Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–16.
ISSN-0037-8712 in press
doi:10.5269/bspm.47787

Applications of New Iterative Method to Fractional Non Linear Coupled ITO System

Rashid Nawaz, Samreen Farid and Samia Bushnaq

abstract: In this article, New Iterative Method (NIM) is tested upon fractional non linear coupled ITO system. The results obtained by the proposed method are compared with that of Homotopy Perturbation Method (HPM) [1]. It is shown that the proposed method is accurate for strongly nonlinear fractional coupled system of PDEs.

Key Words: New Iterative Method (NIM), Time Fractional Coupled ITO system, Homotopy Perturbation Method (HPM), Caputo derivative.

Contents

1. Introduction

Nonlinear fractional partial differential equations arise in modeling of complicated physical phenomena. Unfortunately in such situation we have no analytical method that solve these nonlinear fractional models exactly and therefore researchers knock the door of approximate methods for finding solution of these problems. Varieties of methods in literature have been used to find solutions of these types of problems. These methods include numerical and semi analytical techniques. Some of well-known semi analytical techniques such as Adomian decomposition method (ADM) [2,3], Variational Iteration Method (VIM) [4,5], Homotopy Perturbation Method (HPM) [6,7,8,9], Homotopy Analysis Method (HAM) [10,11] and Optimal Homotopy Asymptotic Method (OHAM) [12,13,14], etc. [15,16,17,18]
New Iterative Method (NIM) proposed by Daftardar-Gejji and Jafari is one of the most realistic techniques for solving Differential equations, Integro-differential equations and Differential Difference equations of both classical and fractional order. By means of more reliable polynomials called Jafari polynomials (NIM) handles nonlinear partial differential equations in accurate way.

Our aim in this paper is to find reliable solutions of time fractional nonlinear coupled ITO systems by using (NIM). The concept of nonlinear generalized coupled ITO system was first introduced in the literature by Masaaki ito in 1980 [19]. This system has various applications in science and engineering i.e. coupled ITO equations are used for continuous quantum state measurement and estimation [20]. Time fractional Coupled ITO systems under consideration in this paper are as follows:

$$\begin{cases} D_t^{\alpha} u = v_x, \\ D_t^{\beta} v = -2(v_{xxx} + 3uv_x + 3vu_x) - 12ww_x, \\ D_t^{\gamma} w = w_{xxx} + 3uw_x \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification:* 35A25, 35C05.
Submitted May 04, 2019. Published November 20, 2019

with initial conditions:

$$
\begin{aligned}
u(x,0) &= \frac{ax}{3c},\\
v(x,0) &= \frac{-a^2x^2}{2c^2},\\
w(x,0) &= 0.
\end{aligned}
$$

and

$$
\begin{cases}
D_t^{\alpha} u = v_x,\\
D_t^{\beta} v = -2v_{xxx} - 6(uv)_x - 6(wp)_x,\\
D_t^{\gamma} w = w_{xxx} + 3uw_x\\
D_t^{\eta} p = p_{xxx} + 3up_x
\end{cases}
\tag{1.2}
$$

withe initial conditions:

$$
\begin{aligned}
u(x,0) &= r_1 - 2\mu^2 \tanh^2(\mu x),\\
v(x,0) &= r_2 + b_2 \tanh^2(\mu x),\\
w(x,0) &= r_3 + f_1 \tanh(\mu x),\\
p(x,0) &= t_0 + t_1 \tanh(\mu x).
\end{aligned}
$$

where $0 < \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,\ 0 < \eta \leq 1,\ r_1 = \dfrac{-b_2 + 4\mu^4}{6\mu^2},\ r_2 \dfrac{-b_2^2 + 4f_1t_1\mu^2 - 8b_2\mu^4}{8\mu^4},\ r_3 = \dfrac{-f_1t_0}{t_1},\ \mu,\ a,\ c,\ b_2,\ t_0,\ t_1,\ f_1$ are constants.

Different techniques in literature have been used in literature for solution of coupled ITO systems. Al-Sawoor et al. employed Reduced Differential Transform Method (RDTM) for finding solution to generalized ITO system [21]. Jaradat et al. employed A Residual Power Series Method (RPSM) for finding solution to time fractional ITO system [22]. Aslan et al. employed Homotopy Analysis Method (HAM) for finding approximate solution of time fractional ITO coupled system [23].

2. Preliminaries

In this section we state some definitions and results from the literature which are relevant to our work.

Definition 2.1. *A real function $f(x),\ x > 0$ is said to be in space $C_\mu,\ \mu \in R$ if there a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0,\infty)$, and it is said to be in the space C_μ^m if only if $f^{(m)} \in C_\mu,\ m \in N$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu,\ \mu \geq -1$ is defined as*

$$
\begin{aligned}
I_a^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-\mu)^{\alpha-1} f(\mu)\, d\mu,\ x > 0\\
I_a^0 f(x) &= f(x)
\end{aligned}
$$

When we formulate the model of real world problems with fractional calculus, the Riemann-Liouville have certain disadvantages. Caputo proposed a modified fractional differential operator D_a^{α} in his work on the theory of viscoelasticity

Definition 2.3. *The fractional derivative of $f(x)$ in Caputo sense is defined as*

$$
D_a^{\alpha} f(x) = I_a^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-a)} \int_a^x (x-\eta)^{m-\alpha-1} f^{(m)}(\eta)\, d\eta,
$$

$m - 1 < \alpha \leq m,\ m \in N,\ x > 0,\ f \in C_{-1}^m$

Definition 2.4. *If* $m-1<\alpha\le m,\ m\in N$ *and* $f\in C_\mu^m,\ \mu\ge -1,$ *then* $D_a^\alpha I_a^\alpha f(x)=f(x)$ *and* $I_a^\alpha D_a^\alpha f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}\frac{(x-a)^k}{k!},\ x>0$

One can find the properties of the operator I_a^α in [24,25]. We mention the following:
For $f\in C_\mu^m, \alpha,\beta>0,\ \mu\ge -1,\ \gamma\ge -1,$

(a) $I_a^\alpha f(x)$ exists for almost every $x\in[a,b]$.

(b) $I_a^\alpha I_a^\beta f(x)=I_a^{\alpha+\beta} f(x)$.

(c) $I_a^\alpha I_a^\beta f(x)=I_a^\beta I_a^\alpha f(x)$.

(d) $I_a^\alpha (x-a)^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

3. Basic theory of (NIM) [26,27]

Consider the following nonlinear equation:

$$v(x)=f(x)+\xi\Big(u(x)\Big)+\psi(v(x)\Big) \tag{3.1}$$

Where f is the given function, $\xi,\ \psi$ are given linear and nonlinear functions of $u(x),\ v(x)$ respectively. Assume eq. (3.1) has solution of the form $v(x)=\sum_{i=0}^{\infty} v_i(x)$.

Since ξ is linear, we can write $\xi\Big(\sum_{i=0}^{\infty} v_i(x)\Big)=\sum_{i=0}^{\infty}\xi\Big(v_i(x)\Big)$.
The nonlinear function ψ can be decomposed as

$$\begin{aligned}\psi\left(\sum_{i=0}^{\infty} v_i(x)\right) &= \psi\Big(v_0(x)\Big)+\left(\psi\Big(v_0(x)+v_1(x)\Big)-\psi(v_0(x))\right)\\ &\quad+\left(\psi\Big(v_0(x)+v_1(x)+v_2(x)\Big)-\psi\Big(v_0(x)+v_1(x)\Big)\right)\\ &\quad+\left(\psi\Big(v_0(x)+v_1(x)+v_2(x)+v_3(x)\Big)-\psi\Big(v_0(x)+v_1(x)+v_2(x)\Big)\right)+...\\ &= \psi\Big(v_0(x)\Big)+\sum_{i=1}^{\infty}\left(\psi\Big(\sum_{j=0}^{i} v_j(x)\Big)-\psi\Big(\sum_{j=0}^{i-1} v_j(x)\Big)\right)\end{aligned}$$

The general solution can be written as

$$v(x)=\sum_{i=0}^{\infty}=f(x)+\xi\Big(\sum_{i=0}^{\infty} v_i(x)\Big)+\psi\Big(v_0(x)\Big)+\sum_{i=1}^{\infty}\left(\psi\Big(\sum_{j=0}^{i} v_j(x)\Big)-\psi\Big(\sum_{j=0}^{i-1} v_j(x)\Big)\right)$$

