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Existence and Multiplicity of Solutions for Anisotropic Elliptic Equations

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ABSTRACT: In this article we study the nonlinear problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)\ |u|^{P_+^+ -2}u = \lambda f(x,u) & in \quad \Omega \\ u = 0 \quad on \quad \partial\Omega \end{cases}$$

Using the variational method, under appropriate assumptions on f, we obtain a result on existence and multiplicity of solutions.

Key Words: $\overrightarrow{p}(.)$-Laplace type operator, variable exponent Lebesgue space, anisotropic space, Ricceri's variational principle.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary. In this paper we will study the existence and the multiplicity of weak solutions of the anisotropic problem :

$$(P) \begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)\ |u|^{P_+^+ -2}u = \lambda f(x,u) & in \quad \Omega \\ u = 0 \quad on \quad \partial\Omega \end{cases}$$

where $b \in L^\infty(\Omega)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\quad a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling some natural hypotheses, and $0 < \lambda \in \mathbb{R}$. The anisotropic differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(.)$-Laplace type operator, where $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$ and $P_+^+ = \max_{i \in \{1,2,...,N\}} \sup_{\Omega} p_i(x)$ for $i = 1, ..., N$, we assume that p_i is a continuous function on $\overline{\Omega}$. We denote by $a_i(x,\eta)$ the continuous derivative with respect to η of the mapping $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $A_i = A_i(x,\eta)$, that means $a_i(x,\eta) = \frac{\partial}{\partial \eta} A_i(x,\eta)$. We make the following assumptions on the mapping A_i :

(A_0) $A_i(x, 0) = 0$ for a.e. $x \in \Omega$.

(A_1) There exists a positive constant $\overline{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x,\eta)| \leq \overline{c}_i(1 + |\eta|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $\eta \in \mathbb{R}$.

(A_2) The inequalities

$$|\eta|^{p_i(x)} \leq a_i(x,\eta)\eta \leq p_i(x)A_i(x,\eta),$$

are verified for all $x \in \Omega$ and $\eta \in \mathbb{R}$.

(A_3) Assume that $p_i : \overline{\Omega} \to [2, \infty)$, and there exists $k_i > 0$ such that

$$A_i(x, \frac{\eta + \xi}{2}) \leq \frac{1}{2} A_i(x,\eta) + \frac{1}{2} A_i(x,\xi) - k_i|\eta - \xi|^{p_i(x)},$$

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for all $x \in \Omega$ and $\eta, \xi \in \mathbb{R}$, with equality if and only if $\eta = \xi$.

Examples

1) If we take $a_i(x,\eta) = |\eta|^{p_i(x)-2}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x,\eta) = \frac{1}{p_i(x)}|\eta|^{p_i(x)}$ for all $i \in \{1, ..., N\}$. Obviously, (A_0) - (A_3) are verified, and we obtain the $\overrightarrow{p}(x)$ -Laplace operator

$$\triangle_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u).$$

2) If we take $a_i(x,\eta) = (1+\eta^2)^{\frac{p_i(x)-2}{2}}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x,\eta) = \frac{1}{p_i(x)}[(1+|\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1, ..., N\}$, then (A_0) - (A_3) are verified, and we find the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i}(1+|\partial_{x_i} u|^2)^{\frac{p_i(x)-2}{2}}\partial_{x_i} u).$$

We use in our work the Ricceri's theorem which is the main tool to study the boundary problems. We infer to some references ([16],[13],[21]), for example, in [21] the authors studied the operator $p(x)-$Laplace, then they showed the existence of at least three solutions under appropriate conditions. In our case, we use the more general operator which called $\overrightarrow{p}(x)$-Laplace type operator with Dirichlet boundary condition on a bounded domain under conditions more weak and obtain three solutions. The problems related to the $\overrightarrow{p}(x)$-Laplace type operator are called anisotropic problems. Let us recall some articles wherein the authors studied this kind of problems :

In [1], the authors considered problem (P). First, they consider the case when $f(x,u) = \lambda(|u|^{q(x)-2}u + |u|^{\gamma(x)-2}u)$ in which the parameter λ is positive and $q(x)$, $\gamma(x)$ are continuous functions on $\overline{\Omega}$, , and they obtained the existence of two nontrivial weak solutions. Their arguments are based on the mountain pass theorem and Ekeland's variational principle [8]. Next, they considered $f(x,u) = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u$ and they established the existence of two unbounded sequence of weak solutions, their proof is based on fountain theorem [22].

