matrix. Assume that the following conditions hold:*

$$st_3-\lim\left\|\sum_{(r,s,t)\in\mathbb{N}^3}a_{ij\ell,m,n,k}S_{rst,p,q}\left(f_{rst}\right)-f_{rst}\right\|=0, r,s,t=0,1,2,3\cdots. \quad (3.1)$$

Then, for any $f\in H_\omega\left(K\right)$,

$$st_3-\lim\left\|\sum_{(r,s,t)\in\mathbb{N}^3}a_{ij\ell,m,n,k}S_{rst,p,q}\left(f\right)-f\right\|=0. \quad (3.2)$$

Proof. Assume that (3.1) holds. Let $S_{rst,p,q}\left(f,(x,y,z)\right)\in H_\omega\left(K\right)$ and $f\left(x,y,z\right)\in K$ be fixed. Since $S_{rst,p,q}\left(f,(u,v,w)\right)\in H_w\left(K\right)$ for all $f\left(u,v,w\right)\in K$ be fixed, we write

$$\left|S_{rst}\left(f,(u,v,w)\right)-S_{rst,p,q}\left(f,(x,y,z)\right)\right|$$
$$\leq r+\epsilon+\frac{2N}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2+\left(\frac{v}{1+v}-\frac{y}{1+y}\right)^2+\left(\frac{w}{1+w}-\frac{z}{1+z}\right)^2\right\}$$

where $N:=\|f\|$. Using the linearity of Bernstein Stancu polynomials of $S_{rst,p,q}\left(f,(x,y,z)\right)$, we obtain

$$\left|\sum_{(r,s,t)\in\mathbb{N}^3}a_{ij\ell,m,n,k}S_{rst,p,q}\left(f,(x,y,z)\right)-f\left(x,y,z\right)\right|$$
$$\leq r+\epsilon+C\left|\sum_{(r,s,t)\in\mathbb{N}^3}a_{ij\ell,m,n,k}S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f\left(x,y,z\right)\right|$$
$$+C\left|\sum_{(r,s,t)\in\mathbb{N}^3}a_{ij\ell,m,n,k}S_{rst,p,q}\left(f_{111},(x,y,z)\right)-f\left(x,y,z\right)\right|$$

$$+C\left|\sum_{(r,s,t)\in\mathbb{N}^3} a_{ij\ell,m,n,k}S_{rst,p,q}\left(f_{222},(x,y,z)\right)-f\left(x,y,z\right)\right|$$

$$+C\left|\sum_{(r,s,t)\in\mathbb{N}^3} a_{ij\ell,m,n,k}S_{rst,p,q}\left(f_{333},(x,y,z)\right)-f\left(x,y,z\right)\right|,$$

where

$$C:=\max\left\{r+\epsilon+N+\frac{2N}{\delta^2}\left(\left(\frac{A}{1+A}\right)^2+\left(\frac{B}{1+B}\right)^2+\left(\frac{C}{1+C}\right)^2\right),\right.$$
$$\left.\frac{6N}{\delta^2}\left(\frac{A}{1+A}\right),\frac{6N}{\delta^2}\left(\frac{B}{1+B}\right),\frac{6N}{\delta^2}\left(\frac{C}{1+C}\right),\frac{2N}{\delta^2}\right\}.$$

Then, taking supremum over $f\left(x,y,z\right)\in K$ we get

$$\left\|\sum_{(r,s,t)\in\mathbb{N}^3} a_{ij\ell m,n,k}S_{rst,p,q}(f)-f\right\|\leq r+\epsilon+C\sum_{r,s,t=0}^{3}\left\|\sum_{(m,n,k)\in\mathbb{N}^3} a_{ij\ell m,n,k}S_{rst}(f_{rst})-f_{rst}\right\| \tag{3.3}$$