We write

$$G_0=\psi\Big(v_0(x)\Big),$$

$$G_m=\psi\Big(\sum_{i=0}^{m} v_i(x)\Big)-\psi\Big(\sum_{i=0}^{m} v_i(x)\Big).$$

The recurrence relation is defined as

$$\begin{cases} v_0(x) = f(x), \\ v_1(x) = \xi\Big(v_0(x)\Big) + G_0, \\ v_{m+1}(x) = \xi\Big(v_m(x)\Big) + G_m, \;\; m = 1, 2, ... \end{cases} \tag{3.2}$$

Hence k-terms approximate solution of eq. (3.1) is of the form $v(x) = v_0(x) + v_1(x) + ... + v_{k-1}(x)$

Convergence Criteria for (NIM):

Theorem 3.1. *If ξ is C^∞ in a neighborhood of v_0 and $||\xi^{(m)}(v_0)|| = \sup\{\xi^{(m)}(v_0)(k_1, k_2, ..., k_n) : ||k_j|| \leq 1, \; 1 \leq j \leq m\} \leq L$, for any m and for some real $L > 0$ and $||v_j|| \leq M < \dfrac{1}{e}, \; j = 1, 2, ...,$ then the series $\sum_{m=0}^{\infty} G_m$ is absolutely convergent and $||G_m|| \leq M^m e^{m-1}(e-1), \; m = 1, 2, ...$*

Proof: See [22].

Sufficient condition for convergence is as follows:

Theorem 3.2. *If ξ is C^∞ and $||\xi^{(m)}(v_0)|| \leq M \leq e^{-1}, \forall m$, then the series $\sum_{m=0}^{\infty} G_m$ is absolutely convergent.*

Proof: See [22].

For more detail of convergence we refer [23] where illustrative example is solved for convergence analysis.

4. Applications of (NIM)

ILLUSTRATION 1:

Time Fractional Coupled ITO system [1].

$$\begin{cases} D_t^\alpha u = v_x, \\ D_t^\beta v = -2(v_{xxx} + 3uv_x + 3vu_x) - 12ww_x, \\ D_t^\gamma w = w_{xxx} + 3uw_x \end{cases} \tag{4.1}$$

where $0 < \alpha \leq 1, \; 0 < \beta \leq 1, \; 0 < \gamma \leq 1$ with conditions:

$$\begin{cases} u(x,0) = \dfrac{ax}{3c}, \\ v(x,0) = \dfrac{-a^2x^2}{2c^2}, \\ w(x,0) = 0 \end{cases} \tag{4.2}$$

where a, c are any constants.

For $\alpha = \beta = \gamma = 1$, the exact solution og the above system is:

$$u(x,t) = \frac{ax}{3(3at+c)},$$
$$v(x,t) = \frac{-a^2x^2}{2(3at+c)^2},$$
$$w(x,t) = 0$$

Applying $I^\alpha, \; I^\beta$ and I^γ to both sides of system (4.1) and using definition 2.4. we have

$$u(x,t)\frac{ax}{3c}+I^{\alpha}\big((v(x,t))_x\big),$$
$$v(x,t)=\frac{-a^2x^2}{2c^2}+I^{\beta}\Big(-2\big(v(x,t)\big)_{xxx}+3u(x,t)\big(v(x,t)\big)_x+3v(x,t)\big(u(x,t)\big)_x-12w(x,t)\big(w(x,t)\big)_x\Big),$$
$$w(x,t)=I^{\alpha}\Big(\big(w(x,t)\big)_{xxx}+3u(x,t)\big(w(x,t)\big)\Big).$$

By using (NIM) and eq. (3.2) we have:

$$\begin{aligned}
u_0(x,t) &= \frac{ax}{3c},\\
v_0(x,t) &= \frac{-a^2x^2}{2c^2},\\
w_0(x,t) &= 0,\\
u_1(x,t) &= I^{\alpha}\left(\Big(v_0(x,t)\Big)_x\right)=\frac{a^2xt^{\alpha}}{c^2\Gamma(\alpha+1)},\\
v_1(x,t) &= I^{\beta}\left(-2\Big(v_0(x,t)\Big)\right)\\
&\quad +I^{\beta}\left(-6u_0(x,t)\Big(v_0(x,t)\Big)_x-6v_0(x,t)\Big(u_0(x,t)\Big)_x-12w_0(x,t)\Big(w_0(x,t)\Big)_x\right)\\
&= \frac{3a^3t^{\beta}x^2}{c^3\Gamma(1+\beta)},\\
w_1(x,t) &= I^{\gamma}\left(\Big(w_0(x,t)\Big)_{xxx}\right)+I^{\gamma}\left(3u_0(x,t)\Big(w_0(x,t)\Big)_x\right)=0,\\
u_2(x,t) &= I^{\alpha}\left(\Big(v_1(x,t)\Big)_x\right)=\frac{6a^3xt^{\alpha+\beta}}{c^3\Gamma(\alpha+\beta+1)},
\end{aligned}$$

$$\begin{aligned}
v_2(x,t) &= I^{\beta}\left(-2\Big(v_1(x,t)\Big)_{xxx}\right)\\
&\quad +I^{\beta}\left(-6\Big(u_0(x,t)+u_1(x,t)\Big)\Big(v_0(x,t)+v_1(x,t)\Big)_x\right.\\
&\quad -6\Big(v_0(x,t)+v_1(x,t)\Big)\Big(u_0(x,t)+u_1(x,t)\Big)_x\\
&\quad \left.-12\Big(w_0(x,t)+w_1(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)\Big)_x\right)\\
&\quad -I^{\beta}\left(-6u_0(x,t)\Big(v_0(x,t)\Big)_x-6v_0(x,t)\Big(u_0(x,t)\Big)_x-12w_0(x,t)\Big(w_0(x,t)\Big)_x\right)\\
&= \frac{9a^4t^{\beta}x^2}{c^5}\left(-\frac{ct^{\alpha}}{\Gamma(\alpha+\beta+1)}-\frac{2ct^{\beta}}{\Gamma(2\beta+1)}+\frac{6at^{\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+2\beta+1)}\right),
\end{aligned}$$

$$\begin{aligned}
w_2(x,t) &= I^{\gamma}\left(\Big(w_1(x,t)\Big)\right)_{xxx}\\
&\quad +I^{\gamma}\left(3\Big(u_0(x,t)+u_1(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)\Big)_x\right)-I^{\gamma}\left(3u_0(x,t)\Big(w_0(x,t)\Big)_x\right)=0,
\end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= I^{\alpha}\Big(\Big(v_2(x,t)\Big)_x\Big) \\
&= \frac{1}{c^5\Gamma(\alpha)}\Big(18a^4t^{\alpha+\beta}x\Big(\frac{32^{1-2\alpha-2\beta}a\sqrt{\pi}t^{\alpha+\beta}}{\alpha\Gamma(1+\beta)\Gamma(0.5+\alpha+\beta)}+c\Gamma(\alpha)\Big(-\frac{t^{\alpha}}{\Gamma(2\alpha+\beta+1)}-\frac{2t^{\beta}}{\Gamma(\alpha+2\beta+1)}\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
v_3(x,t) &= I^{\beta}\Big(-2\Big(v_2(x,t)\Big)_{xxx}\Big) \\
&+I^{\beta}\Big(-6\Big(u_0(x,t)+u_1(x,t)+u_2(x,t)\Big)\Big(v_0(x,t)+v_1(x,t)+v_2(x,t)\Big)_x \\
&-6\Big(v_0(x,t)+v_1(x,t)+v_2(x,t)\Big)\Big(u_0(x,t)+u_1(x,t)+u_2(x,t)\Big)_x \\
&-12\Big(w_0(x,t)+w_1(x,t)+w_2(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)+w_2(x,t)\Big)_x\Big) \\
&-I^{\beta}\Big(-6\Big(u_0(x,t)+u_1(x,t)\Big)\Big(v_0(x,t)+v_1(x,t)\Big)_x \\
&-6\Big(v_0(x,t)+v_1(x,t)\Big)\Big(u_0(x,t)+u_1(x,t)\Big)_x \\
&-12\Big(w_0(x,t)+w_1(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)\Big)_x\Big) \\
&= \frac{54a^5t^{2\beta}x^2}{c^8}\Big(-\frac{6ac^2t^{\alpha+\beta}\Big(\Gamma(\beta+1)\Gamma(\alpha+\beta+1)+\Gamma(\alpha+1)\Gamma(2\beta+1)\Big)\Gamma(\alpha+2\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(2\beta+1)\Gamma(\alpha+3\beta+1)} \\
&+c\Big(\frac{-\frac{3act^2\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}-\frac{6act^{\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha+3\beta+1)}}{\Gamma(\alpha+1)} \\
&+2t^{\beta}\Big(\frac{c^2}{\Gamma(3\beta+1)}+\frac{9a^2t^{\alpha}}{\Gamma(\alpha+\beta+1)}\Big(\frac{4^{\alpha+\beta}t^{\alpha}\Gamma(\alpha+\beta+0.5)}{\sqrt{\pi}\Gamma(2\alpha+3\beta+1)}+\frac{2t^{\beta}\Gamma(\alpha+3\beta+1)}{\Gamma(2\beta+1)\Gamma(\alpha+4+1)}\Big)\Big)\Big) \\
&+\frac{2t^{\alpha}}{\Gamma(\alpha+2\beta+1)}\times \\
&\Big(c^3-\frac{9a^2t^{\alpha+\beta}\Big(6at^{\beta}\Gamma(\alpha+1)(\Gamma(2\alpha+3\beta+1))^2-c\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)\Gamma(2\alpha+4\beta+1)\Big)}{(\alpha+0)^2\Gamma(\beta+1)\Gamma(2\alpha+3\beta+1)\Gamma(2\alpha+4\beta+1)}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
w_3(x,t) &= I^{\gamma}\Big(\Big(w_2(x,t)\Big)_{xxx}\Big) \\
&+I^{\gamma}\Big(3\Big(u_0(x,t)+u_1(x,t)+u_2(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)+w_2(x,t)\Big)_x\Big) \\
&-I^{\gamma}\Big(3\Big(u_0(x,t)+u_1(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)\Big)_x\Big)=0.
\end{aligned}
$$