In [15], the authors established the existence and uniqueness of a weak energy solution to the following boundary value problem :

$$(S)\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = f(x,u) & in \quad \Omega \\ u = 0 \qquad on \qquad \partial\Omega \end{cases}$$

In [18], the authors considered (S) where $f = \lambda|u|^{q(x)-2}u$, and established the existence of a continuous spectrum in several distinct situations. But in [17], the authors took the same problem with λ depends on the variable x, using the mountain-pass theorem of Ambrosetti and Rabinowitz [2] and the Ekeland's variational principle, they proved that under suitable conditions, problem (S) has two nontrivial weak solutions. In [5], Boureanu proved that problem (S) has a sequence of weak solutions by means of the symmetric mountain-pass theorem.

Given $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad and \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

Let $p \in C_+(\overline{\Omega})$, then $L^{p(x)}(\Omega)$ is called variable exponent Lebesgue space which is defined as follow

$$L^{p(x)}(\Omega) = \left\{ u : \begin{array}{c} u \text{ is a measurable real-valued function such that} \\ \int_\Omega |u(x)|^{p(x)}\, dx < \infty \end{array} \right\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\}$$

is a separable and reflexive Banach space (see [12]).
We say that p is logarithmic Hölder continuous if

$$|p(x) - p(y)| \leq -\frac{M}{log(|x-y|)} \qquad \forall x,\, y \in \Omega \text{ such that } |x-y| \leq 1/2. \tag{1.1}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in [L^{p(x)}(\Omega)]^N\}.$$

For all $u \in W^{1,p(x)}(\Omega)$, we have $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. If p satisfies (1.1), the space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\|_{p(x)} = |\nabla u|_{p(x)}$.
Now, we introduce a natural generalization of the function space $W_0^{1,p(x)}(\Omega)$, which will allow us to study the problem (P), which is called anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$. If $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$; $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$, and for each $i \in \{1, 2, ..., N\}$, we have $p_i \in C_+(\overline{\Omega})$, and satisfy (1.1), the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|u\|_{\overrightarrow{p}(.)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)},$$

and it's a reflexive Banach space (see[9, 18]). From now on, we put $X = W_0^{1,\overrightarrow{p}(x)}(\Omega)$.
In order to study the problem (P) we have to introduce the vectors $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbb{R}^N$ which are defined in the following way

$$\overrightarrow{P}_+ = (p_1^+, p_2^+, ..., p_N^+),\ \ \overrightarrow{P}_- = (p_1^-, p_2^-, ..., p_N^-),$$

and the positive real numbers P_+^+, P_-^+, P_-^- as the following

$$P_+^+ = \max\{p_1^+, ..., p_N^+\},\ P_-^+ = \max\{p_1^-, ..., p_N^-\},\ P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1. \tag{1.2}$$

Define $P_-^*, P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1},\ P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

Suppose that the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions :

(F_1) $|f(x,t)| \leq c(x) + d|t|^{\alpha(x)-1}$, for all $(x,t) \in \Omega \times \mathbb{R}$ where c is in $L^{\alpha'(x)}(\Omega)$ with $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$, $d \geq 0$ is a constant, $\alpha(x) \in C_+(\Omega)$ such that $\alpha^+ = \sup\limits_{x\in\overline{\Omega}} \alpha(x) < P_-^- < P_{-,\infty}$, and $P_-^- > N$.

(F_2) there exists a constant $0 < \theta < 1$, for $0 < t < 1$, we have $F(x,tu) > t^\theta |u|^\theta$.

(F_3) $f(x,t) < 0$, when $|t| \in (0,1)$, $f(x,t) \geq m > 0$, when $t \in (t_0, \infty), t_0 > 1$.

And assume that
(B) $b \in L^{\infty}(\Omega)$ and there exist $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.
We give now the main results of this paper .