For a given $\rho>0$, choose $r+\epsilon>0$ such that $(r+\epsilon)<\rho$. Then, for each $r,s,t=0,1,2,3$, setting

$$U:=\left\{(i,j,\ell):\left\|\sum_{(r,s,t)\in\mathbb{N}^3} a_{ij\ell,m,n,k}S_{rst,p,q}\left(f\right)-f\right\|\geq\rho\right\},$$

$$U^{'}_{rst}:=\left\{(i,j,\ell):\left\|\sum_{(r,s,t)\in\mathbb{N}^3} a_{ij\ell,m,n,k}S_{rst,p,q}(f_{rst})-f_{rst}\right\|\geq\frac{\rho-(r+\epsilon)}{6C}\right\},(r,s,t=0,1,2,3),$$

it follows that (3.3) that

$$U\subset\bigcup_{r,s,t=0}^{3}U_{rst}$$

which gives, for all $(m,n,k)\in\mathbb{N}^3$,

$$\delta_3\left(U\right)\leq\sum_{r,s,t=0}^{3}\delta_3\left(U_{rst}\right)$$

From (3.1), we obtain (3.2). This completes the proof. □

If we take $A=I$, which is the identity matrix we get the follwing statistical version of theorem 3.1.

Corollary 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$ and we take $A = I$, which is the identity matrix. A triple sequence of Bernstein Stancu polynomials of $(S_{rst,p,q}(f,x))$ of real numbers from $H_w(K)$ into $C_B(K)$. Assume that the following conditions hold:*

$$st_3 - \lim \|S_{rst,p,q}(f_{rst}) - f_{rst}\| = 0, r,s,t = 0,1,2,3,\dots. \tag{3.4}$$

Then, for any $f \in H_w(K)$

$$st_3 - \lim \|S_{rst,p,q}(f) - f\| = 0. \tag{3.5}$$

In Corollary 3.2, if the statistical convergence ($[C,1,1]$ statistical convergence) replace with Pringsheim convergence, we obtain the following classical version of Theorem 3.1.

Corollary 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$ and we take $A = I$, which is the identity matrix. A triple sequence of Bernstein Stancu polynomials of $(S_{rst,p,q}(f,x))$ of real numbers from $H_w(K)$ into $C_B(K)$. Assume that the following conditions hold:*

$$P - \lim \|S_{rst,p,q}(f_{rst}) - f_{rst}\| = 0, r,s,t = 0,1,2,3,\dots. \tag{3.6}$$

Then, for any $f \in H_w(K)$,

$$P - \lim \|S_{rst,p,q}(f) - f\| = 0. \tag{3.7}$$

Remark 3.4. *We now show that our result Theorem 3.1 is stronger than its classical version Corollary 3.3 and statistical version Corollary 3.2. To see this first consider the following Bleimann, Butzer and Hahn operators of three variables of Bernstein Stancu polynomials is*

$$S_{rst,p,q}(f,(x,y,z)) = \frac{1}{(1+x)^r(1+y)^s(1+x)^t}$$
$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} S_{rst,p,q}\left(f,\left(\frac{m}{r-m+1},\frac{n}{s-n+1},\frac{k}{t-k+1}\right)\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^m y^n z^k,$$

where $f \in H_\omega(K)$, and $K = [0,\infty)\times[0,\infty)\times[0,\infty)$. We have

$$S_{mnk}(f_{000},(x,y,z)) = 1, \tag{3.8}$$
$$S_{mnk}(f_{111},(x,y,z)) = \frac{r}{r+1}\frac{x}{1+x},$$