Therefore four terms approximate solution of $u(x,t)$, $v(x,t)$ and $w(x,t)$ can be written in the form:

$$
\begin{gathered}
u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t) \\
u(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t) \\
w(x,t)=w_0(x,t)+w_1(x,t)+w_2(x,t)+w_3(x,t)
\end{gathered}
$$

ILLUSTRATION 2:
Time fractional Coupled ITO system [1].

$$\begin{cases} D_t^{\alpha} u = v_x, \\ D_t^{\beta} v = -2v_{xxx} - 6(uv)_x - 6(wp)_x, \\ D_t^{\gamma} w = w_{xxx} + 3uw_x \\ D_t^{\eta} p = p_{xxx} + 3up_x \end{cases} \tag{4.3}$$

where $0 < \alpha \leq 1,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,\ 0 < \eta \leq 1.$
with

$$\begin{cases} u(x,0) = r_1 - 2\mu^2 \tanh^2(\mu x), \\ v(x,0) = r_2 + b_2 \tanh^2(\mu x), \\ w(x,0) = r_3 + f_1 \tanh(\mu x) \\ p(x,0) = t_0 + t_1 \tanh(\mu x) \end{cases} \tag{4.4}$$

where $r_1 = \dfrac{-b_2 + 4\mu^4}{6\mu^2},\ r_2 = \dfrac{-b_2^2 + 4f_1t_1\mu^2 - 8b_2\mu^4}{8\mu^4},\ r_3 = \dfrac{-f_1t_0}{t_1}$
The exact solutions for $\alpha = \beta = \gamma = 1$ are given as:

$$\begin{aligned} u(x,t) &= r_1 - 2\mu^2 \tanh^2\Big(\mu(x - \frac{b_2 t}{2\mu^2})\Big), \\ v(x,t) &= r_2 + b_2 \tanh^2\Big(\mu(x - \frac{b_2 t}{2\mu^2})\Big), \\ w(x,t) &= r_3 + f_1 \tanh^2\Big(\mu(x - \frac{b_2 t}{2\mu^2})\Big), \\ p(x,t) &= t_0 + t_1 \tanh^2\Big(\mu(x - \frac{b_2 t}{2\mu^2})\Big) \end{aligned}$$

Applying $I^{\alpha},\ I^{\beta},\ I^{\gamma}$ and I^{η} to both sides of system (4.3) and using definition 2.4. we have

$$\begin{aligned} u(x,t) &= r_1 - 2\mu^2 \tanh^2(\mu x) + I^{\alpha}(v_x), \\ v(x,t) &= r_2 + b_2 \tanh^2(\mu x) + I^{\beta}\Big(-2v_{xxx} - 6(uv)_x - 6(wp)_x\Big), \\ w(x,t) &= r_3 + f_1 \tanh(\mu x) + I^{\gamma}(w_{xxx} + 3uw_x), \\ p(x,t) &= t_0 + t_1 \tanh(\mu x) + I^{\eta}(p_{xxx} + 3up_x) \end{aligned}$$

By using (NIM) and eq. (3.2) we have:

$$\begin{aligned} u_0(x,t) =& r_1 - 2\mu^2 \tanh^2(\mu x), \\ v_0(x,t) =& r_2 + b_2 \tanh^2(\mu x), \\ w_0(x,t) =& r_3 + f_1 \tanh(\mu x), \\ p_0(x,t) =& t_0 + t_1 \tanh(\mu x), \\ u_1(x,t) =& I^{\alpha}\Big((v_0(x,t))_x\Big) \\ =& \frac{2t^{\alpha}\mu sech^2(\mu x) b_2 \tanh(\mu x)}{\Gamma(\alpha+1)}, \\ v_1(x,t) =& I^{\beta}\bigg(-2\Big(v_0(x,t)\Big)_{xxx}\bigg) + I^{\beta}\bigg(-6\Big(u_0(x,t)v_0(x,t)\Big)_x - 6\Big(w_0(x,t)p_0(x,t)\Big)_x\bigg) \\ =& \frac{2t^{\beta} sech^2(\mu x)}{\Gamma(\beta+1)}\bigg(-3(f_1t_0 + r_3t_1) + 2\Big(6r_2\mu^2 + (-3r_1 + 8\mu^2)b_2\Big)\tanh(\mu x)\bigg), \end{aligned}$$

$$w_1(x,t) = I^{\gamma}\left(\left(w_0(x,t)\right)_{xxx}\right) + I^{\gamma}\left(3u_0(x,t)\left(w_0(x,t)\right)_x\right) = \frac{f_1 t^{\gamma}\mu(3r_1 - 2\mu^2)sech^2(\mu x)}{\Gamma(\gamma+1)},$$

$$p_1(x,t) = I^{\eta}\left(\left(p_0(x,t)\right)_{xxx}\right) + I^{\eta}\left(3u_0(x,t)\left(p_0(x,t)\right)_x\right) = \frac{t_1 t^{\eta}\mu(3r_1 - 2\mu^2)sech^2(\mu x)}{\Gamma(\eta+1)},$$

$$\begin{aligned} u_2(x,t) =& I^{\alpha}\left(\left(v_1(x,t)\right)_x\right) \\ =& \frac{2t^{\alpha+\beta}\mu^2 sech^4(\mu x)}{\Gamma(\alpha+\beta+1)}\Big(6(f_1t_1 - 2r_2\mu^2)(-2+\cosh(2\mu x)) + 3(f_1t_0 + r_3t_1)\sinh(2\mu x) \\ &+ 2(3r_1 - 8\mu^2)(-2+\cosh(\mu x))b_2\Big), \end{aligned}$$