Theorem 1.1. *Under the assumptions* $(A_0)-(A_3)$, (B), (F_1) *and* (F_2), *the problem* (P) *has at least one nontrivial weak solution in* X.

Theorem 1.2. *If* $(A_0)-(A_3)$, (B), (F_1) *and* (F_3) *hold, then there exists an open interval* $\Lambda \subset (0,\infty)$ *and a positive real number* $\rho > 0$ *such that each* $\lambda \in \Lambda$, (P) *has at least three solutions whose norms are less than* $\rho > 0$.

This paper is divided into two sections. In the first section we will give some known results, in the second we will give the proof of our main results.

2. Preliminaries

First, we recall some important definitions and proprieties of the Lebesgue and Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Proposition 2.1. *(see [6, 12, 11])*

1. *The space* $(L^{p(x)}(\Omega), |u|_{p(x)})$ *is a separable, uniformly convex Banach space and its dual space is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*
$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}$$
2. *If* $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$, $\forall x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

Proposition 2.2. (*see*[10]) *Denote* $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$. *Then for* $u \in L^{p(x)}(\Omega)$, $(u_n) \subset L^{p(x)}(\Omega)$ *we have*

1. $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1; > 1)$,
2. $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$,
3. $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$,
4. $|u|_{p(x)} \to 0 (\to \infty) \Leftrightarrow \rho_{p(x)}(u) \to 0 (\to \infty)$,
5. $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

We recall now some results which concerning the embedding theorem.

Proposition 2.3. (*see*[18]) *Suppose that* $\Omega \subset \mathbb{R}^N (N > 3)$ *is a bounded domain with smooth boundary and relation (1.2) is fulfilled.*

1. *For any* $q \in C(\overline{\Omega})$ *verifying*
$$1 < q(x) < P_{-,\infty} \ \ \forall x \in \overline{\Omega},$$
the embedding
$$W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$
is continuous and compact.

2. *Assume that* $P_-^- > N$*, then the embedding*

$$W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$$

is continuous and compact.

If A_i satisfies the conditions (A_0), (A_1), (A_2), and (A_3) we have the proposition below .

Proposition 2.4. (*cf.*[15, 17, 5]) *Let*

$$\mathcal{A}_i(u) = \int_\Omega A_i(x, \partial_{x_i} u)\, dx$$

For $i \in \{1, 2, ..., N\}$*, we have :*

- $\mathcal{A}_i$ *is well defined on* X*,*
- *the functional* $\mathcal{A}_i \in C^1(X, \mathbb{R})$ *and*

$$\langle \mathcal{A}_i^{'}(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi\, dx,$$

for all $u, \varphi \in X$ *: In addition* $\mathcal{A}_i^{'}$ *is continuous, bounded and strictly monotone.*

- $\mathcal{A}_i$ *is weakly lower semi-continuous.*
- *Let*

$$\mathcal{A}(u) = \int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u)\, dx,$$

then $\mathcal{A}^{'}$ *is an operator of type* (S_+).

The main theorem that we use here is the one which proved by Ricceri in [19, 20, 14, 4]. Based on [3], it can be equivalently stated as follows

Lemma 2.5. *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *is a continuous Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on* X^**;* $\Psi : X \to \mathbb{R}$ *is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that :*

1. $\lim\limits_{\|u\|_X \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty \ \forall \lambda > 0,$
2. *there exist* r *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$*,*
3. $\inf\limits_{u \in \Phi^{-1}(-\infty, r]} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)},$

then there exist an open interval $\Lambda \subset (0, \infty)$ *and a positive constant* $\rho > 0$ *such that for any* $\lambda \in \Lambda$ *the equation* $\Phi^{'}(u) + \lambda \Psi^{'}(u) = 0$ *has at least three solutions in* X *whose norms are less than* ρ*.*

And we have also the known following result.

