Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.46206

Existence of Multiple Solutions for a Nonhomogeneous p-Laplacian Elliptic Equation with Critical Sobolev-Hardy Exponent

Atika Matallah and Sara Litimein and Sofiane Messirdi

ABSTRACT: This paper concerns the existence of multiple nontrivial solutions for nonhomogeneous p-Laplacain elliptic problems involving the critical Hardy-Sobolev exponent. The method used here is based on Ekeland variational principale on Nehari manifold.

Key Words: Variational methods, Critical Hardy-Sobolev exponent, Nehari manifold, p-Laplacain equations.

Contents

1. Introduction and main results

In this paper, we consider the following nonhomogeneous elliptic problem

$$(\mathcal{P}_{\mu,s}) \begin{cases} -\triangle_p u - \mu \dfrac{|u|^{p-2}}{|x|^p} u = \dfrac{|u|^{p_*(s)-2}}{|x|^s} u + f(x) \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing 0 in its interior, $\triangle_p u$ denotes the p-Laplace operator defined as $div\left(|\nabla u|^{p-2} \nabla u\right)$ with $1 < p < N$, $-\infty < \mu < \overline{\mu}$, $\overline{\mu} := \left[(N-p)/p\right]^p$, $0 \leq s < p$, $p_*(s) = p(N-s)/(N-p)$ is the critical Sobolev-Hardy exponent, note that $p_*(0) = p_* = pN/(N-p)$ is the critical Sobolev exponent and f is a given measurable function different than 0.
The problem is related to the following Sobolev-Hardy inequality [4]:

$$\left(\int_\Omega \frac{|u|^{p_*(s)}}{|x|^s} dx\right)^{1/p_*(s)} \leq C_s \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p} \text{ for all } u \in C_0^\infty(\Omega), \tag{1.1}$$

for some positive constant C_s. If $s = p$ in (1), then $p_*(s) = p$, $C_s = 1/\overline{\mu}$ and we have the following Hardy inequality [7] :

$$\int_\Omega \frac{|u|^p}{|x|^p} dx \leq \frac{1}{\overline{\mu}} \int_\Omega |\nabla u|^p \, dx, \text{ for all } u \in C_0^\infty(\Omega).$$

We shall work with the space $W_\mu^{1,p} := W_\mu^{1,p}(\Omega)$ for $-\infty < \mu < \overline{\mu}$ endowed with the norm

$$\|u\|_\mu^p := \int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}\right) dx,$$

which is equivalent to the norm $\|.\|_p$.

2010 *Mathematics Subject Classification:* 35J20, 35J70, 47J30, 58E30.
Submitted January 12, 2019. Published May 15, 2019

Elliptic problems involving the Hardy inequality or Hardy–Sobolev inequality has been studied by some authors either in bounded domain or in the whole space $\mathbb{R}^N$, see [1, 2, 6, 8-12] and the references therein. Furthermore, by the Pohozaev identity, the problem $(\mathcal{P}_{\mu,s})$ has no nontrivial solution in the case $f \equiv 0$ and Ω is a star-shaped domain.

When the problem $(\mathcal{P}_{\mu,s})$ has no singular term $(s = \mu = 0)$, Tarantello in [13] proved the existence of two nontrivial solutions for $p = 2$ and $f \in H^{-1}$ $\left(\text{the dual of } H_0^1\right)$ such that

$$\int_\Omega fu\, dx < \frac{4}{N-2}\left[\frac{N-2}{N+2}\int_\Omega |\nabla u|^2\, dx\right]^{(N+2)/4}.$$

A natural interesting question is whether the results concerning the solutions of $(\mathcal{P}_{0,0})$ with $p = 2$ remain true for the problem $(\mathcal{P}_{\mu,s})$. As in [13], we study in this paper the problem $(\mathcal{P}_{\mu,s})$ and give some positive answers. To the best of our knowledge, the results are new in the case when $p \neq 2$ and $s \neq 0$.

In the sequel, we denote the norms of $L^{p_*(s)}\left(\Omega,\ |x|^{-s}\right)$ and W_μ^* $\left(\text{the dual of } W_\mu^{1,p}\right)$ by $\|u\|_{p_*(s)}$ and $\|u\|_{\_}$ respectively, $B(x_0, r)$ denotes a ball in Ω of radius r centred at x_0.
Furthermore, set

$$\Lambda_f =: \inf_{u \in W_\mu^{1,p}} \left\{ \frac{p_*(s) - p}{p-1}\left[\frac{(p-1)\|u\|_\mu^p}{(p_*(s)-1)\|u\|_{p_*(s)}^p}\right]^{\frac{p_*(s)-1}{p_*(s)-p}} - \frac{\int_\Omega fu\, dx}{\|u\|_{p_*(s)}} \right\}$$

Here are the main results of this paper.

Theorem 1.1. *Let $1 < p < N$, $-\infty < \mu < \overline{\mu}$, $0 \le s < p$ and $f \not\equiv 0$ satisfying $\Lambda_f > 0$. Then, $(\mathcal{P}_{\mu,s})$ has at least one positive solution which is a ground state solution.*

Theorem 1.2. *Suppose $2 \le p < N$, $f(x) \ge a_0 > 0$ in a small neighborhood of 0 and satisfies $\Lambda_f > 0$. Then, problem $(\mathcal{P}_{0,s})$ has at least two different solutions.*

This paper is organized as follows. In Section 2, we give some preliminary results. The proof of our main results is contained in Section 3.

