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Spectral Inclusions Between C_0-quasi-semigroups and Their Generators

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ABSTRACT: In this paper, we show a spectral inclusion of a different spectra of a C_0-quasi-semigroup and its generator $A(t)$. Precisely, we focus for ordinary spectrum, point spectrum, approximate spectrum, residual spectrum and essential spectrum.

Key Words: C_0-quasi-semigroup, C_0-semigroup, Ordinary spectrum, Point spectrum, Approximate spectrum, Residual spectrum, Essential spectrum.

Contents

1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. An unbounded operator T on X is a linear application partially defined on a subspace $D(T) \subseteq X$ called domain of T, and it's closed if its graph $\Gamma(T) = \{(x; T(x))/x \in D(T)\}$ is closed in X^2, we denote the space of these operators by $\mathcal{C}(X)$.
Let T be a closed linear operator on X with domain $D(T)$, we denote by $Rg(T)$, $Rg^\infty(T) := \cap_{n\geq 1} Rg(T^n)$, $N(T)$, $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$ respectively the range, the hyper range, the kernel, the resolvent and the spectrum of T, where $\sigma(T) = \{\lambda \in \mathbb{C} \setminus \lambda - T \text{ is not bijective}\}$.
For a closed operator T we define the point spectrum, the approximate point spectrum and the residual spectrum by

- $\sigma_p(T) = \{\lambda \in \mathbb{C} \setminus \lambda - T \,\text{is not injective}\,\}$,
- $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} \setminus \lambda - T \text{ is not injective or } Rg(\lambda - T) \text{ is not closed in } X\}$,
- $\sigma_r(T) = \{\lambda \in \mathbb{C} \setminus\, Rg(\lambda - T) \text{ is not dense in } X\}$.

From [1, p.79], we have $\lambda \in \sigma_{ap}(T)$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(T)$, such that $\|x_n\| = 1$ and $lim_{n\to\infty}\|(T - \lambda)x_n\| = 0$.
A closed operator T is called Fredholm if $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{co}\dim Rg(T)$ are finite. The essential spectrum is defined by,

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}.$$

The family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a C_0-semigroup if it has the following properties :

1. $T(0) = I$,
2. $T(t)T(s) = T(t + s)$,
3. The map $t \to T(t)x$ from $[0, +\infty[$ into X is continuous for all $x \in X$.

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In this case, its generator A is defined by

$$\mathcal{D}(\mathcal{A}) = \{x \in X \,/\, \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists}\},$$

with

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

The theory of quasi-semigroups of bounded linear operators, as a generalization of semigroups of operators, was introduced by H. Leiva and D. Barcenas [2], [3], [4].
Recall that a two parameter commutative family $\{R(t,s)\}_{t,s\geq 0} \subseteq \mathcal{B}(X)$ is called a strongly continuous quasi-semigroup (or C_0-quasi-semigroup) of operators [9] if for every $t, s, r \geq 0$ and $x \in X$, we have

1. $R(t, 0) = I$, the identity operator on X,
2. $R(t, s + r) = R(t + r, s)R(t, r)$,
3. $lim_{s\longrightarrow 0} ||R(t, s)x - x|| = 0$,
4. there exists a continuous increasing mapping $M : [0, +\infty[\longrightarrow [0, +\infty[$ such that,

$$||R(t, s)|| \leq M(t + s).$$

For a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist,

$$\lim_{s\to 0^+} \frac{R(0,s)x - x}{s}, \ \lim_{s\to 0^+} \frac{R(t,s)x - x}{s} \text{ and } \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}$$

and

$$\lim_{s\to 0^+} \frac{R(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}.$$

In this case, for $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

The family $\{A(t)\}_{t\geq 0}$ is called infinitesimal generator of the C_0-quasi-semigroups $\{R(t,s)\}_{t,s\geq 0}$.

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X with its generator A. Through this family, Sutrima and al built concrete examples of C_0-quasi-semigroup.