$$\begin{aligned} v_2(x,t) =& I^{\beta}\left(-2\left(v_1(x,t)\right)_{xxx}\right) \\ &+ I^{\beta}\Big(-6\left(u_0(x,t)+u_1(x,t)\right)\left(v_0(x,t)+v_1(x,t)\right)_x \\ &- 6\left(w_0(x,t)+w_1(x,t)\right)\left(p_0(x,t)+p_1(x,t)\right)_x\Big) \\ &- I^{\beta}\left(-6\left(u_0(x,t)v_0(x,t)\right)_x - 6\left(w_0(x,t)p_0(x,t)\right)_x\right) \\ =& 2t^{\beta}\mu^2 sech^2(\mu x)\Bigg(\frac{3t_1t^{\eta}(3r_1-2\mu^2)sech^2(\mu x)\left(f_1(-2+\cosh(2\mu x)) + r_3\sinh(2\mu x)\right)}{\Gamma(\beta+\eta+1)} \\ &+ \frac{3f_1t^{\gamma}(3r_1-2\mu^2)sech^2(\mu x)\left(t_1(-2+\cosh(2\mu x)) + t_0\sinh(2\mu x)\right)}{\Gamma(\beta+\gamma+1)} \\ &+ \frac{6t^{\alpha}b_2\left(r_2(2-3sech^2(\mu x))\right)\left(t_1(-2+\cosh(2\mu x)) + t_0\sinh(2\mu x)\right)}{\Gamma(\beta+\gamma+1)} \\ &+ \frac{6t^{\alpha}b_2\left(r_2(2-sech^2(\mu x))\right) + \left(2 - 7sech^2(\mu x) + 5sech^4(\mu x)b_2\right)}{\Gamma(\alpha+\beta+1)} \\ &+ \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\beta+\gamma+\eta+1)}\Big(12t^{\alpha+\beta}\mu\Gamma(\alpha+\beta+1)sech^5(\mu x)b_2(6(f_1t_0+r_3t_1)\cosh(\mu x) \\ &- 3(f_1t_0+r_3t_1)\cosh(3\mu x) - 6(f_1t_1 - 2r_2\mu^2)(-5\sinh(\mu x) + \sinh(\mu x)) \\ &- 2(3r_1 - 8\mu^2)(-5\sinh(\mu x) + \sinh(3\mu x))b_2)\Big) \\ &+ \frac{1}{\Gamma(\gamma+1)\Gamma(\eta+1)\Gamma(\beta+\gamma+\eta+1)}\left(12f_1t_1t^{\gamma+\eta}\mu(3r_1-2\mu^2)^2\Gamma(\gamma+\eta+1)sech^2(\mu x)\tanh(\mu x)\right) \\ &+ \frac{1}{\Gamma(2\beta+1)}4t^{\beta}\Big((-6r_1 + 4\mu^2 + 9(r_1+2\mu^2)sech^(\mu x) - 30\mu^2 sech^4(\mu x))(3f_1t_1 - 6r_2\mu^2 + (3r_1 - 8\mu^2)b_2) \\ &- 3(f_1t_0 + r_3t_1)(3r_1 - 2\mu^2)\tanh(\mu x)\Big)\Bigg), \end{aligned}$$

$$
\begin{aligned}
w_2(x,t) =& I^{\gamma}\left(\Big(w_1(x,t)\Big)_{xxx}\right)\\
&+ I^{\gamma}\left(3\Big(u_0(x,t)+u_1(x,t)\Big)\Big(w_0(x,t)+w_1(x,t)\Big)_x\right)\\
&- I^{\gamma}\left(3u_0(x,t)\Big(w_0(x,t)\Big)_x\right)\times\\
& f_1 t^{\gamma}\mu^2 sech^4(\mu x)\tanh(\mu x)\Bigg(-\frac{t^{\gamma}(3r_1-2\mu^2)\Big(3r_1-14\mu^2+(3r_1-2\mu^2)\cosh(2\mu x)\Big)}{\Gamma(2\gamma+1)}\\
&+\frac{6t^{\alpha}b_2}{\Gamma(\alpha+\gamma+1)}\Big(1+\frac{2t^{\gamma}\mu(-3r_1+2\mu^2)(\Gamma(\alpha+\gamma+1))^2\tanh(\mu x)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+2\gamma+1)}\Big)\Bigg),\\
p_2(x,t) =& I^{\eta}\left(\Big(p_1(x,t)\Big)_{xxx}\right)+I^{\eta}\left(3\Big(u_0(x,t)+u_1(x,t)\Big)\Big(p_0(x,t)+p_1(x,t)\Big)_x\right)\\
&- I^{\eta}\left(3u_0(x,t)\Big(p_0(x,t)\Big)_x\right)\\
=& t_1 t^{\eta}\mu^2 sech^4(\mu x)\tanh(\mu x)\Bigg(-\frac{t^{\eta}(3r_1-2\mu^2)\Big(3r_1-14\mu^2+(3r_1-2\mu^2)\cosh(2\mu x)\Big)}{\Gamma(2\eta+1)}\\
&+\frac{6t^{\alpha}b_2}{\Gamma(\alpha+\eta+1)}\Big(1+\frac{2t^{\eta}\mu(-3r_1+2\mu^2)(\Gamma(\alpha+\eta+1))^2\tanh(\mu x)}{\Gamma(\alpha+1)\Gamma(\eta+1)\Gamma(\alpha+2\eta+1)}\Big)\Bigg)
\end{aligned}
$$

The three terms approximate solution of $u(x,t),\ v(x,t),\ w(x,t)$ and $p(x,t)$ takes the form:

$$
\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)\\
v(x,t) &= v_0(x,t)+v_1(x,t)+v_2(x,t)\\
w(x,t) &= w_0(x,t)+w_1(x,t)+w_2(x,t)\\
p(x,t) &= p_0(x,t)+p_1(x,t)+p_2(x,t)
\end{aligned}
$$

5. Tables and figures

TABLES 1-3 show the comparison of the absolute errors of 3rd order (NIM) solution of parts of $u(x,t)$ and $v(x,t)$ coupled ITO system (4.1) with 3rd order (HPM). TABLE 2 shows comparison of 3rd order (NIM) of $v(x,t)$ using fractional values of α keeping $\beta=1$ with 3rd order (HPM) solution while TABLE 4 shows comparison of 3rd order (NIM) of $v(x,t)$ using fractional values of β keeping $\alpha=1$ with 3rd order (HPM) solution. TABLES 5-8 display the solution of $u(x,t),\ v(x,t),\ w(x,t)$ and $p(x,t)$ parts of time fractional coupled ITO system (4.3).

FIGURE 1(a) shows the 2D plot of $u(x,t)$ part of system (4.1) using fractional values of α keeping $\beta=1$ at $t=0.1$, while FIGURE 1(b) shows the 2D plot of $v(x,t)$ part of system (4.1) using fractional values of β keeping $\alpha=1$ at $t=0.1$. FIGURE 2 displays the 2D plots of absolute errors of $u(x,t),\ v(x,t)$ parts of coupled ITO system (4.1) at $t=0.1$. Similarly FIGURE 3 displays the 3D plots of $u(x,t),\ v(x,t)$ parts of system (4.1) using values of $\alpha=\beta=1$. FIGURES 4 and 5 display the 2D plots of $u(x,t),\ v(x,t),\ w(x,t),\ p(x,t)$ using different fractional values of $\alpha,\ \beta,\ \gamma,\ \eta$ of coupled ITO system (4.3). FIGURES 6 and 7 display the 3D plots of $u(x,t),\ v(x,t),\ w(x,t),\ p(x,t)$ parts of system (4.3) using $\alpha=\beta=\gamma=\eta=1$. Throughout for computations of system (4.1) we used $a=1,\ c=9$ while for computations of system (4.3) we choose values of parameters to be $\mu=0.5,\ f_1=0.6,\ b_2=0.03,\ t_0=-0.4$ and $t_1=-0.1$.

Table 1: Comparison of 3rd order (NIM) solution of $u(x,t)$ with 3rd order (HPM) solution using $a = 1,\ \alpha = \beta = 1$ and $c = 9$ of ITO System (4.1).

(x,t)	$u(x,t)$ at $\alpha = 1$	Exact Solution	Absolute Error of 3rd (HPM)[1]	Absolute Error of 3rd order (NIM)
(0.1,0.1)	0.00358423	0.00358423	4.42497×10^{-9}	2.90082×10^{-9}
(0.2,0.1)	0.00716845	0.00716846	8.84995×10^{-9}	5.80163×10^{-9}
(0.3,0.1)	0.01075270	0.01075270	1.32749×10^{-8}	8.70245×10^{-9}
(0.4,0.1)	0.01433690	0.01433690	1.76999×10^{-8}	1.16033×10^{-8}
(0.5,0.1)	0.01792110	0.01792110	2.21249×10^{-8}	1.45041×10^{-8}
(0.1,0.2)	0.00347218	0.00347222	6.85871×10^{-8}	4.42006×10^{-8}
(0.2,0.2)	0.00694436	0.00694444	1.37174×10^{-7}	8.84012×10^{-8}
(0.3,0.2)	0.01041650	0.01041670	2.05761×10^{-7}	1.32602×10^{-7}
(0.4,0.2)	0.013888870	0.01388890	2.74348×10^{-7}	1.76802×10^{-7}
(0.5,0.2)	0.01736090	0.01736110	3.42936×10^{-7}	2.21003×10^{-7}
(0.1,0.3)	0.00336679	0.00336700	3.367×10^{-7}	2.13244×10^{-7}
(0.2,0.3)	0.00673358	0.00673401	6.73401×10^{-7}	4.26487×10^{-7}
(0.3,0.3)	0.01010040	0.01010100	1.0101×10^{-6}	6.39731×10^{-7}
(0.4,0.3)	0.01346720	0.01346800	1.3468×10^{-6}	8.52974×10^{-7}
(0.5,0.3)	0.01683400	0.01683500	1.6835×10^{-6}	1.06622×10^{-6}

Table 2: Comparison of 3rd order (NIM) solution with 3rd order (HPM) solution of $u(x,t)$ using $a = 1,\ \beta = 1$ and $c = 9$ for different values of α of ITO System. (4.1).