$$
\begin{aligned}
S_{mnk}\left(f_{222},(x,y,z)\right) &= \frac{s}{s+1}\frac{y}{1+y},\\
S_{mnk}\left(f_{333},(x,y,z)\right) &= \frac{t}{t+1}\frac{z}{1+z},\\
S_{mnk}\left(f_{444},(x,y,z)\right) &= \frac{r\left(r-2\right)\left(r-1\right)}{\left(r+1\right)^{2}}\left(\frac{x}{1+x}\right)^{3}+\frac{m\left(m-1\right)}{\left(m+1\right)^{2}}\left(\frac{x}{1+x}\right)^{2}\\
&+\frac{r}{\left(r+1\right)^{2}}\frac{x}{1+x}+\frac{s\left(s-2\right)\left(s-1\right)}{\left(s+1\right)^{2}}\left(\frac{y}{1+y}\right)^{3}+\frac{t\left(t-1\right)}{\left(t+1\right)^{2}}\left(\frac{y}{1+y}\right)^{2}+\frac{s}{\left(s+1\right)^{2}}\frac{y}{1+y}\\
&+\frac{t\left(t-2\right)\left(t-1\right)}{\left(t+1\right)^{2}}\left(\frac{z}{1+z}\right)^{3}+\frac{t\left(t-1\right)}{\left(t+1\right)^{2}}\left(\frac{z}{1+z}\right)^{2}+\frac{t}{\left(t+1\right)^{2}}\frac{z}{1+z}.
\end{aligned}
$$

Now take $A=[C,1,1,1]$ and define a triple sequence $u:=(u_{rst})$ by

$$u_{rst}=(-1)^{r+s+t} \tag{3.9}$$

we observe that

$$st_3-\lim C\left[1,1,1\right](u)=0. \tag{3.10}$$

However, the Bernstein Stancu polynomials of triple sequence of u is not P-convergent and statistical convergent. Now using (3.9) and (3.10), we define the following double positive linear operators on $H_w(K)$ as follows:

$$S_{rst,p,q}\left(f,(x,y,z)\right)=\left(1+u_{rst}\right)S_{rst,p,q}\left(f,(x,y,z)\right) \tag{3.11}$$

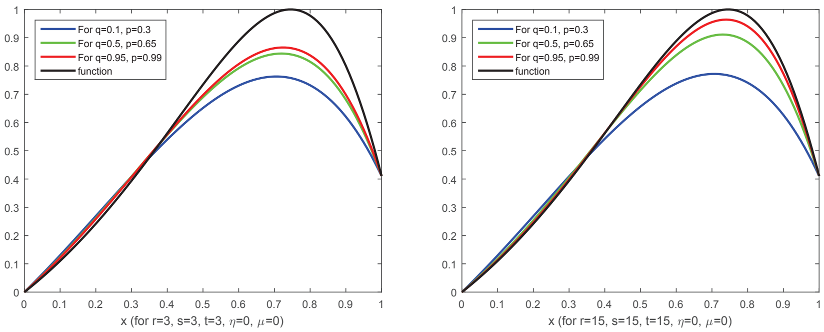
Then, observe that the Bernstein Stancu polynomials of triple sequence of $(S_{rst,p,q})$ defined by (3.11) satisfy all hypotheses of Theorem 3.1. Hence by (3.8) and (3.10), we have, for all $f\in H_w(K)$,

$$st_3-\lim\left\|S_{rst,p,q}(f)-f\right\|=0.$$

Since u is not $P-$convergent and statistical convergent, the sequence $\left(S_{rst,p,q}(f)\right)$ can not uniformly convergence to f on K or statistical sense.

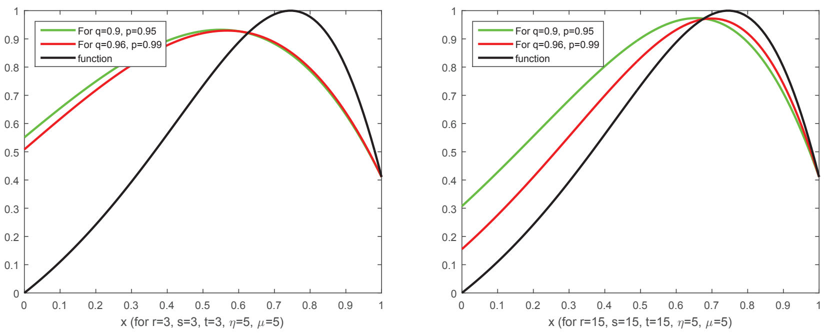
Example 3.5. *With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (2.3) in (3.1) to the function $f(x)=\sin(xe^x)$ under different parameters.*