2. Preliminary results

In this section, we give some preliminary results which will be used later.

We define for $0 \le \mu < \overline{\mu}$

$$S_{\mu,s} := \inf_{u \in W_\mu^{1,p}\setminus\{0\}} \frac{\|u\|_\mu^p}{\|u\|_{p_*(s)}^{p_*(s)}}$$

and

$$S_{0,s} := \inf_{u \in W_\mu^{1,p}\setminus\{0\}} \frac{\|u\|_0^p}{\|u\|_{p_*(s)}^{p_*(s)}}$$

From [9], $S_{\mu,s}$ is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that $S_{\mu,s}(\Omega) = S_{\mu,s}\left(\mathbb{R}^N\right) = S_{\mu,s}$. In addition, the constant $S_{\mu,s}$ is achieved by a family of functions

$$V_\varepsilon(x) := \varepsilon^{(p-N)/p}\tilde{u}_{p,\mu}\left(\frac{x}{\varepsilon}\right),\ \varepsilon > 0,$$

where $\tilde{u}_{p,\mu}(x) = \tilde{u}_{p,\mu}(|x|)$ is the unique radial solution for the problem

$$\begin{cases} -\Delta_p u - \mu\dfrac{|u|^{p-1}u}{|x|^p} = \dfrac{|u|^{p_*(s)-2}}{|x|^s}u & \text{in } \mathbb{R}^N\setminus\{0\} \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases}$$

In the other hand, from [8] $S_{0,s}$ is independent of any $\Omega \subset \mathbb{R}^N$ and it is achieved by a family of functions

$$U_\varepsilon(x) := \left[\varepsilon (N-s)\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{N-p}{p(p-s)}} \left(\varepsilon + |x|^{\frac{p-s}{p-1}}\right)^{\frac{p-N}{p-s}}, \ \varepsilon > 0,$$

Moreover the functions U_ε solve the equation

$$\begin{cases} -\Delta_p u = \dfrac{|u|^{p_*(s)-2}}{|x|^s} u & \text{in } \mathbb{R}^N \setminus \{0\} \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases}$$

Remark 2.1. *The accurate form of the solutions V_ε for the first limiting problem is not clear, different from the second one U_ε, which leads to some clear differences between the proofs of Theorem 1 and Theorem 2. For $0 \le \mu < \overline{\mu}$ we can prove the existence of one solution, but in the case $\mu = 0$ we use the accurate form of U_ε to prove the existence of two solutions.*

Now, we shall give some estimates for the extremal functions U_ε which we will use later. Set $\delta > 0$ small enough such that $B(0,\delta) \subset \Omega$, $\varphi \in C_0^\infty(\Omega)$ such that for

$$0 \le \varphi(x) \le 1, \varphi(x) = \begin{cases} 0 & \text{if } |x| \ge 2\delta \\ 1 & \text{if } |x| \le \delta \end{cases} ; \text{ and } |\nabla \varphi(x)| \le C.$$

Put $u_\varepsilon = \varphi(x) U_\varepsilon(x)$.

By [8] we have the following estimates.

Lemma 2.1. *Assume $2 \le p < N$, $0 \le s < p$ and $\varepsilon > 0$ small enough. By taking*

$$v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p_*(s)}}$$

so that $\|u_\varepsilon\|_{p_*(s)}^{p_*(s)} = 1$, we have the following estimates:

(1) $\|v_\varepsilon\|_0^p = S_{0,s} + O\left(\varepsilon^{\frac{N-p}{p-s}}\right)$,

(2) $\int_\Omega |\nabla v_\varepsilon|^\alpha \, dx = O\left(\varepsilon^{\frac{\alpha(N-p)}{p(p-s)}}\right)$ *for* $\alpha = 1, .., \ p-1$,

(3) $\int_\Omega \dfrac{v_\varepsilon^{p_*(s)-1}}{|x|^s} \, dx = O\left(\varepsilon^{\frac{(p-1)(N-p)}{p(p-s)}}\right)$,

(4) $\int_\Omega \dfrac{v_\varepsilon}{|x|^s} \, dx = O\left(\varepsilon^{\frac{N-p}{p(p-s)}}\right)$.

Now, we define the Euler-Lagrange functional associated to the problem $(\mathcal{P}_{\mu,s})$ by:

$$I(u) = \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*(s)}\|u\|_{p_*(s)}^{p_*(s)} - \int_\Omega f u \, dx, \quad \text{for all } u \in W_\mu^{1,p},$$

we have $I \in C^1\left(W_\mu^{1,p}, \ \mathbb{R}\right)$. A critical point u of I satisfies

$$\int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \frac{|u|^{p-2}}{|x|^p} uv - \frac{|u|^{p_*(s)-2}}{|x|^s} uv - fv\right) dx = 0$$

for all $v \in W_\mu^{1,p}$, and correspond to weak solution of problem $(\mathcal{P}_{\mu,s})$.