(x,t)	3rd order (HPM) at $\alpha = 0.5$[1]	3rd order (HPM) at $\alpha = 0.75$[1]	3rd order (NIM) at $\alpha = 0.5$	3rd order (NIM) at $\alpha = 0.75$
(0.1,0.1)	0.00328086	0.00347345	0.00328091	0.00347346
(0.2,0.1)	0.00656173	0.00694689	0.00656182	0.00694691
(0.3,0.1)	0.00984259	0.01042030	0.00984273	0.01042040
(0.4,0.1)	0.01312350	0.013893880	0.01312360	0.01389380
(0.5,0.1)	0.01640430	0.01736720	0.01640450	0.01736730
(0.1,0.2)	0.00312764	0.0032961	0.00312801	0.00332971
(0.2,0.2)	0.00625529	0.00665923	0.00625602	0.00665942
(0.3,0.2)	0.00938293	0.00998884	0.00938403	0.00998914
(0.4,0.2)	0.01251060	0.01331850	0.01251200	0.01331880
(0.5,0.2)	0.01563820	0.01664810	0.01564010	0.01664860
(0.1,0.3)	0.00302194	0.00321282	0.00302318	0.00321323
(0.2,0.3)	0.00604389	0.00642565	0.00604636	0.00642646
(0.3,0.3)	0.00906583	0.00963847	0.00906953	0.00963969
(0.4,0.3)	0.01208780	0.01285130	0.01209270	0.01285290
(0.5,0.3)	0.01510970	0.01606410	0.01511590	0.01606610

Table 3: Comparison of 3rd order (NIM) solution with 3rd order (HPM) solution of $v(x,t)$ using $a = 1,\ \beta = 1$ and $c = 9$ of ITO System. (4.1).

(x,t)	$v(x,t)$ at $\beta = 1$	Exact Solution	Absolute Error of 3rd (HPM)[1]	Absolute Error of 3rd order (NIM)
(0.1,0.1)	-0.0000578100	-0.0000578102	3.66369×10^{-10}	1.29409×10^{-10}
(0.2,0.1)	-0.0002312400	-0.0002312410	1.46548×10^{-9}	5.17637×10^{-10}
(0.3,0.1)	-0.0005202900	-0.0005202910	3.29732×10^{-9}	1.16468×10^{-9}
(0.4,0.1)	-0.0009249600	-0.0009249620	5.8619×10^{-9}	2.07055×10^{-9}
(0.5,0.1)	-0.0014452500	-0.0014452500	9.15922×10^{-9}	3.23523×10^{-9}
(0.1,0.2)	-0.0000542516	-0.0000542535	5.64415×10^{-9}	1.92144×10^{-9}
(0.2,0.2)	-0.0002170060	-0.0002170140	2.25766×10^{-8}	7.68575×10^{-9}
(0.3,0.2)	-0.0004882640	-0.0004882810	5.07973×10^{-8}	1.72929×10^{-8}
(0.4,0.2)	-0.0008680250	-0.0008680560	9.03064×10^{-8}	3.0743×10^{-8}
(0.5,0.2)	-0.0013562900	-0.0013563400	1.41104×10^{-7}	4.8036×10^{-8}
(0.1,0.3)	-0.0000510062	-0.0000510152	2.75842×10^{-8}	9.04341×10^{-9}
(0.2,0.3)	-0.0002040250	-0.0002040610	1.10193×10^{-7}	3.61736×10^{-8}
(0.3,0.3)	-0.0004590550	-0.0004591370	2.47934×10^{-7}	8.13907×10^{-8}
(0.4,0.3)	-0.0008160990	-0.0008162430	4.40771×10^{-7}	1.44695×10^{-7}
(0.5,0.3)	-0.0012751500	-0.0012753800	6.88705×10^{-7}	2.26085×10^{-7}

Table 4: Comparison of 3rd order (NIM) solution with 3rd order (HPM) solution of $v(x,t)$ using $a = 1$, $\alpha = 1$ and $c = 9$ for different values of β of ITO System. (4.1).

(x,t)	3rd order (HPM) at $\beta = 0.5$	3rd order (HPM) at $\beta = 0.75$	3rd order (NIM) at $\beta = 0.5$	3rd order (NIM) at $\beta = 0.75$
(0.1,0.1)	-0.0000495189	-0.0000544972	-0.0000495483	-0.000054000
(0.2,0.1)	-0.0001980760	-0.0002179890	-0.0001981930	-0.0002180000
(0.3,0.1)	-0.0004456700	-0.0004904750	-0.0004459340	-0.0004905000
(0.4,0.1)	-0.0007923030	-0.0008719560	-0.0007927720	-0.0008720010
(0.5,0.1)	-0.0012379700	-0.0013624300	-0.0012387100	-0.0013625000
(0.1,0.2)	-0.0000455010	-0.0000503167	-0.0000456639	-0.0000503432
(0.2,0.2)	-0.0001820040	-0.0002012670	-0.0001826560	-0.0002013730
(0.3,0.2)	-0.0004095090	-0.0004528500	-0.0004109750	-0.0004530890
(0.4,0.2)	-0.0007280170	-0.0008050670	-0.0007306220	-0.0008054910
(0.5,0.2)	-0.0011375300	-0.0012579200	-0.0011416000	-0.0012585800
(0.1,0.3)	-0.0000425196	-0.0000470196	-0.0000429608	-0.0000471172
(0.2,0.3)	-0.0001700790	-0.0001880790	-0.0001718430	-0.0001884690
(0.3,0.3)	-0.0003826770	-0.0004231770	-0.0003866470	-0.0004240540
(0.4,0.3)	-0.0006803140	-0.0007523140	-0.0006873730	-0.0007538750
(0.5,0.3)	-0.0010629900	-0.0011754900	-0.0010740200	-0.0011779300

Table 5: 2nd order (NIM) solution of $u(x,t)$ using $\beta = \gamma = \eta = 1$ for different values of α for ITO System. (4.3).

(x,t)	$u(x,t)$ at $\alpha = 0.5$	$u(x,t)$ at $\alpha = 0.75$	$u(x,t)$ at $\alpha = 1$	Exact Solution	Absolute Error
(-2,0.2)	-0.148169	-0.146458	-0.145260	-0.145260	1.1822×10^{-8}
(-2,0.4)	-0.150141	-0.148562	-0.147162	-0.147162	9.33867×10^{-8}
(-2,0.6)	-0.151635	-0.150393	-0.149053	-0.149053	3.11189×10^{-7}
(-2,0.8)	-0.152880	-0.152060	-0.150933	-0.150932	7.28229×10^{-7}
(-1,0.2)	0.0343715	0.0363332	0.0377049	0.0377049	7.10506×10^{-8}
(-1,0.4)	0.0320613	0.0338919	0.0355090	0.0355096	5.67671×10^{-7}
(-1,0.6)	0.0302720	0.0317384	0.0333030	0.0333049	1.91335×10^{-6}
(-1,0.8)	0.0287498	0.0297486	0.0310868	0.0310913	4.52919×10^{-6}
(1,0.2)	0.0453753	0.0434290	0.0420660	0.0420660	7.1227×10^{-8}
(1,0.4)	0.0476230	0.0458255	0.0442314	0.0442308	5.70493×10^{-7}
(1,0.6)	0.0493312	0.0479133	0.0463865	0.0463846	1.92763×10^{-6}
(1,0.8)	0.0507574	0.0498185	0.0485315	0.0485269	4.57431×10^{-6}
(2,0.2)	-0.138485	-0.140213	-0.141421	-0.141421	1.21221×10^{-8}
(2,0.4)	-0.136445	-0.138060	-0.139486	-0.139486	9.81878×10^{-8}
(2,0.6)	-0.134862	-0.136157	-0.137539	-0.137539	3.35494×10^{-7}
(2,0.8)	-0.133511	-0.134397	-0.135580	-0.135581	8.05044×10^{-7}

Table 6: 2nd order (NIM) solution of $v(x,t)$ using $\alpha = \gamma = \eta = 1$ for different values of β for ITO System. (4.3).