Example 1.1. *[9, Examples 2.2, 2.4 and 2.5]*

1. *Let for $t, s \geq 0$,*

$$R(t, s) = T(s).$$

Then $\{R(t,s)\}_{t,s\geq 0}$ is a C_0-quasi-semigroup with $\mathcal{D} = \mathcal{D}(\mathcal{A})$ and its generator for all $t \geq 0$

$$A(t) = A.$$

2. *Let for $t, s \geq 0$,*

$$R(t, s) = T(g(t + s) - g(t)).$$

where $g(t) = \int_0^t a(u)du$ and $a \in \mathcal{C}([0, +\infty[)$ where $\mathcal{C}([0, +\infty[)$ is the set of all continuous functions defined on $[0, +\infty[\to [0, +\infty[$. Then $\{R(t,s)\}_{t,s\geq 0}$ is a C_0-quasi-semigroup with $\mathcal{D} = \mathcal{D}(\mathcal{A})$ and its generator for all $t \geq 0$

$$A(t) = a(t)A.$$

3. *Let for $t, s \geq 0$,*
$$R(t,s) = e^{T(s+t)-T(t)}.$$
Then $\{R(t,s)\}_{t,s\geq 0}$ is a C_0-quasi-semigroup with $\mathcal{D} = \mathcal{D}(\mathcal{A})$ and its generator for all $t \geq 0$
$$A(t) = AT(t).$$

Theorem 1.2. *[9, Theorems 3.1 and 3.2] Let $\{R(t,s)\}_{t,s\geq 0}$ be a C_0-quasi-semigroup on X with generator $A(t)$. Then we have*

1. *For each $t \geq 0$, $R(t,.)$ is strongly continuous on $[0,+\infty[$.*
2. *For each $t \geq 0$ and $x \in X$,*
$$\lim_{s\to 0^+} \frac{1}{s}\int_0^s R(t,h)x dh = x.$$
3. *If $x \in \mathcal{D}$, $t \geq 0$ and $t_0, s_0 \geq 0$, then $R(t_0,s_0)x \in \mathcal{D}$ and*
$$R(t_0,s_0)A(t)x = A(t)R(t_0,s_0)x.$$
4. *For each $s > 0$ and $x \in \mathcal{D}$,*
$$\frac{\partial}{\partial s}(R(t,s)x) = A(t+s)R(t,s)x = R(t,s)A(t+s)x.$$
5. *If $A(.)$ is locally integrable, then for every $x \in \mathcal{D}$ and $s \geq 0$,*
$$R(t,s)x = x + \int_0^s A(t+h)R(t,h)x dh.$$
6. *If $f : [0,+\infty[\longrightarrow X$ is a continuous, then for every $t \in [0,+\infty[$*
$$\lim_{r\to 0^+} \frac{1}{h}\int_s^{s+r} R(t,h)f(h)dh = R(t,s)f(s).$$

Contrary to C_0-semigroup, the generator $A(t)$ of C_0-quasi-semigroup is not necessary closed or densely defined [9, Examples 2.3 and 3.3].

Theorem 1.3. *[9, Theorem 3.4] Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that the resolvent $\mathcal{R}(\lambda, A(t)) = (\lambda - A(t))^{-1}$ exists in $S = \{\lambda \in \mathbb{C},\ -\theta \leq \arg(\lambda) \leq \theta$ with $\theta \in]\frac{\pi}{2}, \pi[\}$. If $\lambda \in \rho(A(t))$, then for all $s > 0$ we have*
$$\mathcal{R}(\lambda, A(t))R(t,s) = R(t,s)\mathcal{R}(\lambda, A(t)).$$

For all $t \geq s \geq 0$, $\lambda \in \mathbb{C}$ and $\{R(t,s)\}_{t,s\geq 0} \in \mathcal{B}(X)$, we define in this paper an integral giving by for all $x \in X$
$$D_\lambda(t,s)x = \int_0^s e^{\lambda(s-h)}R(t-h,h)x dh,$$
where the integral is understood in the sense of Bochner [5].
It's clear that for each $t, \geq s \geq 0$, we have $D_\lambda(t,s) \in \mathcal{B}(X)$. Indeed, there exists a continuous increasing mapping $M : [0,+\infty[\longrightarrow [0,+\infty[$ satisfying $\|R(t,s)\| \leq M(t+s)$ if $\lambda = 0$, then $\|D_0(t,s)x\| \leq M(t+s)s\|x\|$ and if $\lambda \neq 0$, then $\|D_\lambda(t,s)x\| \leq \frac{M(t+s)\|x\||1-e^{\lambda s}|}{|\lambda|}$.