Lemma 2.6. (*see*[7]) *Let* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a Carathéodory function with primitive* $F(x, u) = \int_0^u f(x, t)\, dt$*. If* f *satisfies* (F_1) *: then,*

$$\Psi(u) = -\int_\Omega F(x, u)\, dx \in C^1(X, \mathbb{R})$$

and

$$\langle \Psi'(u), \varphi \rangle = -\int_\Omega f(x, u) \varphi\, dx,$$

furthermore the operator $\Psi' : X \longrightarrow X^*$ *is compact.*

3. Proof of main results

We are interested to prove the existence of weak solutions. Let us define the functional I associated with the problem (P) then $I : X \longrightarrow \mathbb{R}$

$$I(u) = \int_\Omega \left[\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+} |u|^{P_+^+} - \lambda F(x,u) \right] dx,$$

where $F(x,t) = \int_0^t f(x,s)\, ds$. Using the notations of the Lemma (2.5), Φ and Ψ are defined as following :

$$\Phi(u) = \int_\Omega \left[\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+} |u|^{P_+^+} \right] dx,$$

$$\Psi(u) = -\int_\Omega F(x,u)\, dx,$$

and

$$I(u) = \Phi(u) + \lambda \Psi(u).$$

It should be noticed that, in this present paper, we have

$$P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^* \text{ and } P_+^+ < P_-^*, \tag{3.1}$$

then the compact embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{p_+^+}(\Omega)$ holds. Under the conditions $(A_0) - (A_3)$, $\Phi \in C^1(X, \mathbb{R})$ and

$$\langle \Phi'(u), \varphi \rangle = \int_\Omega \left[\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi + b(x) |u|^{P_+^+ - 2} u \varphi \right] dx.$$

and we have already

$$\langle \Psi'(u), \varphi \rangle = -\int_\Omega f(x,u) \varphi\, dx.$$

Then, I is well defined and $I \in C^1(X, \mathbb{R})$, so let us now give the definition of a weak solution.

Definition 3.1. *A function u is a weak solution of the problem (P) if and only if*

$$\int_\Omega \left[\sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi + b(x) |u|^{P_+^+ - 2} u \varphi - \lambda f(x,u) \varphi \right] dx = 0,$$

for all $\varphi \in X$.

Obviously the weak solutions of (P) are the critical points of I.

3.1. Existence of a nontrivial weak solution

In this section, we prove our result Theorem1.1.

Lemma 3.2. *Under the conditions (A_i), $i = 0,1,2,3$ and (F_1) the functional I is weakly lower semi-continuous, and coercive.*

Proof. The functional I is obviously weakly lower semi-continuous. Let us prove that I is coercive. For $u \in X$ such that $\|u\| \geq 1$, we have

$$\Phi(u) = \int_\Omega \left[\sum_{i=1}^N A_i(x, \partial_{x_i} u) + b(x) \frac{|u|^{P_+^+}}{P_+^+} \right] dx.$$

From (A_2) we deduce

$$\begin{aligned}\Phi(u) &\geq \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\,dx + \frac{b_0}{P_+^+}\int_\Omega |u|^{P_+^+}\,dx\\ &\geq \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u|^{p_i(x)}}{P_+^+}\,dx\end{aligned}$$

Let for $i\in\{1,2,...,N\}$

$$r_i=\begin{cases}P_+^+ & \text{if } |\partial_{x_i}u|_{p_i(x)}\leq 1.\\ P_-^- & \text{if } |\partial_{x_i}u|_{p_i(x)}> 1.\end{cases}$$

Using the Proposition (2.2), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}\,dx &\geq \sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(x)}^{r_i}\\ &\geq \sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(x)}^{p_-^-}-\sum_{i:r_i=p_+^+}\left(|\partial_{x_i}u|_{p_i(x)}^{p_-^-}-|\partial_{x_i}u|_{p_i(x)}^{p_+^+}\right)\\ &\geq \sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(x)}^{p_-^-}-N.\end{aligned}$$

Applying the Jensen inequality to the convex function $g:\mathbb{R}^+\to\mathbb{R}^+$ which is defined as following $g(t)=t^{P_-^-}$, $P_-^-\geq 2$, we find that

$$\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}\,dx \geq \frac{\|u\|^{P_-^-}}{N^{P_-^- -1}}-N, \tag{3.2}$$

so,

$$\Phi(u)\geq \frac{1}{P_+^+}\left(\frac{\|u\|^{P_-^-}}{N^{P_-^- -1}}-N\right),$$