(x,t)	$v(x,t)$ at $\beta = 0.5$	$v(x,t)$ at $\beta = 0.75$	$v(x,t)$ at $\beta = 1$	Exact Solution	Absolute Error
(-2,0.2)	-0.133983	-0.134183	-0.134284	-0.134284	3.18723×10^{-7}
(-2,0.4)	-0.133786	-0.134020	-0.134168	-0.134170	2.54971×10^{-6}
(-2,0.6)	-0.133644	-0.133873	-0.134048	-0.134057	8.60504×10^{-6}
(-2,0.8)	-0.133536	-0.133735	-0.133924	-0.133944	2.03966×10^{-5}
(-1,0.2)	-0.145050	-0.145177	-0.145262	-0.145262	4.0942×10^{-7}
(-1,0.4)	-0.144899	-0.145023	-0.145127	-0.145131	3.27532×10^{-6}
(-1,0.6)	-0.144774	-0.144881	-0.144987	-0.144998	1.1054×10^{-5}
(-1,0.8)	-0.144656	-0.144740	-0.144839	-0.144865	2.62018×10^{-5}
(1,0.2)	-0.145713	-0.145604	-0.145524	-0.145524	4.09431×10^{-7}
(1,0.4)	-0.145847	-0.145749	-0.145657	-0.145654	3.27549×10^{-6}
(1,0.6)	-0.145961	-0.145883	-0.145794	-0.145783	1.10549×10^{-5}
(1,0.8)	-0.146072	-0.146015	-0.145938	-0.145912	2.62045×10^{-5}
(2,0.2)	-0.134568	-0.134560	-0.134515	-0.134515	3.18741×10^{-7}
(2,0.4)	-0.134626	-0.134660	-0.134633	-0.134631	2.55×10^{-6}
(2,0.6)	-0.134697	-0.134756	-0.134756	-0.134748	8.6065×10^{-6}
(2,0.8)	-0.134789	-0.134875	-0.134886	-0.134865	2.04012×10^{-5}

Table 7: 2nd order (NIM) solution of $w(x,t)$ using $\alpha = \beta = \eta = 1$ for different values of γ for ITO System. (4.3).

(x,t)	$w(x,t)$ at $\gamma = 0.5$	$w(x,t)$ at $\gamma = 0.75$	$w(x,t)$ at $\gamma = 1$	Exact Solution	Absolute Error
(-2,0.2)	-2.86022	-2.85928	-2.85846	-2.85849	4.55402×10^{-7}
(-2,0.4)	-2.86143	-2.86078	-2.85995	-2.85995	3.64338×10^{-6}
(-2,0.6)	-2.86240	-2.86207	-2.86142	-2.86143	1.2297×10^{-5}
(-2,0.8)	-2.86328	-2.86325	-2.86287	-2.86289	2.91497×10^{-5}
(-1,0.2)	-2.68330	-2.68162	-2.68009	-2.68009	5.58514×10^{-7}
(-1,0.4)	-2.68557	-2.68443	-2.68290	-2.68290	4.46913×10^{-6}
(-1,0.6)	-2.68742	-2.68687	-2.68568	-2.68569	1.50868×10^{-5}
(-1,0.8)	-2.68912	-2.68912	-2.68843	-2.68847	3.57694×10^{-5}
(1,0.2)	-2.13097	-2.12759	-2.12557	-2.12557	5.58258×10^{-7}
(1,0.4)	-2.13458	-2.13104	-2.12842	-2.12842	4.46504×10^{-6}
(1,0.6)	-2.13721	-2.13407	-2.13128	-2.13129	1.5066×10^{-5}
(1,0.8)	-2.13925	-2.13681	-2.13414	-2.13418	3.57038×10^{-5}
(2,0.2)	-1.94740	-1.94563	-1.94456	-1.94456	4.55359×10^{-7}
(2,0.4)	-1.94932	-1.94747	-1.94606	-1.94610	3.64269×10^{-6}
(2,0.6)	-1.95072	-1.94909	-1.94763	-1.94764	1.22935×10^{-5}
(2,0.8)	-1.95182	-1.95056	-1.94917	-1.94920	2.91386×10^{-5}

Table 8: 2nd order (NIM) solution of $p(x,t)$ using $\alpha = \beta = \gamma = 1$ for different values of η for ITO System. (4.3).

(x,t)	$p(x,t)$ at $\eta = 0.5$	$p(x,t)$ at $\eta = 0.75$	$p(x,t)$ at $\eta = 1$	Exact Solution	Absolute Error
(-2,0.2)	-0.323297	-0.323453	-0.323590	-0.323590	7.59003×10^{-8}
(-2,0.4)	-0.323095	-0.323203	-0.323342	-0.323341	6.0723×10^{-7}
(-2,0.6)	-0.322933	-0.322988	-0.323097	-0.323095	2.04949×10^{-6}
(-2,0.8)	-0.322787	-0.322791	-0.322856	-0.322851	4.85828×10^{-6}
(-1,0.2)	-0.352783	-0.353064	-0.353318	-0.353318	9.30857×10^{-8}
(-1,0.4)	-0.352405	-0.352595	-0.352851	-0.352850	7.44856×10^{-7}
(-1,0.6)	-0.352097	-0.352189	-0.352387	-0.352385	2.51446×10^{-6}
(-1,0.8)	-0.351813	-0.351814	-0.351928	-0.351922	5.96156×10^{-6}
(1,0.2)	-0.444838	-0.445401	-0.445739	-0.445739	9.3043×10^{-8}
(1,0.4)	-0.444236	-0.444826	-0.445264	-0.445263	7.44173×10^{-7}
(1,0.6)	-0.443798	-0.444321	-0.444787	-0.444784	2.511×10^{-6}
(1,0.8)	-0.443459	-0.443865	-0.444309	-0.444303	5.95063×10^{-6}
(2,0.2)	-0.475433	-0.475728	-0.475906	-0.475906	7.58931×10^{-8}
(2,0.4)	-0.475114	-0.475421	-0.475651	-0.475651	6.07115×10^{-7}
(2,0.6)	-0.474880	-0.475151	-0.475395	-0.475393	2.04891×10^{-6}
(2,0.8)	-0.474697	-0.474906	-0.475138	-0.475133	4.85644×10^{-6}

(a) 2D plot of 3rd order (NIM) solution of $u(x,t)$ using different values of α with exact solution at $t = 0.1$

(b) 2D plot of 3rd order (NIM) solution of $v(x,t)$ using different values of β with exact solution at $t = 0.1$

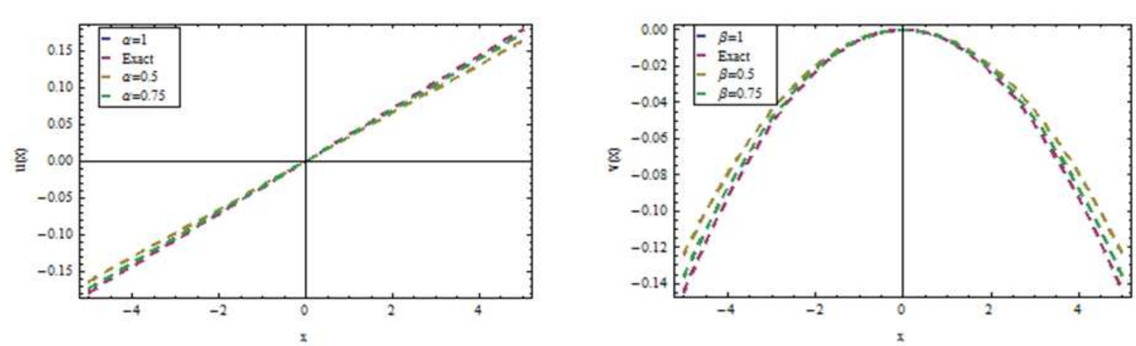

Figure 1: 2D plot of 3rd order (NIM) solutions

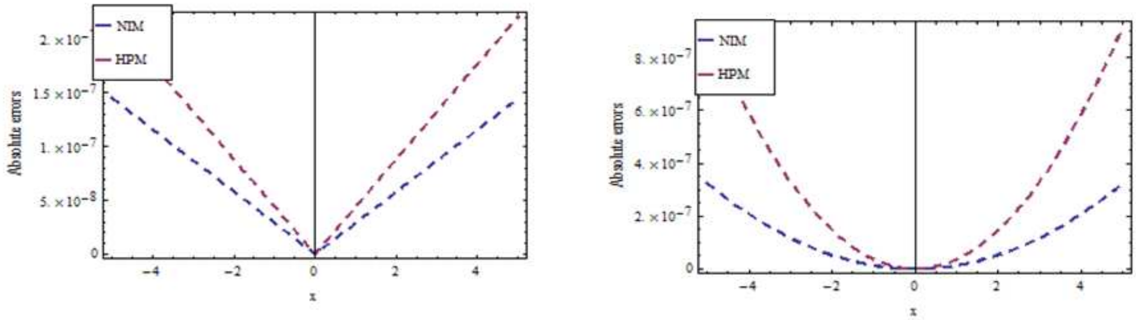

Figure 2: Comparison of absolute errors of (NIM) and (HPM) solution of $u(x,t),\ v(x,t)$ parts of ITO system (4.1)