Inspired by the spectral study of C_0-semigroup, in this work, we show that the spectral inclusion of different spectra for C_0-quasi-semigroup and its generator.

2. Main results

We start with the important result.

Theorem 2.1. *Let $A(t)$ be the generator of the C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined.*
Then for all $t \geq s \geq 0$ and all $\lambda \in \mathbb{C}$, we have

1. *For all $x \in \mathcal{D}$,*
$$D_\lambda(t,s)(\lambda - A(t))x = [e^{\lambda s} - R(t-s,s)]x,$$
where $D_\lambda(t,s)x = \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh$.

2. *For all $x \in X$, we have $D_\lambda(t,s)x \in \mathcal{D}$ and*
$$(\lambda - A(t))D_\lambda(t,s)x = [e^{\lambda s} - R(t-s,s)]x.$$

Proof. 1. By Theorem 1.2, we know for all $h > 0$ and for all $x \in \mathcal{D}$,
$$\frac{\partial}{\partial h}(R(t-h,h)x) = A(t)R(t-h,h)x = R(t-h,h)A(t)x.$$
Therefore, we conclude that
$$\begin{aligned} D_\lambda(t,s)[A(t)x] &= \int_0^s e^{\lambda(s-h)} R(t-h,h)[A(t)x]dh \\ &= \int_0^s e^{\lambda(s-h)} \Big[\frac{\partial}{\partial h}(R(t-h,h))\Big] xdh \\ &= \Big[e^{\lambda(s-h)} R(t-h,h)x\Big]_0^s + \lambda \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh \\ &= R(t-s,s)x - e^{\lambda s}x + \lambda D_\lambda(t,s)x. \qquad (*) \end{aligned}$$
Finally, we obtain for all $x \in \mathcal{D}$
$$D_\lambda(t,s)(\lambda - A(t))x = [e^{\lambda s} - R(t-s,s)]x.$$

2. Let $\mu \in \rho(A(t))$. From Theorem 1.3, we have for all $x \in X$
$$R(\mu, A(t))R(t,s)x = R(t,s)R(\mu, A(t))x.$$
Hence, for all $x \in X$ we conclude
$$\begin{aligned} \mathcal{R}(\mu, A(t))D_\lambda(t,s)x &= \mathcal{R}(\mu, A(t)) \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh \\ &= \int_0^s e^{\lambda(s-h)} \mathcal{R}(\mu, A(t))R(t-h,h)xdh \\ &= \int_0^s e^{\lambda(s-h)} R(t-h,h)\mathcal{R}(\mu, A(t))xdh \\ &= D_\lambda(t,s)\mathcal{R}(\mu, A(t))x. \end{aligned}$$