On the other hand we have for $u\in X$ such that $\|u\|\geq 1$, by the Hölder inequality and the embedding theorem, we have

$$\begin{aligned}\Psi(u)=-\int_\Omega F(x,u)\,dx &\leq \int_\Omega [c(x)|u(x)|+\frac{d}{\alpha(x)}|u|^{\alpha(x)}]\,dx,\\ &\leq 2|c|_{\alpha'(x)}|u|_{\alpha(x)}+\frac{d}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx,\\ &\leq 2M|c|_{\alpha'(x)}\|u\|+\frac{d}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx,\end{aligned}$$

By the embedding theorem, we have $u\in L^{\alpha(x)}(\Omega)$; therefore,

$$\int_\Omega |u|^{\alpha(x)}\leq \max\{|u|_{\alpha(x)}^{\alpha^+},|u|_{\alpha(x)}^{\alpha^-}\}\leq M'\|u\|^{\alpha+}.$$

Then

$$|\Psi(u)|\leq 2M|c|_{\alpha'(x)}\|u\|+\frac{d}{\alpha^-}M'\|u\|^{\alpha+}.$$

From relation (3.2) above , we have

$$\Phi(u)\geq \frac{1}{P_+^+}\left(\frac{\|u\|^{P_-^-}}{N^{P_-^- -1}}-N\right),$$

this implies that for any $\lambda > 0$ that

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{P_+^+}\left(\frac{\|u\|^{P_-^-}}{N^{P_-^- -1}} - N\right) - 2\lambda M|c|_{\alpha'(x)}\|u\| - \frac{\lambda d M'}{\alpha^-}\|u\|^{\alpha+}.$$

Under the condition $1 < \alpha^+ < P_-^-$, we obtain

$$\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty.$$

finally the functional I is coercive. □

In order to demonstrate Theorem 1.1, it remains to verify that the solution is not trivial, because we have already proved that I is weakly lower semi-continuous, and coercive. Since I is weakly lower semi-continuous functional and coercive in X which is a reflexive Banach space, then I admits a global minimum. As it's differentiable, this minimum is a critical point, then a weak solution of (P). Let's prove that this solution is nontrivial. In the fact, it's sufficient to prove that there exists a function u_1 such that $I(u_1) < 0$ because $I(0) = 0$. To get this result, we use the assumption (F_1). By (A_0) and (A_1), we have

$$A_i(x,\eta) = \int_0^1 a_i(x,t\eta)\,dt \leq C\left(|\eta| + \frac{|\eta|^{p_i(x)}}{p_i(x)}\right), \forall x \in \overline{\Omega},\ x \in \mathbb{R},\ C = \max_{i\in\{1,2,\ldots,N\}} \overline{c}_i.$$

Then

$$\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}u)\,dx \leq C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}u| + \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\right)dx.$$

Let $0 \neq \varphi \in C_0^\infty(\Omega)$, and $0 < \theta < 1$. For $t > 0$ is small enough, we have

$$\begin{aligned}
I(t\varphi) &= \int_\Omega \left\{\sum_{i=1}^N A_i(x,\partial_{x_i}(t\varphi)) + \frac{b(x)}{P_+^+}|t\varphi|^{P_+^+} - \lambda F(x,t\varphi)\right\}dx, \\
&\leq C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(t\varphi)| + \frac{|\partial_{x_i}(t\varphi)|^{p_i(x)}}{p_i(x)}\right)dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\varphi|^{P_+^+}\,dx \\
&\quad - \int_\Omega \lambda F(x,t\varphi)\,dx, \\
&\leq C\sum_{i=1}^N \int_\Omega \left(t|\partial_{x_i}\varphi| + \frac{t^{P_-^-}|\partial_{x_i}\varphi|^{p_i(x)}}{p_i(x)}\right)dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\varphi|^{P_+^+}\,dx \\
&\quad - \int_\Omega \lambda F(x,t\varphi)\,dx, \\
&\leq t\left\{C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\varphi| + \frac{|\partial_{x_i}\varphi|^{p_i(x)}}{P_-^-}\right)dx + \frac{1}{P_+^+}\int_\Omega b(x)|\varphi|^{P_+^+}\,dx\right\} \\
&\quad -\lambda t^\theta|\varphi|^\theta, \\
&< 0.
\end{aligned}$$

3.2. Existence of three solutions

In this section, we prove our result Theorem 1.2 by using Lemma 2.5. First we need to verify that the precondition of Φ in Lemma 2.5 are fulfilled.