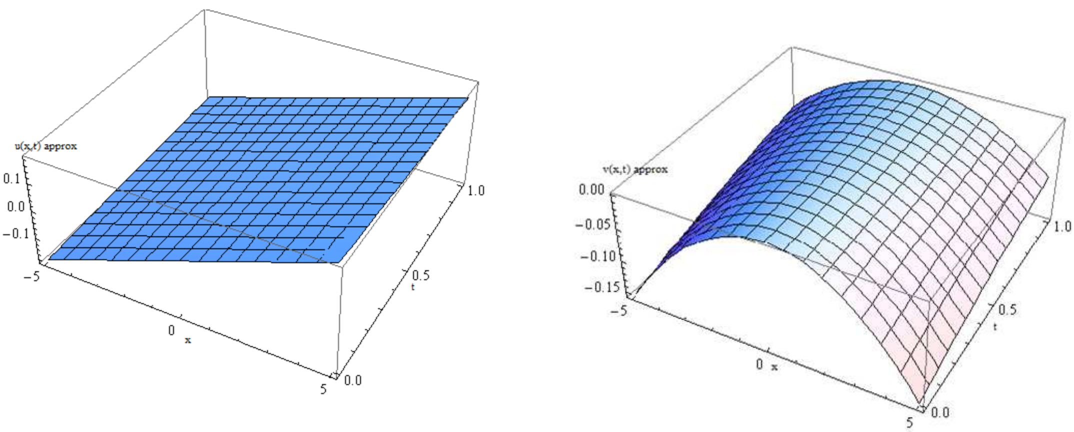

Figure 3: 3D plots of $u(x,t),\ v(x,t)$ parts of system (4.1) using values of $\alpha=\beta=1$

(a) 2D plot of $u(x,t)$ for different values of α keeping $\beta=\gamma=\eta=1$ at $t=5$ (yellow for $\alpha=0.2$, dashed for $\alpha=0.4$, green for $\alpha=0.6$, dots for $\alpha=0.8$ and blue for $\alpha=1$)

(b) 2D plot of $v(x,t)$ for different values of β keeping $\alpha=\gamma=\eta=1$ at $t=5$ (yellow for $\beta=0.2$, dashed for $\beta=0.4$, green for $\beta=0.6$, dots for $\beta=0.8$ and blue for $\beta=1$)

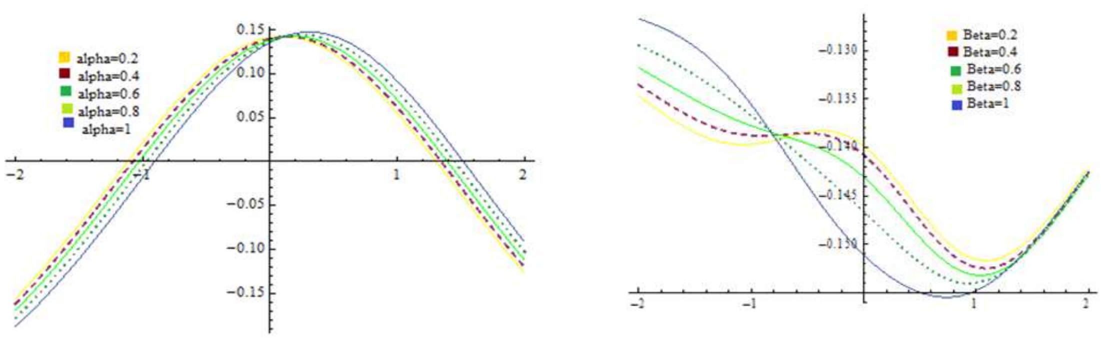

Figure 4: 2D plot of $u(x,t),\ v(x,t)$

(a) 2D plot of $w(x,t)$ for different values of γ keeping $\alpha = \beta = \eta = 1$ at $t = 5$ (yellow for $\gamma = 0.2$, dashed for $\gamma = 0.4$, green for $\gamma = 0.6$, dots for $\gamma = 0.8$ and blue for $\gamma = 1$)

(b) 2D plot of $p(x,t)$ for different values of η keeping $\alpha = \beta = \gamma = 1$ at $t = 5$ (yellow for $\eta = 0.2$, dashed for $\eta = 0.4$, green for $\eta = 0.6$, dots for $\eta = 0.8$ and blue for $\eta = 1$)

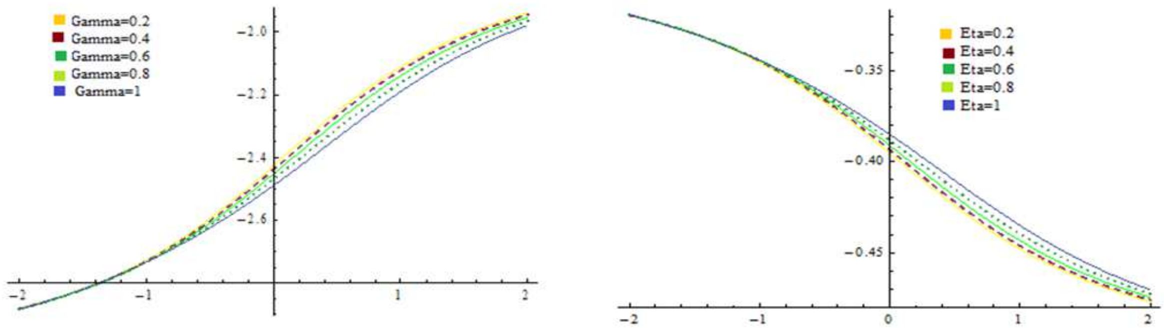

Figure 5: 2D plot of $w(x,t)$, $p(x,t)$

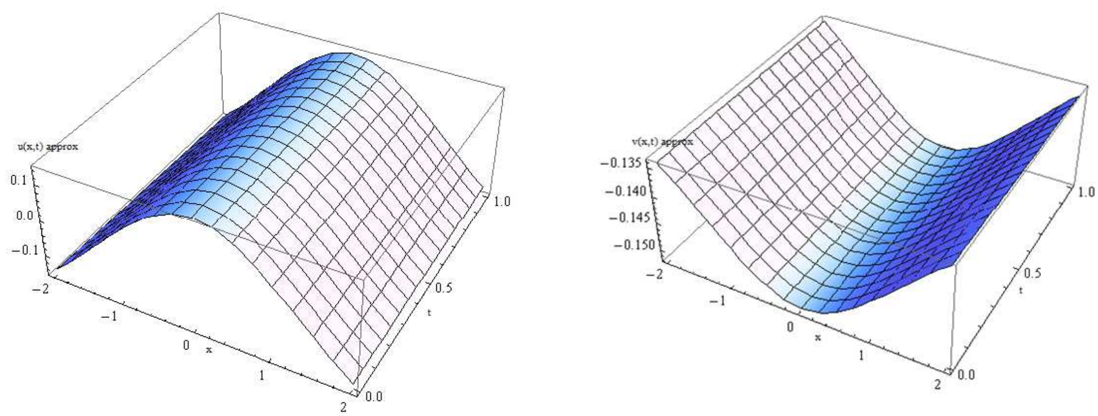

Figure 6: 3D plots of $u(x,t)$, $v(x,t)$ parts of system (4.3) using values of $\alpha = \beta = 1$