We consider the Nehari manifold

$$\mathcal{N}=\left\{u\in W_{\mu}^{1,p}\backslash\{0\},\ \ \langle I'(u),u\rangle=0\right\}.$$

Thus, $u\in\mathcal{N}$ if and only if

$$\langle I'(u),u\rangle=\|u\|_{\mu}^{p}-\|u\|_{p_*(s)}^{p_*(s)}-\int_{\Omega}fu\ dx=0.$$

Denote

$$J(u)=\langle I'(u),u\rangle$$

Then

$$\langle J'(u),u\rangle=(p-1)\|u\|_{\mu}^{p}-(p_*(s)-1)\|u\|_{p_*(s)}^{p_*(s)}.$$

Obviously, $\mathcal{N}$ can be divided into the following three parts:

$$\begin{array}{l}\mathcal{N}^{+}=\{u\in\mathcal{N}:\langle J'(u),u\rangle>0\},\\ \mathcal{N}^{-}=\{u\in\mathcal{N}:\langle J'(u),u\rangle<0\},\\ \mathcal{N}^{0}=\{u\in\mathcal{N}:\langle J'(u),u\rangle=0\}.\end{array}$$

Denote

$$t_u^{\max}:=\left[\|u\|_{\mu}^{p}(p-1)/(p_*(s)-1)\|u\|_{p_*(s)}^{p_*(s)}\right]^{\frac{1}{p_*(s)-p}}.$$

Lemma 2.2. *Assume that $\Lambda_f>0$, then $\mathcal{N}^0=\varnothing$ and $\mathcal{N}^{\pm}\neq\varnothing$.*

Proof. Suppose that $\mathcal{N}^0\neq\varnothing$. For $u\in\mathcal{N}^0$, we have

$$\begin{aligned}(p-1)\|u\|_{\mu}^{p}&=(p_*(s)-1)\|u\|_{p_*(s)}^{p_*(s)},\\ (p-1)\int_{\Omega}fu\ dx&=(p_*(s)-p)\|u\|_{p_*(s)}^{p_*(s)},\end{aligned}$$

and

$$(p_*(s)-1)\int_{\Omega}fu\ dx=(p_*(s)-p)\|u\|_{\mu}^{p}.$$

Using the definition of S_μ we get

$$\begin{aligned}\|u\|_{p_*(s)}^{p_*(s)}&=\frac{p-1}{p_*(s)-1}\|u\|_{\mu}^{p}\\ &\geq\left[\frac{(p-1)}{(p_*(s)-1)}S_{\mu}\right]^{p_*(s)/(p_*(s)-p)}.\end{aligned}$$

Thus $u\neq0$ and

$$\frac{\|u\|_{\mu}^{p}}{\|u\|_{p_*(s)}^{p_*(s)}}=\frac{p_*(s)-1}{p-1}.$$

Therefore,

$$\begin{aligned}0&=\frac{p_*(s)-p}{p_*(s)-1}\|u\|_{\mu}^{p}-\int_{\Omega}fu\ dx\\ &=\|u\|_{p_*(s)}\left[\frac{p_*(s)-p}{p_*(s)-1}\frac{\|u\|_{\mu}^{p}}{\|u\|_{p_*(s)}}-\frac{\int_{\Omega}fu\ dx}{\|u\|_{p_*(s)}}\right]\\ &=\|u\|_{p_*(s)}\left[\frac{p_*(s)-p}{p-1}\left[\frac{p-1}{p_*(s)-1}\frac{\|u\|_{\mu}^{p}}{\|u\|_{p_*(s)}^{p}}\right]^{\frac{p_*(s)-1}{p_*(s)-p}}-\frac{\int_{\Omega}fu\ dx}{\|u\|_{p_*(s)}}\right]\\ &\geq\left[\frac{(p-1)}{(p_*(s)-1)}S_{\mu}\right]^{\frac{1}{p_*(s)-p}}\Lambda_f>0,\end{aligned}$$

which is impossible.

Now, we prove that $\mathcal{N}^{\pm} \neq \varnothing$. Define

$$\varphi_u(t) = \frac{t^p}{p}\|u\|_\mu^p - \frac{t^{p_*(s)}}{p_*(s)}\|u\|_{p_*(s)}^{p_*(s)} - t\int_\Omega fu\,dx,$$

$$\bar{\varphi}_u(t) = t^{p-1}\|u\|_\mu^p - t^{p_*(s)-1}\|u\|_{p_*(s)}^{p_*(s)}$$

for $u \in W_\mu^{1,p}\backslash\{0\}$, then

$$\varphi_u'(t) = \bar{\varphi}_u(t) - \int_\Omega fu\,dx.$$

Easy computations show that $\bar{\varphi}_u$ is concave and achieves its maximum at $t_u^{\max}$. Moreover,

$$\bar{\varphi}_u(t_u^{\max}) = (p_*(s)-p)\left(\frac{\|u\|_\mu^p}{p_*(s)-1}\right)^{\frac{p_*(s)-1}{p_*(s)-p}}\left(\frac{p-1}{\|u\|_{p_*(s)}^{p_*(s)}}\right)^{\frac{p-1}{p_*(s)-p}}.$$

Then, there exist constants $t_{u_0}^-$ and $t_{u_0}^+$ such that

$$0 < t_{u_0}^- < t_u^{\max} < t_{u_0}^+,\ t_{u_0}^- u_0 \in \mathcal{N}^+ \text{ and } t_{u_0}^+ u_0 \in \mathcal{N}^-.$$