Therefore, we obtain for all $x \in X$

$$\begin{aligned}
D_\lambda(t,s)x &= \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh \\
&= \int_0^s e^{\lambda(s-h)} R(t-h,h)(\mu - A(t))\mathcal{R}(\mu, A(t))xdh \\
&= \mu \int_0^s e^{\lambda(s-h)} R(t-h,h)\mathcal{R}(\mu, A(t))xdh - \int_0^s e^{\lambda(s-h)} R(t-h,h)A(t)\mathcal{R}(\mu, A(t))xdh \\
&= \mu \int_0^s e^{\lambda(s-h)} \mathcal{R}(\mu, A(t))R(t-h,h)xdh - \int_0^s e^{\lambda(s-h)} R(t-h,h)A(t)\mathcal{R}(\mu, A(t))xdh \\
&= \mu R(\mu, A(t)) \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh - \int_0^s e^{\lambda(s-h)} R(t-h,h)[A(t)\mathcal{R}(\mu, A(t))x]dh \\
&= \mu \mathcal{R}(\mu, A(t))D_\lambda(t,s)x - D_\lambda(t,s)[A(t)\mathcal{R}(\mu, A(t))x] \\
&\overset{(*)}{=} \mu \mathcal{R}(\mu, A(t))D_\lambda(t,s)x - \Big[R(t-s,s)\mathcal{R}(\mu, A(t))x - e^{\lambda s}\mathcal{R}(\mu, A(t))x + \lambda D_\lambda(t,s)\mathcal{R}(\mu, A(t))x\Big] \\
&= \mu \mathcal{R}(\mu, A(t))D_\lambda(t,s)x - \mathcal{R}(\mu, A(t))R(t-s,s)x + e^{\lambda s}\mathcal{R}(\mu, A(t))x - \lambda \mathcal{R}(\mu, A(t))D_\lambda(t,s)x \\
&= \mathcal{R}(\mu, A(t))\Big[\mu D_\lambda(t,s)x - R(t-s,s)x + e^{\lambda s}x - \lambda D_\lambda(t,s)x\Big].
\end{aligned}$$

Therefore, for all $x \in X$ we deduce $D_\lambda(t,s)x \in \mathcal{D}$ and we have

$$(\mu - A(t))D_\lambda(t,s)x = \mu D_\lambda(t,s)x - R(t-s,s)x + e^{\lambda s}x - \lambda D_\lambda(t,s)x.$$

Finally, we obtain for all $x \in X$,

$$(\lambda - A(t))D_\lambda(t,s)x = [e^{\lambda s} - R(t-s,s)]x.$$

□

Corollary 2.2. *Let $A(t)$ be the generator of the C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s \geq 0$ and all $\lambda \in \mathbb{C}$, we obtain*

1. *For all $x \in X$,*
$$(\lambda - A(t))^n [D_\lambda(t,s)]^n x = [e^{\lambda s} - R(t-s,s)]^n x.$$
2. *For all $x \in \mathcal{D}^n$,*
$$[D_\lambda(t,s)]^n (\lambda - [A(t)]^n)x = [e^{\lambda s} - R(t-s,s)]^n x.$$
3. *$N[\lambda - A(t)] \subseteq N[e^{\lambda s} - R(t-s,s)]$.*
4. *$Rg[e^{\lambda s} - R(t-s,s)] \subseteq Rg[\lambda - A(t)]$.*
5. *$N[\lambda - A(t)]^n \subseteq N[e^{\lambda s} - R(t-s,s)]^n$.*
6. *$Rg[e^{\lambda s} - R(t-s,s)]^n \subseteq Rg[\lambda - A(t)]^n$.*
7. *$Rg^\infty[e^{\lambda s} - R(t-s,s)] \subseteq Rg^\infty[\lambda - A(t)]$.*

Proof. It's automatic by Theorem 2.1. □

The following theorem characterizes the ordinary, point, approximate point, essential and residual spectra of a C_0-quasi-semigroup.

Theorem 2.3. *Let $A(t)$ be the generator of the C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s \geq 0$, we get*

1. $e^{\sigma(A(t))s} \subset \sigma(R(t-s,s))$
2. $e^{\sigma_p(A(t))s} \subset \sigma_p(R(t-s,s))$
3. $e^{\sigma_{ap}(A(t))s} \subset \sigma_{ap}(R(t-s,s))$
4. $e^{\sigma_e(A(t))s} \subset \sigma_e(R(t-s,s))$
5. $e^{\sigma_r(A(t))s} \subset \sigma_r(R(t-s,s))$.