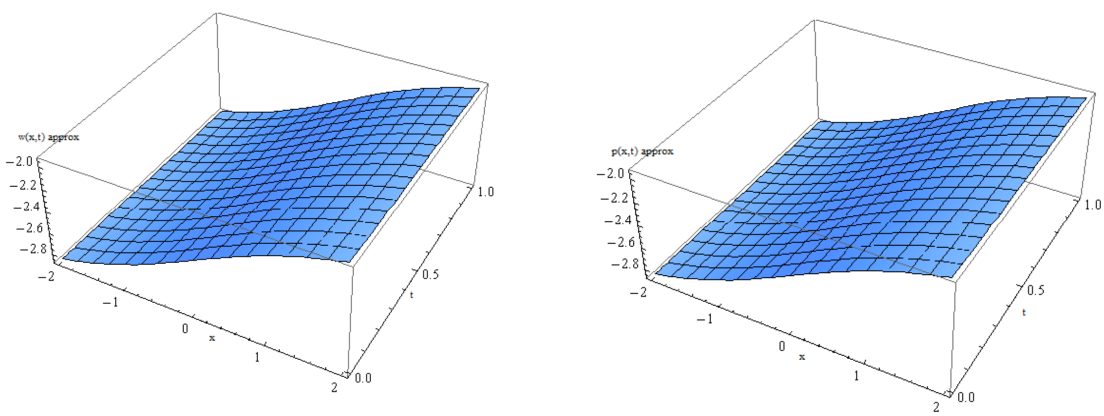

Figure 7: 3D plots of $w(x,t)$, $p(x,t)$ parts of system (4.3) using values of $\gamma = \eta = 1$

6. Conclusion

Approximate solutions of ITO systems with time fractional derivatives have been obtained by successful application of (NIM). In the recent development of fractional order differential equations in some fields of applied mathematics, conceive it necessary to inspect methods of solutions for such types of equations and we anticipate that this work is a step in towards solutions of fractional problems. It has been observed that approximate solutions by extended formulation are in excellent agreement with the exact solutions. Consistent accuracy throughout the domain of the problems is also observed. The accuracy of proposed method can further be increased by taking higher iterations.

References

1. S. Kumar, *A New Efficient Algorithm to Solve Non-Linear Fractional Ito Coupled System and Its Approximate Solution*, Walailak, J, Sci and Tech., 11(12), 1057-1067, (2014).
2. G. Adomian, *Nonlinear Stochastic Differential Equations*, J. Math. Anal. Appl., 55, 441-452, (1976).
3. G. Adomian, *Review of the Decomposition Method and some Recent Results for Nonlinear Equations*, Comput. Math. Appl.,21, 101-127, (1991).
4. J. H. He, *Approximate Solution of Nonlinear Differential Equations with Convolution Product Nonlinearities*, Comput. Meth. Appl. Mech. Eng., 167(6), 69-73, (1998).
5. J. H. He, *Variational Iteration Method a Kind of Non-Linear Analytical Technique: Some Examples*, Int. Nonl. Mech., 34, 699-708, (1999).
6. J. H. He, *Homotopy Perturbation Technique*, Comp. Meth. Appl. Mech. Eng., 178, 257-262, (1999).
7. J. H. He, *Application of Homotopy Perturbation Method to Nonlinear Wave Equations*, Chao. Solit. Fract, 26, 695-700, (2005).
8. M. M. Rashidi, D. D. Ganji and S. Dinarvand, *Explicit Analytical Solutions of the Generalized Burger and Burger Fisher Equations by Homotopy Perturbation Method*, Num. Meth. Part. Diff. Eq., 25, 409-417, (2009).
9. M. M. Rashidi, N. Freidoonimehr, A. Hosseini, B. O. Anwar and T. K. Hung, *Homotopy Simulation of Nano fluid Dynamics from a Nonlinearly Stretching Isothermal Permeable Sheet with Transpiration*, Mecc., 49, 469-482, (2014).
10. S. J. Liao, *On the Homotopy Analysis Method for Nonlinear Problems*, Appl. Math. Comput., 147(4), 499-513, (2004).
11. S. Abbasbandy, *Solitary Wave Solutions to the Modified Form of Camassa Holm Equation by Means of the Homotopy Analysis Method*, Chao. Solit. Fract., 39, 428-435, (2009).
12. V. Marinca, N. Herisanu, *An Optimal Homotopy Asymptotic Method for Solving Nonlinear equations arising in Heat Transfer*, Int. Com. Heat Mass tran., 710-715, (2008).
13. V. Marinca, N. Herisanu, and I. Nemes, *Optimal Homotopy Asymptotic Method with Application to Thin Film Flow*, Cent. Eur. J. Phys., 6, 648-653, (2008).
14. V. Marinca, N. Herisanu, and Gh. Madescu, *An Analytical Approach to non-Linear Dynamical Model of a Permanent Magnet Synchronous Generator*, Wind Energy, 18(9), 1657-1670, (2015).
15. K. Shah, H. Khalil, R. Ali Khan, *A Generalized Scheme Based on Shifted Jacobi Polynomials for Numerical Simulation of Coupled Systems of Multi-Term Fractional-Order Partial Differential Equations*, Cambridge University Press, 20, 11-29, (2017).
16. K. Shah, J. R. Wang, *A Numerical Scheme Based on Non-Discretization of Data for Boundary Value Problems of Fractional Order Differential Equations*, Revista de la Real Academia de Ciencias Exaxtas. 113(3), 2277-2294, (2019).
17. Y. Li, K. Shah, R. Ali Khan, *Iterative Technique for Coupled Integral Boundary Value Problem of Non-Integer Order Differential Equations*, Advances in Difference Equations, 2017(1), DOI: 10.1186/s13662-017-1308-x, (2017).
18. S. Ali, K. Shah, Y. Li, M. Arif, *Optimum Solutions of Space Fractional Order Diffusion Equation*, Thermal Sciences, DOI: 10.2298/TSCI171120036A, (2018).
19. I. Masaaki, *An Extension of Non Linear Evolution Equations K-dv and (mk-dv) Type to Higher Orders*, Journal of Physical Society of Japan, 49(2), 771-778, (1980).
20. L. Diosi, T. Konrad, A. Scherer, Audretsch, *Coupled ITO Equations of Continuous Quantum State Measurement and Estimation*, arXiv:quant-ph/10608097v2, (2006).
21. Ann J. Al-Sawoor, M. O. Al-Amr, *Reduced Differential Transform Method for the Generalized Ito System*, Int. J.Enh. Resea. Sci. Tech. Eng., 2, 135-145, (2013).
22. H. M. Jaradat, I. J. Marwan Alquran, M. M. M. J. Zead Mustafa and K. M. A. Ra'ed Abdelkarim, *Approximate Solutions to the Generalized Time-Fractional ITO System*, Ital. J. Pur. Appl. Math., 37, 699-710, (2017).
23. E. C. Aslan, M. Inc, *Numerical Solutions and Comparisons for Nonlinear Time fractional Ito Coupled System*, J. Comput. Theor. Nano Science, 13, 5426-5431, (2016).

24. K.B. Oldham, J. Spanier, *Fractional Calculus*, Academic Press, New York, (1974).

25. I. Podlubny, *Fractional Differential Equations*, Academic Press New York (1999).

26. V. Daftardar-Gejji, H. Jafari, *An iterative method for solving nonlinear functional equations*, J. Math. Anal. Appl, 316, 753-763, (2006).

27. V. Daftardar-Gejji, S. Bhalekar, *Solving Boundary Value Problems with Dirichlet Boundary Conditions using a New Iterative Method*, Comput. Math. Appl., 59, 1801-1809, (2010).

28. Bhalekar, S. and Daftardar-Gejji, V., *Convergence of New iterative method*, International Journal of Differential Equations, Article ID 989065, 10 pages, (2011).

29. Noor, M.A. and Noor, K.I., *Some iterative schemes for nonlinear equations*, Applied Mathematics and Computation, 183, 774-779, (2006).

Rashid Nawaz, Samreen Farid,
Department of Mathematics,
Abdul Wali Khan University, Mardan,
Pakistan.
E-mail address: rashid_uop@yahoo.com, samahmad139@gmail.com

and

Samia Bushnaq,
Department of Basic Sciences,
Princess Sumaya University for Technology,
Jordan.
E-mail address: s.bushnaq@psut.edu.jo