Thus we can get easily $\mathcal{N}^{\pm} \neq \varnothing$. □

By the previous lemma we conclude that $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$, and we can define

$$m^+ := \inf_{u\in\mathcal{N}^+} I(u) \text{ and } m^- := \inf_{u\in\mathcal{N}^-} I(u).$$

Lemma 2.3. *Suppose that $\Lambda_f > 0$, then we have:*

i) The functional I is coercive and bounded from below on $\mathcal{N}$.

ii) There exist $m_0^+ < 0$ such that

$$\inf_{u\in\mathcal{N}} I(u) \le \inf_{u\in\mathcal{N}^+} I(u) \le m_0^+ < 0.$$

Proof. i) Let $u \in \mathcal{N}$, by Hölder and Young inequalities we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*(s)}\|u\|_{p_*(s)}^{p_*(s)} - \int_\Omega fu\,dx \\ &\ge \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*(s)}\|u\|_{p_*(s)}^{p_*(s)} + \|u\|_{p_*(s)}^{p_*(s)} - \|u\|_\mu^p \\ &\ge -\left(\frac{p-1}{p}\right)\|u\|_\mu^p + \left(\frac{p_*(s)-1}{p_*(s)}\right)S_\mu^{-p_*(s)/p}\|u\|_\mu^{p_*(s)}. \end{aligned}$$

Let $X = \|u\|_\mu^p$ and

$$h(X) = -\left(\frac{p-1}{p}\right)X^p + \left(\frac{p_*(s)-1}{p_*(s)}\right)S_\mu^{-p_*(s)/p}X^{p_*(s)}.$$

Direct calculations show that h is convex and achieves its minimum at

$$X_0 = \left[\frac{p-1}{p_*(s)-1}S_\mu^{p_*(s)/p}\right]^{\frac{1}{p_*(s)-p}},$$

so

$$I(u) \ge -\frac{(p-1)(p_*(s)-p)}{p\,p_*(s)}\left[\frac{p-1}{p_*(s)-1}S_\mu^{p_*(s)/p}\right]^{\frac{p}{p_*(s)-p}}.$$

Then conclusion holds.

ii) Let $u_0 \in W_\mu^{1,p}$ be the unique solution of the following problem

$$\begin{cases} -\triangle_p u - \mu \dfrac{|u|^{p-2} u}{|x|^p} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, as $f \not\equiv 0$ we have $I_f(u_0) = \|u_0\|_\mu^p > 0$ and $\|u_0\|_\mu^p = \|f\|_-^p$. Moreover from the proof of Lemma 2.2, there exists $t_{u_0}^- > 0$ such that $t_{u_0}^- u_0 \in \mathcal{N}^+$. This implies that

$$\begin{aligned} m^+ &\leq I\left(t_{u_0}^- u_0\right) \\ &= \frac{(1-p)\left(t_{u_0}^-\right)^p}{p} \|u_0\|_\mu^p + \frac{1-p_*(s)}{p_*(s)} \left(t_{u_0}^-\right)^{p_*(s)} \|u\|_{p_*(s)}^{p_*(s)} \\ &\leq \frac{(1-p)\left(t_{u_0}^-\right)^p}{p} \|u_0\|_\mu^p \\ &\leq \frac{1-p}{p} \left(t_{u_0}^-\right)^p \|f\|_-^p. \end{aligned}$$

Thus $m^+ \leq m_0^+ < 0$ where

$$m_0^+ = \frac{1-p}{p} \left(t_{u_0}^-\right)^p \|f\|_-^p.$$

□

Lemma 2.4. *Suppose that f satisfies $\Lambda_f > 0$, then for each $u \in \mathcal{N}$, there exist $\varepsilon > 0$ and a differentiable function $\zeta : B(0,\varepsilon) \subset W_\mu^{1,p} \longrightarrow \mathbb{R}^+$ such that $\zeta(0) = 1$, $\zeta(v)(u-v) \in \mathcal{N}$ for $\|v\| < \epsilon$ and*

$$\left(\zeta'(0), v\right) = \frac{\int_\Omega \left[p\left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \dfrac{|u|^{p-2}}{|x|^p} uv \right) - p_*(s) \dfrac{|u|^{p_*(s)-2}}{|x|^s} uv - fv \right] dx}{(p-1)\|u\|_\mu^p - (p_*(s)-1)\|u\|_{p_*(s)}^{p_*(s)}}$$

Proof. Define $\psi : \mathbb{R} \times W_\mu^{1,p} \longrightarrow \mathbb{R}$ such that

$$\psi(\zeta, v) = \zeta \|u-v\|_\mu^p - \zeta^{p_*(s)-1} \|u-v\|_{p_*(s)}^{p_*(s)} - \int_\Omega f(u-v)\, dx.$$

As $u \in \mathcal{N}$ and $\mathcal{N}^0 = \varnothing$, we have

$$\psi(1,0) = 0, \ \frac{\partial \varphi}{\partial \zeta}(1,0) = (p-1)\|u\|_\mu^p - (p_*(s)-1)\|u\|_{p_*(s)}^{p_*(s)} \neq 0.$$

Then by the implicit function Theorem, we get our result. □

3. Proof of our main results

3.1. Proof of Theorem 1.1 (Existence of the first solution when $0 \leq \mu < \overline{\mu}$)

We prove that I can achieve a local minimum on $\mathcal{N}^+$ when $0 \leq \mu < \overline{\mu}$.