From figure 1(a), it can be observed that as the value the q and p approaches towards 1 provided $0<q<p\leq 1$, (p,q)-Bernstein Stancu operators given by (2.3) in (3.1) converges towards the function $f(x)=\sin(xe^x)$. From figure 1(a) and (b), it can be observed that for $\eta=\mu=0$, as the value the (r,s,t) increases, (p,q)-Bernstein Stancu operators given by (2.3) in (3.1) converges towards the function. Similarly from figure 2(a), it can be observed that for $\eta=\mu=5$, as the value the q and p approaches towards 1 or some thing else provided $0<q<p\leq 1$, (p,q)-Bernstein Stancu operators given by (2.3) in (3.1) converges towards the function. From figure 2(a) and (b), it can be observed that as the value the $[r,s,t]$ increases, (p,q)-Bernstein Stancu operators given by $f(x)=\sin(xe^x)$ converges towards the function.



(a) (b)

Figure 1: (p, q)-Bernstein Stancu operators



(a) (b)

Figure 2: (p, q)-Bernstein Stancu operators

Competing Interests:

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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References

1. T. Acar and F. Dirik, *Korovkin-type theorems in weighted L_p-spaces via summation process*, The Scientifc World J. 2013, Article ID 534054, 6 pages (2013).

2. F. Altomare and M. Campiti, *Korovkin type approximation theory and its applications*, Walter de Gruyter Publ., Berlin (1994).

3. S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optimiz., 29(3-4), 291-303 (2008).

4. S. Aytar, *The rough limit set and the core of a real sequence*, Numer. Funct. Anal. Optimiz., 29(3-4), 283-290 (2008).

5. N. L. Braha, H. M. Srivastava and S. A. Mohiuddine, *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Appl. Math. Comput., 228, 162-169 (2014).

6. S. Debnath, B. Sarma and B.C. Das, *Some generalized triple sequence spaces of real numbers*, J. Nonlinear Anal. Optimiz., 6(1), 71-79 (2015).

7. O. Duman, *A Korovkin type approximation throrems via I-convergence*, Czechoslovak Math. J., 75(132), 367-375 (2007).

8. O. Duman, M. K. Khan and C. Orhan, *A-statistical convergence of approximating operators*, Math. Inequa. Appl., 6(4), 689-699 (2003).

9. E. Dundar and C. Cakan, *Rough convergence of double sequences*, Gulf J. Math., 2(1), 45-51 (2014).

10. A. J. Dutta, A. Esi and B.C. Tripathy, *Statistically convergent triple sequence spaces defined by Orlicz function*, J. Math. Anal., 4(2), 16-22 (2013).

11. O. H. H. Edely, S. A. Mohiuddine and A. Noman, *Korovkin type approximation theorems obtained through generalized statistical convergence*, Appl. Math. Letters, 23(11), 1382-1387 (2010).

12. A. Esi, *On some triple almost lacunary sequence spaces defined by Orlicz functions*, Research and Reviews: Discrete Math. Structures, 1(2), 16-25 (2014).

13. A. Esi and M. Necdet Catalbas, *Almost convergence of triple sequences*, Global J. Math. Anal., 2(1), 6-10 (2014).

14. A. Esi and E. Savas, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Appl. Math. Inf. Sci., 9(5), 2529-2534 (2015).

15. A. Esi and N.Subramanian, *On triple sequence spaces of Bernstein operator of χ^3 of rough λ-statistical convergence in probability of random variables defined by Musielak-Orlicz function*, Int. J. Open Probl. Comput. Sci. Math., 11(2), 62-70 (2018).

16. A. Esi and N. Subramanian, *Some triple difference rough Cesaro and lacunary statistical sequence spaces*, J. Math. Appl., 41, 1-13 (2018).

17. A. Esi, S. Araci and M. Acikgoz, *Statistical Convergence of Bernstein Operators*, Appl. Math. Inf. Sci., 10(6), 2083-2086 (2016).