Proof. 1. Let $\lambda \in \mathbb{C}$ such that for all $t \geq s \geq 0$, $e^{\lambda s} \notin \sigma(R(t-s,s))$.
Then there exists $F_\lambda(t,s) \in \mathcal{B}(X)$ satisfying

$$F_\lambda(t,s)[e^{\lambda s} - R(t-s,s)] = [e^{\lambda s} - R(t-s,s)]F_\lambda(t,s) = I.$$

Hence, by Theorem 2.1, we obtain for every $x \in \mathcal{D}$

$$\begin{aligned} x &= F_\lambda(t,s)[e^{\lambda s} - R(t-s,s)]x \\ &= F_\lambda(t,s)[D_\lambda(t,s)(\lambda - A(t))]x \\ &= [F_\lambda(t,s)D_\lambda(t,s)](\lambda - A(t))x. \end{aligned}$$

On the other hand, also from Theorem 2.1, we obtain for every $x \in X$

$$\begin{aligned} x &= [e^{\lambda s} - R(t-s,s)]F_\lambda(t,s)x \\ &= [(\lambda - A(t))D_\lambda(s)]F_\lambda(t,s)x \\ &= (\lambda - A(t))[D_\lambda(t,s)F_\lambda(t,s)]x. \end{aligned}$$

Since we know that $R(t-s,s)F_\lambda(t,s) = F_\lambda(t,s)R(t-s,s)$, then

$$F_\lambda(t,s)D_\lambda(t,s) = D_\lambda(t,s)F_\lambda(t,s).$$

Consequently, we obtain

$$F_\lambda(t,s)[e^{\lambda s} - R(t-s,s)] = [e^{\lambda s} - R(t-s,s)]F_\lambda(t,s).$$

Finally, we conclude that $\lambda - A(t)$ is invertible and hence $\lambda \notin \sigma(A(t))$.

2. Let $\lambda \in \sigma_p(A(t))$, then there exists $x \neq 0$ such that

$$x \in N(\lambda - A(t)).$$

From Corollary 2.2, we deduce that

$$x \in N[e^{\lambda s} - R(t-s,s)].$$

Therefore, we conclude that $e^{\lambda s} \in \sigma_p(R(t-s,s))$.

3. Let $\lambda \in \sigma_{ap}(A(t))$, then there exists $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ satisfying $\|x_n\| = 1$ and

$$\|(\lambda - A(t))x_n\| \to 0 \text{ as } n \to \infty.$$

By Theorem 2.1, we obtain

$$\begin{aligned} \|[e^{\lambda s} - R(t-s,s)]x_n\| &= \|D_\lambda(t,s)(\lambda - A(t))x_n\| \\ &= \|\int_0^s e^{\lambda(s-h)}R(t-h,h)(\lambda - A(t))x_n dh\| \\ &\leq \int_0^s \|e^{\lambda(s-h)}R(t-h,h)(\lambda - A(t))x_n\|dh \\ &\leq \Big[\int_0^s e^{\lambda(s-h)}dh\Big]M(t+s)\|(\lambda - A(t))x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Then $\|[e^{\lambda s} - R(t-s,s)]x_n\| \to 0$, and hence we conclude that

$$e^{\lambda s} \in \sigma_{ap}[R(t-s,s)].$$

4. Let $\lambda \in \mathbb{C}$ such that
$$e^{\lambda s} \notin \sigma_e(R(t-s,s)).$$
Then we have $\alpha[e^{\lambda s} - R(t-s,s)] < +\infty$ and $\beta[e^{\lambda s} - R(t-s,s)] < +\infty$.
Therefore, by Corollary 2.2, we conclude that $\alpha[\lambda - A(t)] < +\infty$ and $\beta[\lambda - A(t)] < +\infty$.
Hence, we deduce that
$$\lambda \notin \sigma_e(A).$$

5. Let $\lambda \in \sigma_r(A(t))$, then $Rg[\lambda - A(t)]$ is not dense in X. We now by Corollary 2.2 that
$$Rg[e^{\lambda s} - R(t-s,s)] \subseteq Rg[\lambda - A(t)].$$
Therefore, we deduce that $Rg[e^{\lambda s} - R(t-s,s)]$ is not dense in X.
Finally, we obtain
$$e^{\lambda s} \in \sigma_r(R(t-s,s)).$$

□

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