It follows from Lemma 2.3 that I is coercive on $\mathcal{N}^+$. Using the Ekeland variational principle [5], we can get a minimizing sequence $(u_n) \subset \mathcal{N}$ such that

$$I(u_n) \leq m^+ + \frac{1}{n} \text{ and } I(u) \geq I(u_n) - \frac{1}{n} \|u - u_n\|_\mu \text{ for all } u \in \mathcal{N}.$$

By Lemma 2.3, we know that (u_n) is bounded in $W_\mu^{1,p}$. As a consequence, there exist a subsequence (still denoted by (u_n)) and u_1 in $W_\mu^{1,p}$ such that $u_1 \not\equiv 0$ and

$$\begin{array}{l} u_n \rightharpoonup u_1 \text{ in } W_\mu^{1,p}, \\ u_n \rightharpoonup u_1 \text{ in } L_{p_*(s)}\left(\Omega,\ |x|^{-s}\right), \\ u_n \to u_1 \text{ a.e.in } \Omega. \end{array}$$

Now we claim that u_1 is a positive solution for the Problem $(\mathcal{P}_{\mu,s})$ and $u_1 \in \mathcal{N}^+$. In order, to prove the claim, we divide the arguments below into five steps.

Step 1. $I'(u_n) \to 0$ in W_μ^*.

Fix n such that $\|I'(u_n)\|_- \neq 0$. Then by Lemma 2.4 there exists $\varepsilon > 0$ and a function $\zeta_n : B(0,\varepsilon) \longrightarrow \mathbb{R}$ such that $w_n = \zeta_n(v_n)(u_n - v_n) \in \mathcal{N}^+$ with

$$v_n = \delta \frac{I'(u_n)}{\|I'(u_n)\|_-} \text{ and } 0 < \delta < \varepsilon.$$

Let $A_n = \|w_n - u_n\|_\mu$. By the Taylor expansion of I, we obtain

$$\begin{aligned} -\frac{1}{n}A_n &\leq I(w_n) - I(u_n) \\ &\leq \langle I'(u_n),\ w_n - u_n\rangle + o(A_n) \\ &= (\zeta_n(v_n) - 1)\langle I'(u_n),\ u_n\rangle - \delta\zeta_n(v_n)\left\langle I'(u_n),\ \frac{I'(u_n)}{\|I'(u_n)\|_-}\right\rangle + o(A_n). \end{aligned}$$

Then

$$\zeta_n(v_n)\|I'(u_n)\|_- \leq \frac{\zeta_n(v_n) - 1}{\delta}\langle I'(u_n), u_n\rangle + \frac{A_n}{n\delta} + \frac{o(A_n)}{\delta}. \tag{3.1}$$

We have

$$\lim_{\delta\to 0}\zeta_n(v_n) = 1, \lim_{\delta\to 0}\frac{|\zeta_n(v_n) - 1|}{\delta} = \lim_{\delta\to 0}\frac{|\zeta_n(v_n) - \zeta_n(0)|}{\delta} \leq \left\|\zeta_n'(0)\right\|_-,$$

and

$$\begin{aligned} \lim_{\delta\to 0}\frac{A_n}{n\delta} &= \lim_{\delta\to 0}\frac{1}{n\delta}\|(\zeta_n(v_n) - 1)u_n - \zeta_n(v_n)v_n\|_\mu \\ &\leq \frac{1}{n}\left(\left\|\zeta_n'(0)\right\|_- \|u_n\|_\mu + 1\right). \end{aligned}$$

Taking $\delta \to 0$ in (2) and since (u_n) is a bounded sequence we get

$$\|I'(u_n)\|_\mu \leq \frac{C_3}{n}\left(\left\|\zeta_n'(0)\right\|_- + 1\right),$$

for a suitable constant $C_3 > 0$. Now, we must show that $\left\|\zeta_n'(0)\right\|_-$ is uniformly bounded in n. From the boundedness of (u_n) we have by Lemma 2.4

$$\left\langle\zeta_n'(0),\ v\right\rangle \leq \frac{C_4\|v\|_\mu}{\left|(p-1)\|u_n\|_\mu^p - (p_*(s) - 1)\|u_n\|_{p_*(s)}^{p_*(s)}\right|},$$

for all $v \in W_\mu^{1,p}$ and some constant $C_4 > 0$. We only need to show that for any sequence $(u_n) \subset \mathcal{N}^+$

$$\left|(p-1)\|u_n\|_\mu^p - (p_*(s) - 1)\|u_n\|_{p_*(s)}^{p_*(s)}\right| > C_5,$$

for some constant $C_5 > 0$. Assume by contradiction that there exists $(u_n) \subset \mathcal{N}^+$ such that

$$\lim_{n\to\infty}\left[(p-1)\|u_n\|_\mu^p - (p_*(s) - 1)\|u_n\|_{p_*(s)}^{p_*(s)}\right] = 0.$$