18. A. Esi, *λ_3-Statistical convergence of triple sequences on probabilistic normed space*, Global J. Math. Anal., 1(2), 29-36 (2013).

19. H. Fast, *Sur la convergence statistique*, Colloq. Math., 2, 241-244 (1951).

20. J. A. Fridy, *On statistical convergence*, Analysis, 5, 301-313 (1985).

21. A. D. Gadjiev, *The convergence problem for a sequence of positive linear operators an unbounded sets, and theorems analogous to that of P.P. Korovkin*, Soviet Math. Dokl., 15, 1433-1436 (1974).

22. A. D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., 32(1), 129-138 (2002).

23. B. Hazarika, N. Subramanian and A. Esi, *On rough weighted ideal convergence of triple sequence of Bernstein polynomials*, Proccedings of the Jangjeon Mathematical Society, 21(3), 497-506 (2018).

24. P. P. Korovkin, *Linear operators and the theory of approximation*, India, Delhi (1960).

25. P. Malik and M. Maity, *On rough statistical convergence of double sequences in normed linear spaces*, Afr. Mat., 27(1), 141-148 (2016).

26. S. A. Mohiuddine, *An application of almost convergence in approximation theorems*, Appl. Math. Letters, 24(11), 1856-1860 (2011).

27. S. A. Mohiuddine and A. Alotaibi, *Statistical convergence and approximation theorems for functions of two variables*, J. Comput. Anal. Appl., 15(2), 218-223 (2013).

28. S. A. Mohiuddine and A. Alotaibi, *Korovkin second theorem via statistical summability* $(C, 1)$, J. Inequa. Appl., 2013, Article 149, 9 pages (2013).

29. M. Mursaleen and A. Kilicman, *Korovkin Second Theorem via B-Statistical A-Summability*, Abstr. Appl. Anal., (2013). 10.1155/2013/598963.

30. M. Mursaleen, F. Khan, A. Khan and A. Kilicman, *Some approximation results for generalized Kantorovich-type operators*. J. Inequal. Appl., 2013:585, 1-14 (2013). 10.1186/1029-242X-2013-585.

31. H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optimiz., 22, 199-222 (2001).

32. H. X. Phu, *Rough continuity of linear operators*, Numer. Funct. Anal. Optimiz., 23, 139-146 (2002).

33. H. X. Phu, *Rough convergence in infinite dimensional normed spaces*, Numer. Funct. Anal. Optimiz., 24, 285-301 (2003).

34. A. Sahiner, M. Gurdal and F. K. Duden, *Triple sequences and their statistical convergence*, Selcuk J. Appl. Math., 8(2), 49-55 (2007).

35. A. Sahiner and B. C. Tripathy, *Some I related properties of triple sequences*, Selcuk J. Appl. Math., 9(2), 9-18 (2008).

36. T. Šalát, *On statistical convergence of real numbers*, Math. Slovaca, 30, 139-150 (1980).

37. E. Savas and A. Esi, *Statistical convergence of triple sequences on probabilistic normed space*, Ann. Uni. Craiova, Math. Compu. Sci. Series, 39(2), 226-236 (2012).

38. I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66, 361-375 (1959).

39. N. Subramanian and A. Esi, *On triple sequence space of Bernstein operator of χ^3 of rough λ-statistical convergence in probability defined by Musielak-Orlicz function of p-metric*, Electron. J. Math. Anal. Appl., 6(1), 198-203 (2018).

40. B. C. Tripathy and R. Goswami, *On triple difference sequences of real numbers in probabilistic normed spaces*, Proyecciones J. Math., 33(2), 157-174 (2014).

41. Ayten Esi, M. Kemal Ozdemir and N. Subramanian, *Korovkin-type approximation theorem for Bernstein Stancu operator of rough statistical convergence of triple sequence*, 2nd International Conference of Mathematical Sciences, 31 July 2018-6 August 2018 (ICMS 2018) Maltepe University, Istanbul, Turkey.

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