Lemma 3.3. *Under the conditions $(A_0) - (A_3)$ and the assumption (3.1), Φ is weakly lower semi-continuous, moreover Φ' admits a continuous inverse.*

Proof. Under the conditions $(A_0)-(A_3)$ and the assumption above (3.1), the functional Φ is well defined and it's of class $C^1(X,\mathbb{R})$, moreover it's weakly lower semi-continuous. The condition (A_3) means that $\Phi^{'}$ is uniformly monotone. Moreover $\Phi^{'}$ is coercive . Let's prove the coercivity of $\Phi^{'}$. For $u \in X$ such that $\|u\| \geq 1$, we have

$$\langle \Phi^{'}(u), u\rangle = \int_\Omega \left[\sum_{i=1}^{N} a_i(x,\partial_{x_i}u)\partial_{x_i}u + b(x)|u|^{P_+^+}\right] dx,$$

by (A_2) and (3.2), we deduce

$$\begin{aligned}\langle \Phi^{'}(u), u\rangle &\geq \sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}\,dx + b_0\int_\Omega |u|^{P_+^+}\,dx\\ &\geq \sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}\,dx\end{aligned}$$

so,

$$\langle \Phi^{'}(u), u\rangle \geq \frac{\|u\|^{P_-^-}}{N^{P_-^- -1}} - N,$$

thus,

$$\frac{\langle \Phi^{'}(u), u\rangle}{\|u\|} \geq \|u\|^{P_-^- -1}\left(\frac{1}{N^{P_-^- -1}} - \frac{N}{\|u\|^{P_-^-}}\right),$$

and for $\|u\|$ big enough, we have $\Phi^{'}$ is coercive.
By a standard argument, we know that $\Phi^{'}$ is hemicontinuous, then $\Phi^{'}$ admits a continuous inverse. □

In following we need to verify that the conditions 2. and 3. in Lemma 2.5 are fulfilled because the condition 1. of Lemma 2.5 is already verified above.
verification of the assumptions 2. and 3. of Ricceri's theorem :

In order to prove the assumptions 2. and 3. of Ricceri's theorem which is the main tool in this paper, we use the condition (F_2), which implies that $F(x,t)$ is increasing for $t \in (t_0,\infty)$ and decreasing for $t \in (0,1)$ uniformly for $x \in \Omega$, and $F(x,0) = 0$ is obvious, $F(x,t) \to \infty$ when $t \to \infty$ because $F(x,t) \geq mt$ uniformly for x. Then, there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau)\ \forall x \in \Omega,\ \ t > \delta,\ \ \tau \in (0,1)$$

The compact embedding from X to $C(\overline{\Omega})$ means that there exists a constant m_1 which satisfies

$$\|u\|_{C(\overline{\Omega})} \leq m_1\|u\|,$$

where $\|u\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)|$. Let a, e be two real numbers such that $0 < a < \min\{1, m_1\}$, we choose $e > \delta$ satisfying $e^{P_-^-} b_0|\Omega| > 1$. When $t \in [0,a]$ we have

$$F(x,t) \leq F(x,0) = 0,$$

then

$$\int_\Omega \sup_{0<t<a} F(x,t)\,dx \leq \int_\Omega F(x,0)\,dx = 0.$$

As $e > \delta$, we have

$$\int_\Omega F(x,e)\,dx > 0,$$

and

$$\frac{1}{m_1^{P_+^+}}\frac{a^{P_+^+}}{e^{P_-^-}}\int_\Omega F(x,e)\,dx>0.$$

Which implies

$$\int_\Omega \sup_{0<t<a} F(x,t)\,dx \leq 0 < \frac{1}{m_1^{P_+^+}}\frac{a^{P_+^+}}{e^{P_-^-}}\int_\Omega F(x,e)\,dx.$$