As $\|u_n\|_\mu \geq C_1 > 0$, then

$$\|u_n\|_\mu^{-p} \|u_n\|_{p_*(s)}^{p_*(s)} = \frac{p-1}{p_*(s)-1} + o_n(1)$$

and

$$(p-1)\int_\Omega f u_n \, dx = (p_*(s)-p)\|u_n\|_{p_*(s)}^{p_*(s)} + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. But this is impossible since, as in the proof of Lemma 2.2 we have

$$\begin{aligned}
o_n(1) &= (p-1)\|u_n\|_\mu^p - (p_*(s)-p)\|u_n\|_{p_*(s)}^{p_*(s)} \\
&= (p_*(s)-p)\|u_n\|_{p_*(s)}^{p_*(s)} - (p-1)\int_\Omega f u_n \, dx \\
&= \|u\|_{p_*(s)} \left[\frac{p_*(s)-p}{p-1} \left[\frac{p-1}{p_*(s)-1} \frac{\|u\|_\mu^p}{\|u\|_{p_*(s)}^p} \right]^{\frac{p_*(s)-1}{p_*(s)-p}} - \frac{\int_\Omega f u \, dx}{\|u\|_{p_*(s)}} \right] \\
&\geq \left[\frac{(p-1)}{(p_*(s)-1)} S_\mu \right]^{\frac{1}{p_*(s)-p}} \Lambda_f > 0.
\end{aligned}$$

At this point we conclude that $I'(u_n) \to 0$ in W_μ^*.

Step 2. $u_n \to u_1$ in $W_\mu^{1,p}$.

Suppose otherwise, so $\|u_1\|_\mu < \underline{\lim}_{n\to\infty} \|u_n\|_\mu$, which implies that

$$\begin{aligned}
m^+ &\leq I(u_1) \\
&= \|u_1\|_\mu^p - \frac{p_*(s)-1}{p_*(s)-2}\int_\Omega f u_1 \, dx \\
&< \underline{\lim}_{n\to\to\infty} \left(\|u_n\|_\mu^p - \frac{p_*(s)-1}{p_*(s)-2}\int_\Omega f u_n \, dx \right) \\
&= m^+.
\end{aligned}$$

This is a contradiction, which led to conclude that $u_n \to u_1$ in $W_\mu^{1,p}$ and $I(u_1) = m^+$.

Step 3. $u_1 \in \mathcal{N}^+$, and u_1 is a nontrivial solution of $(\mathcal{P}_{\mu,s})$.

Suppose that $u_1 \in \mathcal{N}^-$, then by Lemma 2.2, we can find positive numbers $t_{u_1}^-$ and $t_{u_1}^+$ such that $0 < t_{u_1}^- < t_{u_1}^{\max} < t_{u_1}^+ = 1$, $t_{u_1}^- u_{u_1} \in \mathcal{N}^+$, $t_{u_1}^+ u_1 \in \mathcal{N}^-$ and

$$m^+ \leq I\left(t_{u_1}^- u_1\right) < I\left(t_{u_1}^+ u_1\right) = I(u_1) = m^+,$$

which is a contradiction. Hence $u_1 \in \mathcal{N}^+$ and

$$m^+ = \inf_{u\in\mathcal{N}^+} I(u) = \inf_{u\in\mathcal{N}} I(u).$$

By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that

$$\varphi_{u_1}'(1) = I'(u_1) = \lambda \varphi_{u_1}''(1),$$

which implies that

$$0 = \langle I'(u_1), u_1 \rangle = \lambda \langle J'(u_1), u_1 \rangle,$$

Note that $\langle J'(u_1), u_1 \rangle \neq 0$, then $\lambda = 0$ and we conclude that $I'(u_1) = 0$. Therefore, u_1 is a ground state solution of problem $(\mathcal{P}_{\mu,s})$.

3.2. Proof of Theorem 1.2 (Existence of the second solution when $\mu = 0$)

In the following, we prove that problem $(\mathcal{P}_{\mu,s})$ has a second solution u_2.

Lemma 3.1. *Let $1 < p < N$, $\mu = 0$, $0 \leq s < p$ and $f \not\equiv 0$ satisfies $\Lambda_f > 0$. Then $I(u)$ verifies the Palais-Smale condition at level c for all $c < m^+ + \dfrac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}}$.*

Proof. Assume (u_n) is a sequence in $W_0^{1,p}$ satisfying as $n \to \infty$

$$I(u_n) \to c < \frac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}} \text{ and } I'(u_n) \to 0 \text{ in } W_0^*. \tag{3.2}$$

By Lemma 2.3, we know that (u_n) is bounded in $W_0^{1,p}$. Then, there exist a subsequence (still denoted by (u_n)) and u_2 in $W_0^{1,p}$ such that $u_2 \not\equiv 0$ and

$$\begin{aligned} u_n &\rightharpoonup u_2 \text{ in } W_0^{1,p}, \\ u_n &\rightharpoonup u_2 \text{ in } L_{p_*(s)}\left(\Omega, |x|^{-s}\right), \\ u_n &\to u_2 \text{ a.e.in } \Omega. \end{aligned}$$

Denote $v_n = u_n - u_2$, then

$$\begin{aligned} v_n &\rightharpoonup 0 \text{ in } W_0^{1,p}, \\ v_n &\rightharpoonup 0 \text{ in } L_{p_*(s)}\left(\Omega, |x|^{-s}\right), \\ v_n &\to 0 \text{ a.e.in } \Omega. \end{aligned}$$

By the Brezis - Lieb Lemma [3] we have

$$\|u_n\|_0^p = \|v_n\|_\mu^p + \|u_2\|_\mu^p + \circ_n(1),$$

and

$$\|u_n\|_{p_*(s)}^{p_*(s)} = \|v_n\|_{p_*(s)}^{p_*(s)} + \|u_2\|_{p_*(s)}^{p_*(s)} + \circ_n(1).$$

Then, from (3) we deduce that

$$c + \circ_n(1) = I(u_2) + \frac{1}{p}\|v_n\|_0^p - \frac{1}{p_*(s)}\|v_n\|_{p_*(s)}^{p_*(s)}$$

and

$$\|v_n\|_0^p - \|v_n\|_{p_*(s)}^{p_*(s)} = \circ_n(1).$$

Using the fact that $v_n \rightharpoonup 0$ in $W_0^{1,p}$, we can assume that

$$\|v_n\|_0^p \to l \text{ and } \|v_n\|_{p_*(s)}^{p_*(s)} \to l \geq 0.$$

So, by the Sobolev-Hardy inequality, we get $l \geq S_{0,s} l^{p/p_*(s)}$.

Now, assume that $l \neq 0$, then

$$l \geq (S_{0,s})^{p_*(s)/(p_*(s)-p)}$$

and we obtain

$$c = I(u_2) + \left(\frac{1}{p} - \frac{1}{p_*(s)}\right) l \geq I(u_2) + \frac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}}.$$

As $I(u_2) \geq m^+$, we get a contradiction. So again $u_n \to u$ in $W_0^{1,p}$ strongly. □

In order, to prove Theorem 1.2, we need the following key lemma.

Lemma 3.2. *Suppose* $2 \leq p < N$, $\mu = 0$, $f(x) \geq a_0 > 0$ *in a small neighborhood of* 0 *and satisfies* $\Lambda_f > 0$. *Then*

$$m^- < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}}.$$

Proof. Set

$$\mathcal{M}_1 = \{0\} \cup \left\{ u \in W_0^{1,p} : \|u\|_0 < t^+_{u\|u\|_0^{-1}} \right\} \text{ and } \mathcal{M}_2 = \left\{ u \in W_0^{1,p} : \|u\|_0 > t^+_{u\|u\|_0^{-1}} \right\}.$$

We have $W_0^{1,p} \backslash \mathcal{N}^- = \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{N}^+ \subset \mathcal{M}_1$, $u_1 \in \mathcal{M}_1$ and $u_1 + Tv_\varepsilon \in \mathcal{M}_2$ for some real $T > 0$. Let

$$\Gamma = \left\{ h : [0,1] \to W_0^{1,p} \text{ continuous, } h(0) = u_1,\ h(1) = u_1 + Tv_\varepsilon \right\},$$

and

$$\tilde{h}(t) = u_1 + tTv_\varepsilon \quad \text{with } t \in [0,1].$$

It is obvious that $\tilde{h}$ belongs to Γ and the range of any $h \in \Gamma$ intersects $\mathcal{N}^-$. Then

$$m^- \leq \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)).$$

Now, we show that

$$\sup_{t \geq 0} I(u_1 + tv_\varepsilon) < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}}.$$

To this purpose we define $g(t) := I(u_1 + tv_\varepsilon)$, then

$$g(0) = I(u_1) < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}},$$

and by the continuity of g there exists $t_0 > 0$ small enough such that

$$g(t) < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}}$$

for all $t \in (0,\ t_0)$. On the other hand, it is easy to see that $g(t) \to -\infty$ as $t \to +\infty$, that is, there exists $t_1 > 0$ large enough such that

$$g(t) < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}}$$

for all $t \geq t_1$. So we only need to show that

$$\sup_{t_0 \leq t \leq t_1} g(t) < m^+ + \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}}.$$

Let ε be sufficiently small satisfying $f(x) \geq a_0 > 0$ in $B(0,\varepsilon)$. Then, we get from Lemma 2.1

$$\begin{aligned}
\sup_{t_0 \leq t \leq t_1} I(tv_\varepsilon) &\leq \sup_{t \geq 0} \left(\frac{1}{p} \|tv_\varepsilon\|_0^p - \frac{1}{p_*(s)} \|tv_\varepsilon\|_{p_*(s)}^{p_*(s)} \right) - t_0 \int_\Omega f v_\varepsilon dx \\
&\leq \sup_{t \geq 0} \left(\frac{1}{p} \|tv_\varepsilon\|_0^p - \frac{1}{p_*(s)} \|tv_\varepsilon\|_{p_*(s)}^{p_*(s)} \right) - t_0 a_0 \int_\Omega v_\varepsilon dx \\
&\leq \frac{p-s}{p(N-s)} (S_{0,s})^{\frac{N-s}{p-s}} + O\left(\varepsilon^{\frac{N-p}{p-s}} \right) - O(\varepsilon^{\frac{N-p}{p^2}}).
\end{aligned}$$