Let $u_0, u_1 \in X$, $u_0(x)=0$ and $u_1(x)=e$ for any $x\in\overline{\Omega}$. We define $r=\frac{1}{N^{P_+^+-1}P_+^+}(\frac{a}{m_1})^{P_+^+}$. Obviously $r\in(0,1)$, $\Phi(u_0)=\Psi(u_0)=0$,

$$\Phi(u_1)=\int_\Omega \frac{b(x)}{P_+^+}|e|^{P_+^+}\,dx \geq \frac{b_0}{P_+^+}e^{P_-^-}|\Omega| > \frac{1}{P_+^+} > \frac{1}{N^{P_+^+-1}P_+^+}(\frac{a}{m_1})^{P_+^+}=r,$$

and

$$\Psi(u_1)=-\int_\Omega F(x,u_1)\,dx=-\int_\Omega F(x,e)\,dx<0.$$

So we have $\Phi(u_0)<r<\Phi(u_1)$. Then 2. of Ricceri's theorem is fulfilled.
On the other hand, we have

$$\begin{aligned}-\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)} &= -r\frac{\Psi(u_1)}{\Phi(u_1)}\\ &= r\frac{\int_\Omega F(x,e)\,dx}{\int_\Omega \frac{b(x)}{P_+^+}|e|^{P_+^+}\,dx}>0.\end{aligned}$$

Let $u\in X$ be such that $\Phi(u)\leq r<1$. Set

$$J(u)=\int_\Omega\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)}\,dx$$

then

$$\frac{J(u)}{P_+^+}\leq\int_\Omega\left\{\sum_{i=1}^N\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}+\frac{b(x)}{P_+^+}|u|^{P_+^+}\right\}dx,$$

by (A_2) we have

$$\frac{J(u)}{P_+^+}\leq\int_\Omega\left\{\sum_{i=1}^N A_i(x,\partial_{x_i}u)+\frac{b(x)}{P_+^+}|u|^{P_+^+}\right\}dx=\Phi(u)\leq r,$$

which means that

$$J(u)\leq P_+^+r=\frac{1}{N^{P_+^+-1}}(\frac{a}{m_1})^{P_+^+}<1,$$

it follows that

$$\int_\Omega|\partial_{x_i}u|^{p_i(x)}\,dx<1.$$

By Proposition 2.2, we have

$$|\partial_{x_i}u|_{p_i(x)}<1,$$

and

$$\begin{aligned} J(u) &= \sum_{i=1}^{N} \int_\Omega |\partial_{x_i}(u)|^{p_i(x)}\,dx \geq \sum_{i=1}^{N} |\partial_{x_i}(u)|_{p_i(x)}^{P_i^+} \\ &\geq \sum_{i=1}^{N} |\partial_{x_i}(u)|_{p_i(x)}^{P_+^+} \\ &\geq N\left(\frac{\sum_{i=1}^{N}|\partial_{x_i}(u)|_{p_i(x)}}{N}\right)^{P_+^+} \\ &= \frac{\|u\|^{P_+^+}}{N^{P_+^+-1}}. \end{aligned}$$

Consequently

$$\frac{\|u\|^{P_+^+}}{N^{P_+^+-1}} \leq J(u) \leq P_+^+ r,$$

it follows that

$$\frac{\|u\|^{P_+^+}}{N^{P_+^+-1}P_+^+} \leq \frac{J(u)}{P_+^+} \leq \Phi(u) \leq r,$$

then

$$|u(x)| \leq m_1\|u\| \leq m_1(N^{P_+^+-1}P_+^+ r)^{\frac{1}{P_+^+}} = a \qquad \forall u \in X,\ x \in \overline{\Omega},\ \Phi(u) \leq r.$$

This inequality shows that

$$-\inf_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r]} -\Psi(u) \leq \int_\Omega \sup_{0<u<a} F(x,u)\,dx \leq 0.$$

Then

$$\inf_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)},$$

which means that condition 3. is obtained. Since the assumptions of lemma 2.5 are fulfilled, there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\Phi^{'}(u) + \lambda\Psi^{'}(u) = 0$ has at least three solutions in X whose norms are less than ρ.

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