For the second one, we can assume that the first solution u_1 is smooth and $\nabla u_1 \in L_\infty(\Omega)$. Thus we have

$$\begin{aligned}\sup_{t_0\le t\le t_1} g(t) &= \sup_{t_0\le t\le t_1} I(u_1+tv_\varepsilon)\\ &\le I(u_1)+\sup_{t\ge 0} I(tv_\varepsilon)+C_1\int_\Omega\left(|\nabla u_1|^{p-1}|\nabla v_\varepsilon|+|\nabla v_\varepsilon|^{p-1}|\nabla u_1|\right)dx+\\ &\quad \int_\Omega\left(|u_1|^{p_*(s)-1}v_\varepsilon+|v_\varepsilon|^{p_*(s)-1}u_1\right)dx\\ &\le m^+ +\frac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}}+O\left(\varepsilon^{\frac{N-p}{p-s}}\right)-O(\varepsilon^{\frac{N-p}{p^2}})+O\left(\varepsilon^{\frac{N-p}{p(p-s)}}\right)+\\ &\quad O\left(\varepsilon^{\frac{(N-p)(p-1)}{p(p-s)}}\right)\end{aligned}$$

From

$$\frac{N-p}{p-s}>\frac{N-p}{p(p-s)}>\frac{N-p}{p^2}\text{ for all } s>0,$$

we have

$$O\left(\varepsilon^{\frac{N-p}{p-s}}\right)-O(\varepsilon^{\frac{N-p}{p^2}})+O\left(\varepsilon^{\frac{N-p}{p(p-s)}}\right)+O\left(\varepsilon^{\frac{(N-p)(p-1)}{p(p-s)}}\right)=O\left(\varepsilon^{\frac{(N-p)(p-1)}{p(p-s)}}\right)-O(\varepsilon^{\frac{N-p}{p^2}}).$$

Since

$$\frac{(N-p)(p-1)}{p(p-s)}-\frac{N-p}{p^2}=\frac{N-p}{p^2(p-s)}[p(p-2)+s]>0,$$

then

$$\sup_{t_0\le t\le t_1} I(u_1+tv_\varepsilon)<m^+ +\frac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}}$$

for ε small enough.

The proof is now complete. □

Now, we prove that I can achieve a local minimum on $\mathcal{N}^-$.

By using Lemma 2.3, there exists a minimizing sequence $(u_n)\subset\mathcal{N}^-$ such that

$$I(u_n)\to m^- \text{ and } I'(u_n)\to 0 \text{ in } W_0^{-1}.$$

From Lemma 3.2 we have $m^- < m^+ +\frac{p-s}{p(N-s)}(S_{0,s})^{\frac{N-s}{p-s}}$, therefore, by Lemma 3.1 we get $u_n\to u_2$ in $W_0^{1,p}$. This means that $u_2\in\mathcal{N}^-$ and $I(u_2)=m^-$.

References

1. Assunção, R., Carrião, P., Miyagaki, O. Subcritical perturbations of a singular quasilinear elliptic equation involving the critical Hardy–Sobolev exponent. Nonlinear Anal. 66, 1351–1364 (2007).
2. Bouchekif, M., Matallah, A. Multiple positive solutions for elliptic equations involving a concave term and critical Sobolev-Hardy exponent. Appl. Math. Lett. 22, 268-275 (2009).
3. Brezis, H., Lieb, E. A Relation Between Point Convergence of Functions and Convergence of Functionals. Proc. Amer. Math. Soc. 88, 486-490 (1983).
4. Caffarelli, L., Kohn, R., Nirenberg, L. First order interpolation inequality with weights. Compos. Math. 53, 259–275 (1984).
5. Ekeland, I. On the variational principle. J. Math. Anal. Appl. 47, 324–354 (1974).
6. Filippucci, R., Pucci, P., Robert, F. On a p-Laplace equation with multiple critical nonlinearities. J. Math. Pures Appl. 91, 156-177 (2009).

7. Garcia Azorero, J. P., Peral Alonso, I. Hardy Inequalities and Some Critical Elliptic and Parabolic Problems. J. of Di erential Equations. 144, 441-476 (1998).

8. Ghoussoub, N., Yuan, C. Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Amer. Math. Soc. 352, 5703-5743 (2000).

9. Kang, D.S. On the quasilinear elliptic problems with critical Sobolev-Hardy exponents and Hardy terms. Nonlinear Anal. 68, 1973-1985 (2008).

10. Sang, Y. Guo, S. Solutions for the quasi-linear elliptic problems involving the critical Sobolev exponent. Journal of Inequalities and Applications. (2017).

11. Secchi, S., Smets, D., Willem, M. Remarks on a Hardy-Sobolev inequality. C.R. Acad. Sci. Paris. 336, 811-815 (2003).

12. Liang, S.H., Zhang, J.H. Multiplicity of solutions for a class of quasilinear elliptic equation involving the critical Sobolev and Hardy exponents. Nonlinear Differ. Equ. Appl. 17, 55-67 (2010).

13. Tarantello, G. On nonhomogeneous elliptic equations involving critical Sobolev exponent. Ann. Inst. Henri Poincaré 9, 281-304 (1992).

Atika Matallah,
Laboratory of Analysis and Control of Partial Differential Equations of Sidi Bel Abbes,
Higher School of Management of Tlemcen,
Algeria.
E-mail address: atika_matallah@yahool.fr

and

Sara Litimein,
University of Sidi Bel Abbès,
Algeria.
E-mail address: sara_litimein@yahoo.fr

and

Sofiane Messirdi,
Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO).
Department of Mathematics and informatics Faculty of Exact Sciences.
University of Mostaganem.
Algeria
E-mail address: messirdi.sofiane@hotmail.fr